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POSITIVE POINTS IN POLAR LATTICES II

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Abstract

The problem of positive points in polar lattices, discussed by Hossain and Worley for the distance functions $F_t(x_1, x_2) = |x_1| + |tx_2|$ and $G_t(x_1, x_2) = (x_1^2 + t^2 x_2^2)^{\frac{1}{2}}$, is considered for a general distance function F. Best possible results are obtained.

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1. Introduction

Let F be a distance function on $\mathbf{R}^2$ for which the set

$$C_F = \{\mathbf{x} \text{ in } \mathbf{R}^2 : F(\mathbf{x}) < 1\}$$

is a convex body, symmetric in the axes, and let μ_F denote the area of C_F. Let Λ be a lattice in $\mathbf{R}^2$ and let

$$\Lambda^* = \{\mathbf{y} \text{ in } \mathbf{R}^2 : \mathbf{x} \cdot \mathbf{y} \text{ is in } \mathbf{Z} \text{ for all } \mathbf{x} \text{ in } \Lambda\}$$

denote the polar lattice of Λ. By Minkowski's convex body theorem, there is a nonzero point $\mathbf{x}$ in Λ such that $F(\mathbf{x}) \leqslant 2\mu_F^{-\frac{1}{2}} d(\Lambda)^{\frac{1}{2}}$. Since $d(\Lambda^*) d(\Lambda) = 1$, we see that there are nonzero points $\mathbf{x}$ in Λ and $\mathbf{y}$ in Λ^* such that

$$\mu_F F(\mathbf{x}) F(\mathbf{y}) \leqslant 4.$$

Now let P denote the positive quadrant

$$P = \{(x_1, x_2) \text{ in } \mathbf{R}^2 : x_1 \geqslant 0, x_2 \geqslant 0\}$$

and let P° denote the interior of P. We seek a bound β such that, for every lattice Λ, there are nonzero points $\mathbf{x}$ in $\Lambda \cap P$ and $\mathbf{y}$ in $\Lambda^* \cap P^\circ$ for which

$$\mu_F F(\mathbf{x}) F(\mathbf{y}) \leqslant \beta.$$

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We introduce the following notations. For a distance function F and a lattice Λ, we set

$$\beta(F,\Lambda) = \min\{\mu_F F(\mathbf{x})F(\mathbf{y}) : \mathbf{x} \text{ in } \Lambda \cap P,\ \mathbf{x} \neq \mathbf{0},\ \mathbf{y} \text{ in } \Lambda^* \cap P^\circ\}$$

and we write $\beta(F)$ for the maximum of $\beta(F,\Lambda)$ over all lattices Λ. The special distance functions

$$E_t(x_1,x_2) = \max\{|x_1|, |tx_2|\},$$
$$F_t(x_1,x_2) = |x_1| + |tx_2|,$$
$$G_t(x_1,x_2) = (x_1^2 + t^2 x_2^2)^{\frac{1}{2}},$$

$H_t(x_1,x_2) = \max\{|x_1|, t|x_2|, t|x_1(\sqrt{2}-t)+x_2(t\sqrt{2}-1)|/(t^2-t\sqrt{2}+1)$ and the special lattice Λ_t generated by $(1,0)$ and $0,t^{-1})$ will play a particular role in what follows.

Hossain and Worley (1978) have shown that $\beta(F_t) = 2(t+t^{-1})$ and $\beta(G_t) = (t^2+t^{-2})^{\frac{1}{2}}$; in both these cases $\beta(F,\Lambda_t) = \beta(F)$. De Silva (1977) has shown that $2(t+t^{-1}) \leqslant \beta(F) \leqslant 4(t+t^{-1})$ where $t = F(0,1)/F(1,0)$. She has also shown that, for $t \geqslant \sqrt{2}$, $\beta(F,\Lambda_t) \leqslant 4t$, equality being required for the distance function E_t. In the present paper, the following results will be proved.

Theorem 1. *Let F be a convex distance function, symmetric in the axes, and let $t = F(0,1)/F(1,0)$. Then $\beta(F) = \beta(F,\Lambda_t)$.*

Theorem 2. *With F and t defined as in Theorem 1, we have*

$$2(t+t^{-1}) \leqslant \beta(F) \leqslant \begin{cases} 4t^{-1} & \text{if } t \leqslant 1/\sqrt{2}, \\ 8(t+t^{-1}-\sqrt{2}) & \text{if } 1/\sqrt{2} < t < \sqrt{2}, \\ 4t & \text{if } t \geqslant \sqrt{2}. \end{cases}$$

Moreover, $\beta(F_t) = 2(t+t^{-1})$, $\beta(E_t) = 4\max\{t,t^{-1}\}$, and $\beta(H_t) = 8(t+t^{-1}-\sqrt{2})$ for $1/\sqrt{2} < t < \sqrt{2}$.

It should be remarked that H_t satisfies $H_t(0,1)/H_t(1,0) = t$ only for $1/\sqrt{2} \leqslant t \leqslant \sqrt{2}$. A careful analysis of the proof will show that E and H are the only distance functions F for which the upper bound in Theorem 2 is attained. In the case of H_t, Λ_t is the only lattice Λ with $\beta(H_t,\Lambda) = \beta(H_t)$. However, if $t \leqslant 1/\sqrt{2}$, we have $\beta(E_t,\Lambda) = \beta(E_t)$ when Λ is a lattice generated by $(k,0)$ and $(km, kt^{-1}h)$ for any k and m and any $h \geqslant 1$. If $t \geqslant \sqrt{2}$, we have $\beta(E_t,\Lambda) = \beta(E_t)$ when Λ is a lattice generated by $(0,kt^{-1})$ and (kh,km) for any k and m and any $h \geqslant 1$.

Theorem 2 remains valid if we weaken the condition that $\mathbf{y}$ be in P° in the definition of $\beta(F,\Lambda)$ and merely require that $\mathbf{y}$ be in P. This is easily seen by

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considering, in place of Λ_t, lattices generated by $(1, \varepsilon)$ and (ε, t^{-1}) where ε is a small positive number.

2. The proof of Theorem 1

Since $\beta(F, \Lambda) = \beta(k_1 F, k_2 \Lambda)$ for any positive constants k_1 and k_2, we can normalize F and Λ as follows: we take F such that $F(1,0) = 1$ and $F(0,1) = t$ and we take Λ such that $F(\mathbf{x}) \geqslant 1$ for all nonzero $\mathbf{x}$ in $\Lambda \cap P$ and $F(\mathbf{x}^0) = 1$ for some $\mathbf{x}^0$ in $\Lambda \cap P$. Let $\mathbf{x}^0 = (a, am)$. After interchanging the roles of the x_1- and x_2-axes if necessary, we may further suppose that $m \leqslant t$. (Interchanging the axes and renormalizing F and Λ has the effect of replacing t by t^{-1} and m by m^{-1}, or by 0 if $\mathbf{x}^0 = (0, t^{-1})$.) The situation is illustrated in Fig. 1. The curve $F(\mathbf{x}) = 1$ for $\mathbf{x}$ in P lies entirely within the rectangle

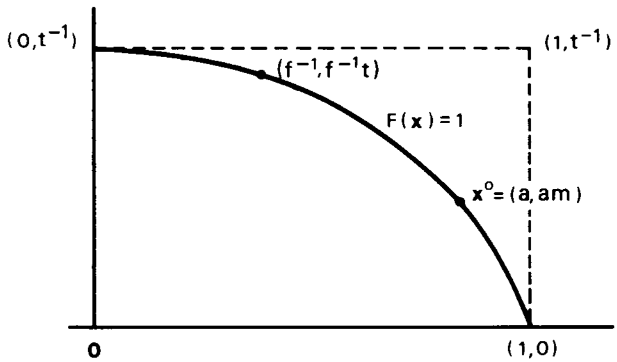

FIGURE 1.

with vertices $(0,0)$, $(1,0)$, $(1,t^{-1})$ and $(0,t^{-1})$ and is decreasing as shown. Indeed, $F(x_1, x_2) = F(-x_1, x_2)$ by symmetry in the axes and, if $|x_1'| \leqslant |x_1|$, the point (x_1', x_2) lies on the line joining (x_1, x_2) and $(-x_1, x_2)$, so $F(x_1', x_2) \leqslant F(x_1, x_2)$ by convexity. Similarly, if $|x_2'| \leqslant |x_2|$, then $F(x_1, x_2') \leqslant F(x_1, x_2)$.

Now consider the special lattice Λ_t generated by $(1,0)$ and $(0,t^{-1})$. The polar lattice Λ_t^* is the lattice generated by $(1,0)$ and $(0,t)$, so by the preceding remarks

$$\min\{F(\mathbf{x}) : \mathbf{x} \text{ in } \Lambda_t \cap P,\ \mathbf{x} \neq \mathbf{0}\} = 1$$

and

$$\min\{F(\mathbf{y}) : \mathbf{y} \text{ in } \Lambda_t^* \cap P^\circ\} = F(1,t).$$

We write $f = F(1,t)$, so that

$$\beta(F, \Lambda_t) = \mu_F f.$$

For a general lattice Λ, the polar lattice is

$$\Lambda^* = \{(-x_2, x_1)/d(\Lambda) : (x_1, x_2) \text{ in } \Lambda\},$$

where $d(\Lambda)$ is the determinant of Λ. Let Q denote the quadrant

$$Q = \{(x_1, x_2) : x_1 < 0, x_2 > 0\}$$

and set $F^*(x_1, x_2) = F(-x_2, x_1)$. Then we have

$$\beta(F, \Lambda) = \mu_F d(\Lambda)^{-1} \min\{F^*(\mathbf{x}) : \mathbf{x} \text{ in } \Lambda \cap Q\}.$$

The points of Λ may be regarded as lying on the lines $x_2 = mx_1 + ke$ $(k = 0, \pm 1, \pm 2, \ldots)$, whence $d(\Lambda) = ae$.

We set $B = \min\{F^*(\mathbf{x})\colon \mathbf{x} \text{ in } \Lambda \cap Q\}$, so that

$$\beta(F, \Lambda) = \mu_F B/ae.$$

Let $(-r, s)$ be the first point of Λ to the left of the x_2-axis lying on the line $x_2 = mx_1 + e$; we will show that this point lies on or outside the curve $F^*(\mathbf{x}) = B$. The situation is illustrated in Fig. 2.

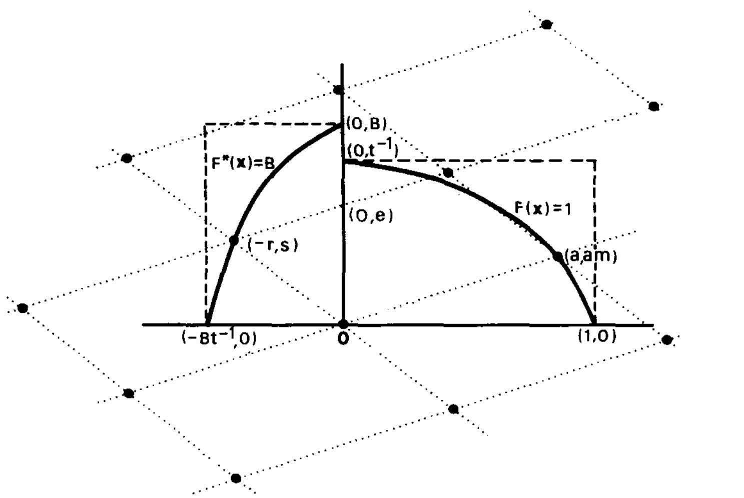

FIGURE 2.

From the definition of the point $(-r, s)$ we have $0 < r \leqslant a$ and $s > -am$. Consequently, the lattice point $(-r+a,\ s+am)$ is in P and it lies on or outside the curve $F(\mathbf{x}) = 1$ and in the strip $0 \leqslant x_1 < a$. In particular, this point lies above the line joining (a, am) and $(0, t^{-1})$, giving the inequality

$$s = e - mr \geqslant rt^{-1}(a^{-1} - mt).$$

Now $am \leqslant t^{-1}$ since (a, am) lies in the rectangle drawn in Figure 1 so

$$r \leqslant aet \quad \text{and} \quad s \geqslant 0.$$

Moreover, if $s = 0$, then $am = t^{-1}$ and Λ contains the point $(r, 0)$ with $0 < r \leqslant a \leqslant 1$. Our normalization of Λ gives $r = a = 1$, so Λ contains the points $(1, 0)$,

$(a, am) = (1, t^{-1})$ and $(1, t^{-1}) - (1, 0) = (0, t^{-1})$ and no other points inside or on the rectangle with vertices $\mathbf{0}$, $(1,0)$, $(1, t^{-1})$ and $(0, t^{-1})$. Thus $s = 0$ implies $\Lambda = \Lambda_t$.

For the remainder of the argument, we may suppose $s > 0$. To complete the proof of Theorem 1, we have to show

$$B \leqslant aef.$$

We shall establish this inequality by several stages.

First suppose $B < e$. Since we have chosen $m \leqslant t$, the point (a, am) lies to the right of the point $Y = (f^{-1}, tf^{-1})$ on the curve $F(\mathbf{x}) = 1$, whence $af \geqslant 1$. We therefore have $B < aef$ in this case.

Now suppose $B \geqslant e$, so that the point $(eB^{-1}, 0)$ lies inside or on the curve $F(\mathbf{x}) = 1$. From what we have already shown, the point (sB^{-1}, rB^{-1}) lies in P and it is on or outside the curve $F(\mathbf{x}) = 1$, since $F(sB^{-1}, rB^{-1}) = F^*(-rB^{-1}, sB^{-1}) \geqslant 1$. If $r \leqslant st$, then the point (sB^{-1}, rB^{-1}) lies to the right of the line joining $(eB^{-1}, 0)$ and Y and it follows that

$$B \leqslant ef/(1 + mt) \leqslant aef.$$

(The second inequality follows from the observation that (a, am) lies above the line joining $(1,0)$ and $(0, t^{-1})$, so that $am \geqslant t^{-1}(1-a)$.) On the other hand, if $r > st$, then the point (sB^{-1}, rB^{-1}) lies above the horizontal line through Y, so that $rB^{-1} \geqslant tf^{-1}$ and

$$B \leqslant rft^{-1} \leqslant aef.$$

3. The proof of Theorem 2

We normalize F and Λ as in the previous section. After interchanging the axes if necessary, we may also suppose that $t \leqslant 1$. Since the point $(1, t)$ is outside the curve $F(\mathbf{x}) = 1$, we have $f = F(1, t) \geqslant 1$. From the previous section, $\beta(F, \Lambda_t) = f\mu_F$, so we need estimates for this quantity.

Consider a tac-line to the curve $F(\mathbf{x}) = 1$ at the point $F = (f^{-1}, tf^{-1})$, as illustrated in Fig. 3. Now $\frac{1}{4}\mu_F$ is at least as large as the area of the quadrilateral with vertices $\mathbf{0}$, $(1,0)$, Y and $(0, t^{-1})$, so

$$f\mu_F \geqslant 4f(\tfrac{1}{2}tf^{-1} + \tfrac{1}{2}t^{-1}f^{-1}) = 2(t + t^{-1}),$$

which is the lower bound in Theorem 2.

On the other hand, $\frac{1}{4}\mu_F$ cannot exceed the area of the pentagon with vertices $\mathbf{0}$, $(1,0)$, R, S and $(0, t^{-1})$. If we set $\eta = TR$, then the area of the triangle RST is

$$\tfrac{1}{2}\eta^2(1 - f^{-1})/(\eta - t^{-1} + tf^{-1}).$$

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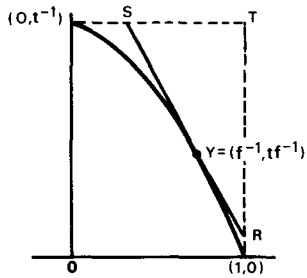

FIGURE 3.

As a function of η, this expression decreases over the interval $t^{-1}-tf^{-1}<\eta<2(t^{-1}-tf^{-1})$ and then increases for $\eta>2(t^{-1}-tf^{-1})$. The admissible range for η is $t^{-1}-tf^{-1}\leqslant\eta\leqslant t^{-1}$, so two cases arise. If $f\geqslant 2t^2$, the area of the triangle RST is a minimum when $\eta=t^{-1}$. This gives the corresponding maximum value for μ_F, so

$$f\mu_F\leqslant 4f\{t^{-1}-\tfrac{1}{2}ft^{-3}(1-f^{-1})\}.$$

Now, for f satisfying $f\geqslant\max\{1,2t^2\}$, the above expression is a decreasing function of f and so its maximum occurs at $f=1$ when $t\leqslant 1/\sqrt{2}$ and at $f=2t^2$ when $t>1/\sqrt{2}$. In both cases, we obtain

$$f\mu_F\leqslant 4t^{-1}$$

On the other hand, if $f<2t^2$, then the maximum value of μ_F occurs at $\eta=2(t^{-1}-tf^{-1})$, giving

$$f\mu_F\leqslant 4f\{t^{-1}-2(1-f^{-1})(t^{-1}-tf^{-1})\}.$$

This expression attains its maximum at $f=t\sqrt{2}$ and so we have

$$f\mu_F\leqslant 8(t+t^{-1}-2).$$

(Note that the constraint $f\geqslant 1$ means that this case can only arise for $1/\sqrt{2}<t\leqslant 1$.) Combining the two estimates for $f\mu_F$ gives the upper bound in Theorem 2.

In order to show that E_t and H_t provide the only cases for which $\beta(F)$ is maximal, as claimed after the statement of Theorem 2, we must investigate when $B=aef$ in the proof of Theorem 1. We must then find when $f\mu_F$ attains the maxima given in the proof of Theorem 2.

Firstly, it is clear that $B=aef$ only if $a(1+mt)=1$. Disregarding the possibility that $F=F_t$ as $\beta(F_t)$ is never large enough, we conclude that $a=1, m=0$ if $\beta(F,\Lambda)$ is maximal.

Secondly, it is clear that $f\mu_F$ attains the required maxima only when either (i) $f=1$, $\eta=t^{-1}$, $\mu_F=4t^{-1}$ and $F=E_t$, or (ii) $f=t\sqrt{2}$, $\eta=2(t^{-1}-\sqrt{2})$, $\mu_F=4\{t^{-1}-2(1-f^{-1})(t^{-1}-tf^{-1})\}$ and $f=H_t$. Taking account of the extra normalization condition used in Theroem 2, case (ii) arises for $1/\sqrt{2} < t < \sqrt{2}$ and case (i) arises for $t \leqslant 1/\sqrt{2}$ (because $f \geqslant t^2$). We consider these cases separately.

If $F = E_t, f = 1, a = 1, m = 0$ and $B = aef$, we have $B = e = s$ and Λ must be generated by $(1,0)$ and $(-r, e)$. Clearly $e \geqslant t^{-1}$ for the point $(1-r, e)$ not to contradict the definition of $\mathbf{x}^0$. On the other hand, it is easy to verify that if Λ is generated by $(1,0)$ and $(-r,e)$ with $e \geqslant t^{-1}$ and $t \leqslant 1/\sqrt{2}$ then $\beta(E_t, \Lambda) = 4t^{-1}$. After allowing for the normalizations made, this justifies the claim concerning the maximum for $t \leqslant 1\sqrt{2}$ and $t \geqslant \sqrt{2}$.

If $F = H_t, f = t\sqrt{2}, a = 1, m = 0, 1/\sqrt{2} < t < \sqrt{2}$, and $B = aef = et\sqrt{2}$, then the point $(-r, s) = (-r, e)$ must lie on the curve $F^*(\mathbf{x}) = B = et\sqrt{2}$. However, if $r \geqslant e\sqrt{2}$, then $et \geqslant 1$ (else the point $(1-r, e)$ contradicts the definition of $\mathbf{x}^0$). Thus

$$F^*(-r, e) = t\{e(\sqrt{2}-t) + r(t\sqrt{2}-1)\}/(t^2 - t\sqrt{2}+1).$$

This equals $et\sqrt{2}$ only when $et = r$. If $r < 1$ the point $(1-r, rt^{-1})$ contradicts the definition of $\mathbf{x}^0$ (as $1/\sqrt{2} < t < \sqrt{2}$) so we have $r = et = 1$. Thus Λ is generated by $(1,0)$ and $(-1, t^{-1})$, that is $\Lambda = \Lambda_t$, showing that for $1/\sqrt{2} < t < \sqrt{2}$, $\beta(F, \Lambda)$ attains the maximum only for $F = H_t$ and $\Lambda = \Lambda_t$.

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