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COVERS AND COMPLEMENTS IN THE SUBALGEBRA LATTICE OF A BOOLEAN ALGEBRA

IVO DÜNTSCH

Section 1 addresses the problem of covers in $\operatorname{Sub} D$, the lattice of subalgebras of a Boolean algebra; we describe those BA's in whose subalgebra lattice every element has a cover, and show that every small and separable subalgebra of $P(\omega)$ has 2^{ω} covers in $SubP(\omega)$. Section 2 is concerned with complements and quasicomplements. As a general result it is shown that $\operatorname{Sub} D$ is relatively complemented if and only if D is a finite- cofinite BA. Turning to $\operatorname{Sub} P(\omega)$, we show that no small and separable $D \leqslant P(\omega)$ can be a quasicomplement. In the final section, generalisations of packed algebras are discussed, and some properties of these classes are exhibited.

0. INTRODUCTION

The set $\operatorname{Sub} D$ of subalgebras of a Boolean algebra D is a complete atomistic [**1**] and dually atomistic [**6**] lattice under set inclusion. This paper is, in a way, a continuation of [**2**] and [**3**]. We shall look in particular at covers, quasi-complements, and complements in $\operatorname{Sub} P(\omega)$; the final section will be concerned with locally packed and diverse Boolean algebras.

If $\mathcal{M} \subseteq \operatorname{Sub} D$, then $\inf \mathcal{M} = \cap \mathcal{M}$, and $\sup \mathcal{M}$ is the subalgebra of D generated by $\cup \mathcal{M}$. If A is a subalgebra of D, we usually write $A \leqslant D$. The subalgebra of D generated by $M \subseteq D$ is denoted by $\langle M \rangle$; if $M = \{a\}$, we just write $\langle a \rangle$ instead of $\langle \{a\} \rangle$. For $A \leqslant D$ and $u \in D \setminus A$, $\langle A \cup \{u\} \rangle$ is called a *simple extension of* A, which is denoted by $A(u)$. It is well known that each $b \in A(u)$ has the form

$$b = x \cdot u + y \cdot -u$$

for some $x, y \in A$. If $A, B \in \operatorname{Sub} D$, then A is a *quasicomplement* of B if A is maximal with respect to the property $A \cap B = \underline{2}$. A Boolean algebra D is called *small* if $|D| < 2^{\omega}$, and *separable* if it has a countable dense subalgebra. The two element Boolean algebra is denoted by $\underline{2}$. Unless stated otherwise, all Boolean algebras mentioned are assumed to be infinite.

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1. Covers in $\operatorname{Sub} P(\omega)$

Our first aim is to give a convenient criterion for when a subalgebra of D is covered:

Proposition 1.1. *If A and B are subalgebras of D, then B covers A if and only if $B = A(u)$ for some element u of $D \setminus A$, and $\{a \in A \mid a \cdot u \in A\}$ is a prime ideal of A.*

Proof: Suppose that B covers A; if $u \in B \setminus A$, then $A \leqslant A(u) \cap B$, and hence $A(u) = B$. Let $a \in A$ such that $a \cdot u \notin A$. Since $A(u)$ covers A and $a \cdot u \in A(u)$, we have $A(a \cdot u) = A(u)$, and thus there are $x, y \in A$ such that

$$\begin{aligned} u &= x \cdot a \cdot u + y \cdot (-a + -u) \\ &= x \cdot a \cdot u + y \cdot -a \end{aligned}$$

which implies $-a \cdot u \in A$.

For the converse, suppose that for all $a \in A$ either $a \cdot u \in A$ or $-a \cdot u \in A$. Let $z \in A(u) \setminus A$; then there are $x, y \in A$ such that $z = x \cdot u + y \cdot -u$. Suppose without loss of generality that $x \cdot u \in A$. Since $z \notin A$, $y \cdot -u$ is not an element of A, and thus $-y \cdot -u \in A$. Now,

$$-z = -x \cdot u + -y \cdot -u,$$

and $y \cdot -z = y \cdot -x \cdot u$ is an element of $A(z)$. Therefore,

$$\begin{aligned} u &= x \cdot u + -x \cdot u \\ &= x \cdot u + y \cdot -x \cdot u + -y \cdot -x \cdot u \end{aligned}$$

is an element of $A(z)$ which implies $A(z) = A(u)$. ∎

Recall that a lattice L is called *weakly atomic* if every proper interval $[c, d]$ contains a jump, that is, there are $a, b \in [c, d]$ such that $a < b$ and b covers a.

Lemma 1.2. *$\operatorname{Sub} D$ is weakly atomic.*

Proof: Let $A < B \leqslant D$. Since $\operatorname{Sub} B$ is dually atomistic, there is a dual atom C of $\operatorname{Sub} B$ containing A. As C is a dual atom of $\operatorname{Sub} B$, it is covered by B. ∎

A Boolean algebra D has the *cover property* if every subalgebra of D has a cover in $\operatorname{Sub} D$, and one may ask which Boolean algebras have the cover property. It turns out that the Boolean algebras with this property are exactly the superatomic ones. An equivalent result has been independently shown by Koppelberg [4] in a different context.

LEMMA 1.3. *If A has the cover property, and if $f\colon A \to B$ is an onto homomorphism, then B has the cover property.*

PROOF: Let $C \leqslant B$, $C \neq B$, and $D = f^{-1}(C)$. By our hypothesis, D is covered in $\operatorname{Sub} A$, say, by $D(x)$. Set $b = f(x)$; then $b \notin C$, and, if $c = f(y) \in C$, then $y \cdot x \in D$ or $-y \cdot x \in D$ by 1.1. It follows that $b \cdot c \in C$ or $-b \cdot c \in C$. ∎

LEMMA 1.4. *If A has a countable dense free subalgebra, then A does not have the cover property.*

PROOF: Let $F \leqslant A$ be freely generataed by $M = \{m_i \mid i < \omega\}$, and let F^* be the completion of F. Since F is dense in A, we may suppose that A is a regular subalgebra of F^*. Let B^* be the subalgebra of F^* completely generated by $M \setminus \{m_0\}$, and set $B = A \cap B^*$. Assume that B is covered in $\operatorname{Sub} A$ by $B(x)$. Since F is dense in A, there is a set $T = \{t_i \mid i < \omega\}$ of elementary products of elements of M such that $x = \sup T$. Since $x \notin B$, m_0 appears in one of the t_i, say $t_0 = \varepsilon_0 m_0 \cdot \ldots \cdot \varepsilon_k m_k$, where $\varepsilon_i = \pm 1$. Now, $m_{k+1} \in B$, and $m_{k+1} \cdot t_0 > 0$ and $-m_{k+1} \cdot t_0 > 0$; thus, m_0 appears in both $m_{k+1} \cdot x$ and $-m_{k+1} \cdot x$. Consequently, none of these elements can be in B which is a contradiction to our assumption that $B(x)$ covers B. ∎

PROPOSITION 1.5. *A has the cover property if and only if A is superatomic.*

PROOF: If A is not superatomic, then, by Sikorski's Extension Theorem, it has a quotient B with a countable free dense subalgebra. By the previous lemma, B does not have the cover property, and, by Lemma 1.3, neither has A.

For the converse, suppose that A is superatomic with ideal sequence $\{I_\beta \mid \beta \leqslant \alpha\}$, and let B be a proper subalgebra of A, that is $I_\alpha \setminus B \neq \emptyset$. Set

$$\beta = \min\{\delta < \alpha \mid I_\delta \nsubseteq B\}.$$

Then β is well-defined and a successor ordinal, say $\beta = \gamma + 1$. Now let $a \in I_\beta \setminus B$, and $\pi\colon A \twoheadrightarrow A/I_\gamma$ be the canonical epimorphism. Then $\pi(a)$ is an atom of A/I_γ.

Suppose that $x \cdot a \notin B$ for some $x \in B$; then, in particular, $x \cdot a \notin I_\gamma$ and hence $\pi(x \cdot a) > 0$. Since $\pi(a)$ is an atom, $\pi(x \cdot a) = \pi(a)$, and thus $\pi(-x \cdot a) = 0$. It follows that $-x \cdot a \in I_\gamma \subseteq B$. Lemma 1.1 now implies that $B(a)$ covers B. ∎

Next, we shall look at covers in $\operatorname{Sub} P(\omega)$. It follows from the previous result that not every subalgebra of $P(\omega)$ has a cover. Nevertheless, covers exist in abundance:

PROPOSITION 1.6. *Let $A \leqslant P(\omega)$ such that $|A| < 2^\omega$ and A has a countably generated prime ideal I; then A has 2^ω covers in $\operatorname{Sub} P(\omega)$.*

PROOF: Let $\{b_i \mid i < \omega\}$ be a set of pairwise disjoint generators of I. Since A is small, there are 2^ω subsets M_α of ω such that $m_\alpha = \bigcup\{b_i \mid i \in M_\alpha\} \notin A$, and we can choose the M_α to be infinite and almost disjoint. Now, since the b_i are pairwise disjoint and the M_α are almost disjoint, we see that for $\alpha < \beta$ we have $m_\alpha \cap m_\beta \in I$ and that, furthermore, $m_\alpha \cap \bigcup\{b_j \mid j \in J\} \in I$ for any finite subset J of ω.

Assume that for some $J \subseteq 2^\omega$ with $|J| = 2^\omega$ and some $m \subseteq \omega$ we have $A(m) \doteq A(m_\alpha)$ for all $\alpha \in J$. Then for each such α there are a_α, $b_\alpha \in A$ such that

$$m_\alpha = a_\alpha \cdot m + b_\alpha \cdot -m.$$

Since A is small we must have $a_\alpha = a_\beta$ and $b_\alpha = b_\beta$ for some $\alpha, \beta \in J$. This implies $m_\alpha = m_\beta$, a contradicction. Thus we may assume that all $A(m_\alpha)$ are different.

Now, suppose $a \in I$; since the b_i generate I as an ideal there is a finite set $J \subseteq \omega$ such that $a \leqslant b = \bigcup\{b_j \mid j \in J\}$. Let $\alpha < 2^\omega$; then

$$a \cap m_\alpha = a \cap (b \cap m_\alpha) \in I.$$

Since I is a prime ideal, $A(m_\alpha)$ covers A by 1.1. ∎

COROLLARY 1.7. *(MA) If A is small and has a free countable dense subalgebra, then A has 2^ω covers in* $\operatorname{Sub} P(\omega)$.

PROOF: Just note that (MA) implies that A has a ultrafilter U which preserves all sups of the dense countable subalgebra. Hence U is countably generated as a filter. ∎

2. COMPLEMENTS AND QUASICOMPLEMENTS

It was shown by Todorčević [8] that every interval algebra $I(C)$ has a sectionally complemented lattice of subalgebras, that is $\operatorname{Sub} A$ is complemented for every $A \leqslant I(C)$.

It seems natural to ask which Boolean algebras have $\operatorname{Sub} D$ not only sectionally complemented but even relatively complemented; it turns out that, in general, the former is all that we can hope for:

PROPOSITION 2.1. *SubD is relatively complemented if and only if D is a finite–cofinite algebra.*

PROOF: The "if" part was shown by Remmel [5], so we need only prove the other direction.

Suppose that D is not a finite–cofinite algebra. Then there exists an element u of D such that below u and $-u$ there are countably infinite sets of pairwise disjoint elements, say, $S = \{a_i \mid i < \omega\}$, and $T = \{b_i \mid i < \omega\}$. Let $A \leqslant D$ be generated by $\{u\} \cup S \cup T$, B be generated by S, C be generated by $\{a_i + b_i \mid i < \omega\}$ and E be generated by $B \cup C$. Since u is not an element of E, the latter is a proper subalgebra of A. The aim now is to show that E has no complement in the interval $[C, A]$; for this, it is enough to prove that, for each $x \in A \setminus E$, $C(x) \cap E$ is not a subalgebra of C. Keeping in mind that $B \cap C = \underline{2}$, this is straightforward and will be left to the reader. ∎

We shall use the construction of 1.3 to shed some light on the problem when in $\mathrm{Sub}\, P(\omega)$ an algebra can be a quasicomplement to another subalgebra of $P(\omega)$. This is connected with a question by Remmel [**5**] where he asks for which pairs (α, β) of cardinals it is possible that A, $B \leqslant D$, $|A| = \alpha$, $|B| = \beta$, and A and B are quasicomplements of each other. In [**3**] an example was given where $|D| = 2^\omega$ and (ω, ω) was realised in $\mathrm{Sub}\, D$. In the same article it was shown that for (α, β) to be realised in $\mathrm{Sub}\, P(\omega)$, one of α or β must in fact be 2^ω. For both results Martin's Axiom was used, and it is unknown to the author if they can be obtained in ZFC proper.

PROPOSITION 2.2. *If $A \leqslant P(\omega)$ has a countably generated ideal then A is not a quasicomplement to any $B \leqslant P(\omega)$.*

PROOF: Let the prime ideal I of A be generated by the set $\{b_i \mid i < \omega\}$ of pairwise disjoint elements. Furthermore, choose the set $\{m_\alpha \mid \alpha < 2^\omega\}$ as in 1.3.

Let $B \leqslant P(\omega)$, and assume that $A(m_\alpha) \cap B \neq \underline{2}$ for all $\alpha < 2^\omega$. Then, for each such α, there are some $b_\alpha \in B$, s_α, $t_\alpha \in A$ such that

$$b_\alpha = (s_\alpha \cap m_\alpha) \cup (t_\alpha \cap -m_\alpha).$$

We may assume that each t_α is an element of I, otherwise we choose $-b_\alpha$. Since A is small, we can also suppose after a simple thinning process that

$$b_\alpha = (s \cap m_\alpha) \cup t$$

for some $s \in A$, $t \in I$, and all $\alpha < 2^\omega$. Observe that $s \notin I$, since otherwise $b_\alpha \in A$.

Now, if $\alpha < \beta < 2^\omega$, then $b_\alpha \cap b_\beta = (s \cap m_\alpha \cap m_\beta) \cup t$ which is an element of A. From $A \cap B = \underline{2}$ it follows that b_α and b_β are disjoint, in particular, $t = 0$. Since $P(\omega)$ has the ccc, there is some $J \subsetneq 2^\omega$ of cardinality 2^ω such that $s \cap m_\alpha = s \cap m_\beta$ for all $\alpha, \beta \in J$. Again since A is small and $-s \in I$, there are $\alpha, \beta \in J$ such that $-s \cap m_\alpha = -s \cap m_\beta$. It follows that $m_\alpha = m_\beta$, which is a contradiction. ∎

COROLLARY 2.3. (MA) *If $A \leqslant \mathrm{Sub}\, P(\omega)$ is small and separable, then it is not a quasicomplement of any $B \leqslant P(\omega)$.*

I have not been able to determine whether $(\alpha, 2^\omega)$ can be realised at all on $\mathrm{Sub}\, P(\omega)$ for an infinite $\alpha < 2^\omega$.

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3. Locally packed and diverse algebras

The final part of this paper generalises the notion of a packed algebra as defined in [9]: D is *packed* if A, $B \leqslant D$ and $|A| = |B| = |D|$ imply that $|A \cap B| = |D|$.

Let λ be a regular uncountable cardinal, and $|D| = \lambda$. D is called:

(1) *diverse* if $D \mid d$ and $D \mid -d$ have no isomorphic subalgebra of cardinality λ for every $d \in D$.

[Note that D is diverse if and only if for every $u \in D$ and every $A \leqslant D$ with $|A| = \lambda$, $\langle u \rangle$ is not independent of A, that is there exists an $a \in A^+$ such that $a \cdot u = 0$ or $a \cdot -u = 0$.]

(2) *locally packed* if, for all $A, B \leqslant D$ and $d \in D$ such that $B \leqslant A(d)$ and $|A| = |B| = \lambda$, we have $|A \cap B| = \lambda$.

Furthermore, let $I_\lambda = \{A \in \operatorname{Sub} D \mid |A| < \lambda\}$. It was shown in [2] that $\operatorname{Sub} D$ is not simple if I_λ is a distributive element in the ideal lattice of $\operatorname{Sub} D$. Observe that we could have defined D to be packed if and only if I_λ is a prime ideal of $\operatorname{Sub} D$.

PROPOSITION 3.1. *If $|D| = \lambda$ is uncountable and regular, each property implies the next:*

(1) *D is packed;*
(2) *I_λ is distributive ideal;*
(3) *D is locally packed;*
(4) *D is diverse.*

PROOF: (1) $\Rightarrow$ (2) follows from 3.9 and (2) $\Rightarrow$ (3) follows from 3.6 of [2].

Suppose that D is locally packed and assume the existence of some $u \in D$ such that there are $A \leqslant D \mid u$, $B \leqslant D \mid -u$, both having cardinality λ, and that $f \colon A \rightarrowtail B$ is an isomorphism.

Set $C = \{x + f(x) \mid x \in A\}$; clearly, C is a subalgebra of D. If A^* is a canonical copy of A in $\operatorname{Sub} D$, then $C \cap A^* = \underline{2}$; on the other hand, $A^* \leqslant C(u)$, contradicting the fact that D is locally packed, so we have (3) $\Rightarrow$ (4). ∎

In general, (4) does not imply (3) as the following example shows:

EXAMPLE 3.2. *There is a Boolean algebra of cardinality ω_1 which is diverse, but not locally packed.*

PROOF: Let $\{a_\alpha \mid \alpha < \omega_1\}$ be a family of infinite almost disjoint subsets of ω, and let $A \leqslant P(\omega)$ be generated by this set. Observe that A is ω_1-like, that is if $a \in A^+$, then $A \mid a$ or $A \mid -a$ is countable.

Let $B \cong FC(\omega_1)$ with atoms $\{b_\alpha \mid \alpha < \omega_1\}$; next, set $D = A \times B$ and assume without loss of generality that $A = D \mid u$ and $B = \mid -u$ for some $u \in D$; let $B^* \leqslant D$

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be generated by the b_α.

(1) D is not locally packed:

Set $e_\alpha = a_\alpha + b_\alpha$ and $E = \langle\{e_\alpha \mid \alpha < \omega_1\}\rangle$. Then, $|E| = \omega_1$, and $B^* \leqslant E(u)$, since $b_\alpha = e_\alpha \cdot -u$.

Assume that $E \cap B^* \neq \underline{2}$, and suppose without loss of generality that $d = b_0 + \ldots + b_n$ is an atom of E. Thus there exist $x_0, \ldots, x_s \in \{\pm e_\alpha \mid \alpha < \omega_1\}$ such that $d = x_0 \cdot \ldots \cdot x_s$. Since each $-e_\alpha \cdot -u$ is cofinite in $D \mid -u = B$, there is at least one $r \leqslant s$ with $x_r \in \{e_\alpha \mid \alpha < \omega_1\}$. On the other hand, since for all α, $\beta < \omega_1$ we have $e_\alpha \cdot e_\beta = a_\alpha \cdot a_\beta < u$, we can have at most one such r. Thus, let us suppose without loss of generality that $d = e_0 \cdot -e_i \cdot \ldots \cdot -e_s$. Since $d < -u$, that is $d \cdot u = 0$, we obtain $0 = a_0 \cdot -a_i \cdot \ldots \cdot -a_s$; in other words, $a_0 \leqslant a_i + \ldots + a_s$ which contradicts the fact that the a_α are infinite and almost disjoint.

(2) D is diverse:

Assume that there is some $c \in D$ such that $D \mid c$ and $D \mid -c$ have isomorphic uncountable subalgebras, that is there is an uncountable $C \leqslant D$ such that c is independent of C. Note that the projections of C to $D \mid c$ and $D \mid -c$ are isomorphic to C.

Since $c = c \cdot u + c \cdot -u$, by possibly taking the complement of c, we may suppose that $c = a + b$ where $a \in A^+$ and $b = 0$ or b is a finite sum of atoms of B. Independence now implies that for all $x, y \in C^+$, $x \neq y$ implies $x \cdot c \neq y \cdot c$. Furthermore, there are only finitely many elements of D below b. Thus, there are only finitely many elements t of C such that $t \cdot a = 0$, and consequently C contains an uncountable subalgebra E such that $e \cdot a > 0$ for every $e \in E^+$. Since $a \leqslant c$, we also have $e \cdot -a > 0$, and hence a is independent of E.

Without loss of generality, let $C = E$ and $c = a$; since A and B have no isomorphic uncountable subalgebras, $a \neq u$. Recall that A is ω_1-like; since $D \mid c$ contains an uncountable subalgebra, it follows that $A \mid c$ is uncountable. Thus $-c \cdot u$ is countable, which, in turn, implies that the projection of C to $(-u]$ is uncountable and thus contains an uncountable set of pairwise disjoint elements. Again we use that $-c \cdot u$ is countable and that $-c = -c \cdot u + -u$, and find that the projection of C to $(-c]$ also contains such a set; hence, by independence, so does the projection of C to $-c$ which is a relative algebra of $A = D \mid u$.

This contradicts the fact that A satisfies the countable chain condition. $\square$

It is not known whether (3) implies (2) in general; however, in some cases the last three properties of 3.1 coincide:

PROPOSITION 3.3. *If $|D| = \lambda$ is regular, D diverse, and* Sub D *sectionally complemented, then I_λ is a distributive ideal of* Sub D.

PROOF: By 3.4 and 3.7 of [**2**], I_λ is distributive if, for every $u \in D$ and every $A \leqslant D$ with $|A| = \lambda$, every subalgebra of $A(u)$ disjoint from A has cardinality less than λ.

Suppose that A and u are as above, and assume the existence of some $B \leqslant A(u)$ such that $|B| = \lambda$ and $A \cap B = \underline{2}$. Since $u \notin A$ and A is the meet of all antiatoms of $\operatorname{Sub} A(u)$ which contain A, there is an antiatom E of $\operatorname{Sub} A(u)$ containing A and $u \notin E$. Then since every subalgebra C of $E(u) = A(u)$ disjoint from E has at most four elements, and since $\operatorname{Sub} D$ is sectionally complemented, $B \cap E$ has cardinality α, and we may suppose without loss of generality that $u \notin A \vee B$ by possibly substituting $B \cap E$ for B.

Let $B_1 \leqslant B$ be generated by $M = \{b \in B \mid b < u\}$, $B_2 \leqslant B$ be generated by $\{b \in B \mid b < -u\}$, and set $B_3 = \langle B_1 \cup B_2 \rangle$. Suppose that C is a complement of B_3 in $\operatorname{Sub} B$; then for each $c \in C^+$ we have $c \cdot u > 0$, hence u is independent of C; therefore $|C| < \lambda$. Thus, we may as well assume that $B = B_1$.

Let $\{b_i \mid i < \lambda\}$ be an enumeration of M^+ and choose $a_i \in A$ such that $b_i = a_i \cdot u$. Because $b_i \notin A$, we see that $a_i \cdot -u > 0$ for all $i < \alpha$. Since $|M| = \lambda$, we may suppose that A is generated by the a_i. Assume that there is an $a \in A^+$ with $a < u$. Since $A \cap (u]$ is an ideal of A, we need only consider the following two cases:

(1)
$$\begin{aligned} a &= a_{i_1} \cdot \ldots \cdot a_{i_n} \cdot -a_{j_1} \cdot \ldots \cdot -a_{j_k} = a \cdot u \\ &= b_{i_1} \cdot \ldots \cdot b_{i_n} \cdot -b_{j_1} \cdot \ldots \cdot -b_{j_k}, \end{aligned}$$

since $b_{i_r} < u$ and $-b_{i_r} \cdot u = -a_{i_r} \cdot u$. This contradicts our assumption that $a \cap B = \underline{2}$.

(2)
$$\begin{aligned} a &= -a_{i_1} \cdot \ldots \cdot -a_{i_n} \\ &= -b_{i_1} \cdot \ldots \cdot -b_{i_n} \cdot u \\ &= -(b_{i_1} + \ldots + b_{i_n}) \cdot u, \end{aligned}$$

and thus, $u = a + b_{i_1} + \ldots + b_{i_n} \in \langle A \cup B \rangle$, a contradiction. This gives us $A \cap (u] = \{0\}$. Let $A_1 \leqslant D$ be generated by $A \cap (-u]$; since u is not independent of A and $A \cap (u] = \{0\}$, every complement of A_1 in $\operatorname{Sub} A$ has cardinality $< \lambda$. But this implies that the projection of A to $(u]$, which includes M, has cardinality less than λ, and again we arrive at a contradiction. ∎

It was mentioned in [**2**] without proof that certain diverse Boolean algebras have very strong rigidness properties.

A Boolean algebra D is called:

(1) *mono-rigid*, if every one-one endomorphism of D is the identity:

(2) *Bonnet-rigid*, if, whenever $f\colon D \rightarrowtail B$ is a one-one homomorphism and $g\colon D \twoheadrightarrow B$ is an onto homomorphism, then $f = g$;

(3) *factor-homogeneous*, if every proper factor of D has the same cardinality as D.

PROPOSITION 3.4. *If D is diverse and factor homogeneous, then D is mono-rigid.*

PROOF: Suppose that $f\colon D \rightarrowtail$ is one-one, and assume the existence of some $d \neq D$ with $f(d) \notin d$. Then there is some $e \in D$ such that $e \cdot f(e) = 0$. The restriction of f to $D \mid e$ is an embedding into $D \mid -e$, and consequently $D \mid e$ and $D \mid -e$ have isomorphic subalgebras of the same cardinality as D, a contradiction. ∎

For locally packed algebras, the following result by Shelah [7] comes in helpful:

> If D is mono-rigid and not Bonnet-rigid, then there exist $d \in D$, $B \leqslant D \mid d$ and an onto homomorphism $f\colon B \twoheadrightarrow D \mid -d$.

PROPOSITION 3.5. *If D is locally packed and factor homogeneous, then D is Bonnet-rigid.*

PROOF: Let $d \in D$, B and f be as in the remark above, and suppose that $|D| = \alpha$. Let E be a canonical copy of $D \mid -d$ in $\operatorname{Sub} D$, and set $C = \{x + f(x) \mid x \in B\}$. Observe that C is a subalgebra of D (see Lemma 7 of [3]), and $E \leqslant C(d)$, since f is onto.

Let $a \in C \cap E$; since $E \cap D \mid -d$ is a prime ideal of E, we may suppose that $a < -d$. By definition of C there exists some $b \in C$ such that $a = b + f(-b)$, which implies that $b = 0$. The fact that f is a homomorphism now shows that $f(b) = 0$. It follows that $C \cap E = \underline{2}$, contradicting the fact that D is locally packed. ∎

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Department of Mathematics
Universiti Brunei Darussalam
Negara Brunei Darussalam