HOMOLOGICAL STABILITY FOR $O_{n,n}$ OVER SEMI-LOCAL RINGS

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Introduction. Let R be a commutative, semi-local ring. Let $O_{n,n}$ be the group of linear automorphisms of R^{2n} which preserve the bilinear form $\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$. The main result of this paper is the following theorem.

THEOREM A. *The natural inclusion of $O_{n,n}$ into $O_{n+1,n+1}$ induces an isomorphism on the* ith *homology group if only n is large enough with respect to i.*

This note is a direct continuation of my [**2**] and the methods are the same. I will use the notation, technical methods and some definitions from [**2**] without writing them down again (especially in Section 2). It was possible to extend my result from local to semi-local rings because of Charney [**4**]. She proved there that, if the poset of isotropic unimodular vectors in R^{2n} (see Section 1 for the definition) is highly connected, then the conclusion of my Theorem A holds for every ring R for which the dimension of $\max(R)$ (the maximal spectrum of R) is finite. So in the following paper I will prove only that the poset of isotropic unimodular vectors is highly connected for semi-local rings.

It is expected that the homological stability theorem for $O_{n,n}$ is true for all rings with finite dimension for the maximal spectrum. Unfortunately it is obvious that the methods used in the present note cannot be applied for rings R which have $\dim(\max(R))$ larger than 0. Moreover my methods do not give any well expressed relation between n and i. Charney in [**4**] proved the homological stability theorem for $O_{n,n}$ over Dedekind domains under the hypothesis $2i+5 \le n$. Panin in [**5**] and [**6**] obtained similar results to Charney's theorem under the assumption $2i+1 \le n$ using a slightly different method from that used in [**4**]. Roughly speaking Charney compared the situation over Dedekind domain with that over its field of fractions and then used well known facts for fields (of course she also used her theorem which is mentioned above). Panin adjusted Suslin's approach to homological stability problems (see [**8**]) to the situation with a bilinear or quadratic form (he also used [**4**]). There is some hope that Charney's or Panin's method might be extended to rings with higher dimension for the maximal spectrum. Panin in [**5**] and [**6**] also claimed homological stability theorem as above for rings which had no finite residue fields but the proof was written only in the case of an infinite field.

Section 1. Let R be a ring and q_n be the bilinear form on R^{2n} which has the matrix $\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$ in the canonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for R^{2n}. A sequence $(v_1, \ldots, v_k)$ of vectors from R^{2n} is called unimodular if it generates a free direct summand of R^{2n} of rank k. A submodule M of R^{2n} is called isotropic if $q_n|_M = 0$.

Let $X(R^{2n})$ be the set of all unimodular sequences $(v_1, \ldots, v_k)$ in R^{2n} which generate isotropic submodules of R^{2n}. This set has a natural partial ordering by inclusion so it forms a poset (see [7] for the definition and the properties of posets). We can consider the topological space $|X(R^{2n})|$ which is the geometric realization of the poset $X(R^{2n})$. In [4] Charney proved that if R is a ring with $\dim(\max(R)) = d < \infty$ and the space $|X(R^{2n})|$ is highly connected for large n, then the natural inclusion $O_{n,n}(R) \to O_{n+1,n+1}(R)$ induces an isomorphism on the ith homology group provided $i \ll n$ (see [4, Theorems 3.2 and 4.4]). Hence our goal is to prove that $X(R^{2n})$ is highly connected for semi-local rings. We will say that a poset is highly connected if its geometric realization is highly connected.

1.1. Definition. Let $Y(R^{2n})$ be a simplicial complex obtained by the following rule:
(a) the set of vertices consists of all unimodular, isotropic vectors of R^{2n};
(b) the set $\{v_1, \ldots, v_k\}$ forms a $(k-1)$-simplex iff $\{v_1, \ldots, v_k\}$ generates a k-dimensional free isotropic direct summand of R^{2n}.

1.2. Theorem. *If R is a semi-local ring and $n \gg i$ then $Y(R^{2n})$ is i-connected (the sign $\gg$ always means "much bigger").*

The proof of this theorem will occupy the next section. Now we will prove that Theorem 1.2 implies Theorem A. From now on R always denotes a commutative semi-local ring. By the previous discussion about Charney's results it is enough to show:

1.3. Proposition. *If $Y(R^{2n})$ is i-connected and n is large enough then $X(R^{2n})$ is i-connected.*

Proof. Let $\bar{Y}(R^{2n})$ be the poset of unordered sets $\{v_1, \ldots, v_k\}$, where each $\{v_1, \ldots, v_k\}$ forms a simplex in $Y(R^{2n})$ and $\bar{Y}(R^{2n})$ is ordered by inclusion. Then $|\bar{Y}(R^{2n})|$ is the same as the barycentric subdivision of $Y(R^{2n})$, so high connectivity of $Y(R^{2n})$ implies that $|\bar{Y}(R^{2n})|$ is highly connected. There is the natural map of posets $f: X(R^{2n}) \to \bar{Y}(R^{2n})$ which takes $(v_1, \ldots, v_k)$ to $\{v_1, \ldots, v_k\}$. We would like now to apply Quillen's machinery from [7, Section 9]. We will need two more lemmas.

1.4. Lemma. *For every element $\{v_1, \ldots, v_k\} = w \in \bar{Y}(R^{2n})$ the poset $\bar{Y}(R^{2n})_{>w}$ is highly connected if $k \ll n$.*

Proof. We can assume that $\{v_1, \ldots, v_k\} = \{e_1, \ldots, e_k\}$, where $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ is the canonical basis for R^{2n}. Then $|\bar{Y}(R^{2n})_{>w}|$ is the barycentric subdivision of a simplicial complex Z which is obtained by the following rule:
(a) the set of vertices consists of the simplices of $\bar{Y}(R^{2n})$ which are of the type $\{e_1, \ldots, e_k, v\}$;
(b) $\{e_1, \ldots, e_k, v_1\}, \ldots, \{e_1, \ldots, e_k, v_s\}$ form an $(s-1)$-simplex in Z iff $\{e_1, \ldots, e_k, v_1, \ldots, v_s\}$ forms an $(s+k-1)$-simplex in $Y(R^{2n})$.

It is obvious that $\{e_1, \ldots, e_k, v\}$ is a vertex in Z if $v = (x_1, \ldots, x_n, 0, \ldots, 0, y_{k+1}, \ldots, y_n)$ in the canonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ and $(x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n)$ is a vertex for $Y(R^{2(n-k)})$. So if $k \ll n$ then $n-k$ is large and our lemma follows in the same way as Theorem 1.2.

1.5. Lemma. *For every $\{v_1, \ldots, v_k\} = w \in \bar{Y}(R^{2n})$ the poset f/w is homotopy equivalent to a wedge of spheres of dimension $k-1$.*

Proof. This is an immediate consequence of [3, Theorems 4.1 and 6.1].

So now we can finish the proof of Proposition 1.3. It is an immediate consequence of the following lemma, which is a weaker version of [**7**, Theorem 9.1].

1.6. LEMMA. *Let $f: X \to Y$ be a map of posets of dimension n and let $s \ll n$. Assume that Y is s-connected and that, for every $y \in Y$ such that $h(y)$ (the height of y) is not larger than s, the poset $Y_{>y}$ is s-connected. Moreover, assume that the poset f/y is $h(y)$-spherical for every $y \in Y$. Then X is s-connected.*

Proof. Proceed just as in the proof of [**7**, Theorem 9.1].

Section 2. In this section R always denotes a commutative, semi-local ring with d maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_d$. We will use the machinery which was developed in [**2**, Sections 1 and 2] and we will use the same notation as in [**2**].

2.1. LEMMA. *Let $\{v_1, \ldots, v_s\}$ be an $(s-1)$-simplex in $Y(R^{2n})$. Then there is a vector $v \in R^{2n}$ such that $\{v, v_1, \ldots, v_s\}$ forms an s-simplex in $Y(R^{2n})$ and v has fewer than $\frac{3}{2}ds + 2$ coordinates not equal to 0.*

We will call v "the special vector for $\{v_1, \ldots, v_s\}$" and the way of finding v "the special construction for $\{v_1, \ldots, v_s\}$".

Proof. For every $i = 1, \ldots, d$ the ring $R_{\mathfrak{m}_i}$ is a local ring. Let $\{v_1^i, \ldots, v_s^i\}$ be the image of $\{v_1, \ldots, v_s\}$ in $Y(R^{2n}_{\mathfrak{m}_i})$. It is well known (see [**1**, Chapter IV]) that $\operatorname{span}(v_1, \ldots, v_s)$ is a free s-dimensional direct summand of R^{2n} iff for every $i = 1, \ldots, d$, $\operatorname{span}(v_1^i, \ldots, v_s^i)$ is a free, s-dimensional direct summand of $R^{2n}_{\mathfrak{m}_i}$.

Let $i_1^l, \ldots, i_s^l$ be the numbers of coordinates used in some special construction for $\{v_1^l, \ldots, v_s^l\}$ (see [**2**, Remark 2.5]), $l = 1, \ldots, d$. Let $\{j_1, \ldots, j_u\}$ be the same set as $\{i_1^1, \ldots, i_s^1, \ldots, i_1^d, \ldots, i_s^d\}$ if we count every coordinate only once. Let $\{j_{i_1}, \ldots, j_{i_k}\}$ be the subset of $\{j_1, \ldots, j_u\}$ which consists of the numbers such that for every $1 \le q \le k$, j_{i_q} and $j_{i_q}^*$ belong to the set $\{j_1, \ldots, j_u\}$ (* denotes the dual coordinate; see [**2**] for the definition) and for every $r \neq q$, $j_{i_r} = j_{i_q}^*$, $r, q = 1, \ldots, k$.

Let $p_1, \ldots, p_k$ be numbers such that:

(1) if $i \neq j$ then $p_i \neq p_j$ and $p_i \neq p_j^*$;

(2) for every $1 \le q \le u$ and $1 \le i \le k$, $p_i \neq j_q$ and $p_i \neq j_q^*$.

Let

$$v_1 = (x_1^1, \ldots, x_{2n}^1), \ldots, v_s = (x_1^s, \ldots, x_{2n}^s).$$

Let r be a number such that $r \neq p_1$, $r \neq p_1^*, \ldots, r \neq p_k$, $r \neq p_k^*$, $r \neq j_1, r \neq j_1^*, \ldots, r \neq j_u, r \neq j_u^*$.

Consider the following system of equations:

$$I\begin{cases} x_{j_1}^1 a_{j_1} + \ldots + x_{j_u}^1 a_{j_u} + x_{p_1}^1 a_{p_1} + x_{p_1^*}^1 a_{p_1^*} + \ldots + x_{p_k}^1 a_{p_k} + x_{p_k^*}^1 a_{p_k^*} = -x_r^1, \\ \ldots, \\ x_{j_1}^s a_{j_1} + \ldots + x_{j_u}^s a_{j_u} x_{p_1}^s a_{p_1} + x_{p_1^*}^s a_{p_1^*} + \ldots + x_{p_k}^s a_{p_k} + x_{p_k^*}^s a_{p_k^*} = -x_r^s, \end{cases}$$

$$II\begin{cases} a_{p_1} = a_{j_{i_1}}, \\ a_{p_1} = -a_{j_{i_1}^*}, \\ \ldots, \\ a_{p_k} = a_{j_{i_k}}, \\ a_{p_k} = -a_{j_{i_k}^*}. \end{cases}$$

It is easy to see that this system of equations has a solution. Let now v be a vector which has a_{j_1} on the coordinate $j_1^*, \ldots, a_{j_u}$ on the coordinate j_u^*, a_{p_1} on the coordinate p_1^*, $a_{p_1^*}$ on the coordinate $p_1, \ldots, a_{p_k}$ on the coordinate p_k^*, $a_{p_k^*}$ on the coordinate p_k, 1 on the coordinate r^* and the rest of the coordinates 0. Then part I of our system of equations can be rewritten as:

$$q_n(v_1, v) = 0, \ldots, q_n(v_s, v) = 0.$$

Part II assures us that v is isotropic. By [**1**, Chapter IV] we know that $\text{span}\{v, v_1, \ldots, v_s\}$ is an $(s+1)$-dimensional, free direct summand of R^{2n} because its image in every $R^{2n}_{m_i}$, $i = 1, \ldots, d$, is of that type.

2.2. Definition. Let $Y(R^{2n})_p$ mean the subcomplex of $Y(R^{2n})$ which is generated by vertices which have fewer than p coordinates not equal to 0, where p is an arbitrary natural number.

2.3. Lemma. *Let K be a finite subcomplex of $Y(R^{2n})$ of dimension m having fewer than t vertices. Let $A = \{v_1, \ldots, v_s\}$ be an $(s-1)$-simplex in K, $s \le m+1$, and let $j_1, \ldots, j_u$ be as in the previous lemma. Then there exist functions $\alpha(t, m)$, $\beta(t)$, $\gamma(t, m)$ such that if $n > \alpha(t, m)$ then the inclusion $K \to Y(R^{2n})$ is homotopic by an inclusion $M \to Y(R^{2n})$ to an inclusion $K' \to Y(R^{2n})$, where M and K' have the following properties:*

(a) $\text{vertices}(M) \setminus (\text{vertices}(M) \cap \text{vertices}(K)) \subset Y(R^{2n})_{\gamma(t,m)}$;

(b) $A \notin K'$;

(c) *M has fewer than $\beta(t)$ vertices.*

Proof. This lemma is almost the same as Lemma 2.8 in [**2**] and the proof is similar. We will point out the differences between these two proofs without going into details.

Step I of the proof is the same for both lemmas. The only new problem which occurs now is to find an extension of an inclusion $K \to Y(R^{2n})$ to an inclusion $i: M \to Y(R^{2n})$.

(i) If $(s-1) = m$ then the proof from Lemma 2.8 of [**2**] is still valid.

(ii) If $(s-1) < m$ then again by an induction we can get an extension of i to M_{s-1} (see [**2**, 2.8] for the definition of M_{s-1}). But now we have to do that extension very carefully using the same method as was used in Lemma 2.19 in [**2**]. It means that if $w_1, \ldots, w_q$ is the full set of vectors which we have added to $i(K)$ during the construction of $i(M_{s-1})$, then for every $1 \le j \le q$ there is a coordinate r_j such that:

(1) the r_jth coordinate of w_j is equal to 1;

(2) the coordinates r_j and r_j^* in w_k are equal to 0 for $k \ne j$;

(3) for $1 \le j \le q$, $r_j \ne j_1, \ldots, r_j \ne j_u$, $r_j \ne j_1^*, \ldots, r_j \ne j_u^*$.

Then $i(v_A)$ is obtained by solving the same system of equations as was used in 2.1 if we take the set $\{v_1, \ldots, v_s, w_1, \ldots, w_q\}$ instead of $\{v_1, \ldots, v_s\}$.

Obviously it is easy to satisfy (1)–(3) because $q < \beta(t)$ and n can be arbitrarily large.

2.4. Proposition. *Let $i: L \to Y(R^{2n})$ be an embedding of some simplicial complex of dimension m into $Y(R^{2n})$. If n is large enough then there is a triangulation of $L \times I$, a function $\gamma'(m)$ and a simplicial map $I: L \times I \to Y(R^{2n})$ such that:*

(a) $L \times \{0\} = L$;

(b) *$I(L \times \{1\}) \subset Y(R^{2n})_{\gamma'(m)}$;*

(c) *$I|_{L \times \{0\}} = i$.*

Proof. This proof is precisely the same as the proof of Proposition 2.9 from [**2**]. We have only to use our Lemma 2.4 instead of Lemma 2.8 from [**2**].

2.5. REMARK. The function $\gamma'(m)$ in 2.4 came from the function $\gamma(m, t)$ of 2.3. But now, because of the construction, the number t is precisely determined by the number m so the function γ' depends only on m.

2.6. DEFINITION. Let X be a simplicial complex spanned by the set of vertices $\{v_i\}_{i\in I}$. We say that the complex X satisfies the condition E_k if for every subset $\{v_j\}_{j=1}^n$ of $\{v_i\}_{i\in I}$, $n \le k$, there exists a vertex $v \in \{v_i\}_{i\in I}$ such that X contains the join of v and the subcomplex spanned by the set $\{v_j\}_{j=1}^n$.

In [2, Proposition 1.2] we proved that if a simplicial complex X satisfies the condition E_k and k is sufficiently large with respect to s then X is s-connected. Now we have the following remark.

2.7. REMARK. If $k \,.\, \gamma'(m) < n$ then the complex $Y(R^{2n})_{\gamma'(m)}$ satisfies the condition E_k.

Proof. Every vertex of $Y(R^{2n})_{\gamma'(m)}$ has at most $\gamma'(m)$ coordinates not equal to 0. Hence if $k \,.\, \gamma'(m) < n$ then for every $v_1, \ldots, v_k \in Y(R^{2n})_{\gamma'(m)}$ there exists a vector v from the canonical basis for R^{2n} such that the join of v and the subcomplex spanned by $\{v_1, \ldots, v_k\}$ is contained in $Y(R^{2n})_{\gamma'(m)}$.

Now we are able to prove our Theorem 1.2.

Proof. If $k \le i$ and n is sufficiently large with respect to i, then by 1.3 of [2] and 2.4 every $f: S^k \to Y(R^{2n})$ is homotopic to $f': S^k \to Y(R^{2n})_{\gamma'(m)}$. But then by 1.2 of [2] and 2.7, f is homotopic to 0.

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