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NONEXISTENCE OF STABLE CURRENTS IN SUBMANIFOLDS OF A PRODUCT OF TWO SPHERES

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Dedicated to Professor Yuen-da Wang on his 68th birthday

By using techniques of the calculus of variations in geometric measure theory, we investigate the nonexistence of stable integral currents in $S^{n_1} \times S^{n_2}$ and its immersed submanifolds. Several vanishing theorems concerning the homology group of these manifolds are established.

0. INTRODUCTION

For any compact Riemannian manifold M, a theorem of Federer and Fleming [2] tells us that any non-trivial integral homology class in $H_p(M, Z)$ corresponds to a stable integral current. By establishing a second variation formula for minimal integral currents and applying it to different situations of M, Lawson and Simons [3] investigated the nonexistence of stable integral currents in M and showed some vanishing theorems concerning the pth singular homology group $H_p(M, Z)$ of M with integer coefficients. For an immersed submanifold M of the unit sphere S^n, they showed the following theorem.

THEOREM. (Lawson and Simons [3]). *Let M^m be a compact submanifold of S^n with the second fundamental form h, and p a given integer, $p \in (0, m)$. If for any $x \in M$ and any orthonormal basis $\{e_i, e_\alpha\}$ $(i = 1, \dots, p; \alpha = p+1, \dots, m)$ of T_xM the following condition is satisfied*

$$B(\xi) = \sum_{i,\alpha} [2\left\|h(e_i, e_\alpha)\right\|^2 - \langle h(e_i, e_i), h(e_\alpha, e_\alpha)\rangle] < p(m-p), \tag{0.1}$$

then there is no stable p-current in M and hence

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

In this paper we shall extend the above theorem. We shall introduce a selfadjoint linear operator Q^A on a p-subspace V of the tangent space T_xM. Replacing $B(\xi)$ in (0.1) by the trace of Q^A, we shall prove the following two theorems.

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Theorem 1. *There is no stable p-current in $S^{m_1} \times S^{m_2}$ and*

$$H_p(S^{m_1} \times S^{m_2}, Z) = 0,$$

where $0 < p < m_1 + m_2$, $p \neq m_1$ and $p \neq m_2$.

Theorem 2. *Let $\phi: M^m \to S^{n_1} \times S^{n_2}$ be an isometric immersion of a compact Riemannian manifold M in $S^{n_1} \times S^{n_2}$, and p a given integer, $p \in (0, m)$. If for any $x \in M$ and any p-subspace V of T_xM*

$$\operatorname{tr} Q^A < 0,$$

then there is no stable p-current in M and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

1. Integral currents

For later convenience, in this section we shall give a brief description of integral currents. We refer the reader to [**2, 3**] for more details.

Let M^m be an m-dimensional compact Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. And let $\mathcal{H}^p$ denote Hausdorff p-measure on M. A subset S of M is called a p-rectifiable set if S is a countable union of disjoint p-dimensional C^1 submanifolds, up to sets of $\mathcal{H}^p$-measure zero. Consider over S an $\mathcal{H}^p$-measurable section $\xi: S \to \wedge^p TM$ with the property that for $\mathcal{H}^p$-almost all $x \in S$, ξ_x is a simple vector of unit length which represents T_xS. Such a pair (S, ξ) is called an oriented, p-rectifiable set.

The set of rectifiable p-currents is defined by

$$\mathcal{R}_p(M) = \{\mathcal{S} = \sum_{n=1}^{\infty} n\mathcal{S}_n;\ \mathcal{S}_n = (S_n, \xi_n),\ M(\mathcal{S}) = \sum_{n=1}^{\infty} n\mathcal{H}^p(S_n) < \infty\}.$$

It can be thought of as the group of infinite, summable chains of oriented p-rectifiable sets.

For an oriented, p-rectifiable set $\mathcal{S} = (S, \xi)$ and a smooth p-form $\omega \in \wedge^p(M)$, define

$$\mathcal{S}(\omega) = \int_S \omega(\xi_x) dS^p(x).$$

This assigns to $\mathcal{S}$ a continuous linear functional on $\wedge^p(M)$. The boundary of $\mathcal{S}$ is defined as the linear functional on $\wedge^{p-1}(M)$ given by

$$(\partial\mathcal{S})(\omega) = \mathcal{S}(d\omega).$$

In the case that $\mathcal{S}$ and $\partial\mathcal{S}$ are both rectifiable currents, $\mathcal{S}$ is called an integral p-current. The space of integral p-currents is denoted by $\mathcal{T}_p(M)$. The direct sum $\mathcal{T}_*(M) = \bigoplus_p \mathcal{T}_p(M)$ together with $\partial\colon \mathcal{T}_*(M) \to \mathcal{T}_*(M)$ forms a differential chain complex. For this complex there is the following theorem.

THEOREM. (Federer and Fleming [2]). *For each $p \geqslant 0$ there is a natural isomorphism*

$$H_p(\mathcal{T}_*(M)) \cong H_p(M, Z).$$

And for each $\alpha \in H_p(\mathcal{T}_*(M))$ there exists a current $\mathcal{S} \in \alpha$ of "least mass", that is,

$$M(\mathcal{S}) \leqslant M(\mathcal{S}')$$

for all $\mathcal{S}' \in \alpha$.

Consider a current $\mathcal{S} \in \mathcal{R}_p(M)$ and a smooth vector field $X \in C(TM)$. Let $\phi_t\colon M \to M$ be the 1-parameter group of local diffeomorphisms generated by X. Then the rectifiable current $\phi_{t*}(\mathcal{S})$ is given by

$$\phi_{t*}(\mathcal{S})(\omega) = \mathcal{S}(\phi_t^*\omega).$$

Its "mass" is

$$M(\phi_{t*}\mathcal{S}) = \int_M \|\phi_{t*}\vec{\mathcal{S}}\| d\,\|\mathcal{S}\|,$$

where $\vec{\mathcal{S}}$ is the field of oriented tangent planes of $\mathcal{S} = \sum_n n\mathcal{S}_n$, for each $x \in \mathcal{S}_n$, $\vec{\mathcal{S}}_x = \xi_n(x)$.

A current $\mathcal{S} \in \mathcal{R}_p(M)$ is said to be stable if for each vector field X there is an $\varepsilon > 0$ such that

$$M(\phi_{t*}\mathcal{S}) \geqslant M(\mathcal{S})$$

for $|t| < \varepsilon$. This implies that for each X we have

$$\frac{d}{dt}M(\phi_{t*}\mathcal{S})\big|_{t=0} = 0, \qquad \frac{d^2}{dt^2}M(\phi_{t*}\mathcal{S})\big|_{t=0} \geqslant 0.$$

The following variation formulas have been derived by Lawson and Simons [3].

$$\text{(1.1)}\qquad \begin{aligned} \frac{d}{dt}M(\phi_{t*}\mathcal{S})\Big|_{t=0} &= \int \langle a^X(\vec{\mathcal{S}}), \vec{\mathcal{S}}\rangle d\,\|\mathcal{S}\|, \\ \frac{d^2}{dt^2}M(\phi_{t*}\mathcal{S})\Big|_{t=0} &= \int \Big\{-\langle a^X(\vec{\mathcal{S}}), \vec{\mathcal{S}}\rangle^2 + \langle a^X a^X(\vec{\mathcal{S}}), \vec{\mathcal{S}}\rangle + \|a^X(\vec{\mathcal{S}})\|^2 \\ &\qquad + \langle \nabla_{X,\vec{\mathcal{S}}}X, \vec{\mathcal{S}}\rangle\Big\} d\,\|\mathcal{S}\|, \end{aligned}$$

where $a^X: \wedge^p T_xM \to \wedge^p T_xM$ is a linear map given by

$$a^X(X_1 \wedge \ldots \wedge X_p) = \sum_j X_1 \wedge \ldots \wedge a^X(X_j) \wedge \ldots \wedge X_p,$$
$$a^X(X_j) = \nabla_{X_j} X,$$

and $\nabla_{X,\cdot} X: \wedge^p T_xM \to \wedge^p T_xM$ is another linear map defined by

$$\nabla_{X, X_1 \wedge \ldots \wedge X_p} X = \sum_j X_1 \wedge \ldots \wedge \left(\nabla_{X, X_j} X\right) \wedge \ldots \wedge X_p,$$
$$\nabla_{X, X_j} X = \nabla_X \nabla_{X_j} X - \nabla_{\nabla_X X_j} X.$$

To any simple p-vector $\xi \in \wedge^p T_xM$ and $X \in C(TM)$, let ϕ_t be the flow generated by X, and define

$$Q_\xi(X) = \frac{d^2}{dt^2} \|\phi_{t*}\xi\| \,|_{t=0} \,.$$

Then the expression (1.1) can be denoted by

$$\text{(1.2)} \qquad \frac{d^2}{dt^2} M(\phi_{t*}S) \,|_{t=0} = \sum_n n \int_{S_n} Q_{\xi_n}(X) d\mathcal{H}^p(x).$$

If $X = \nabla f$ for some $f \in C^3(M)$, from [3, p.436] we have

$$\text{(1.3)} \qquad Q_\xi(X) = \left[\sum_j \langle a^X(e_j), e_j\rangle\right]^2 + 2\sum_{j,\alpha} \langle a^X(e_j), e_\alpha\rangle^2 + \sum_j \langle \nabla_{X, e_j} X, e_j\rangle,$$

where $\{e_i, e_\alpha\}$ $(i = 1, \ldots, p; \alpha = p+1, \ldots, m)$ is an orthonormal basis of T_xM and $\xi = e_1 \wedge \ldots \wedge e_p$.

2. A SELFADJOINT LINEAR OPERATOR

For a p-rectifiable set S in M, we know that at $\mathcal{H}^p$-almost all points $x \in S$, there exists an approximate p-space $T_xS \subset T_xM$, to S. In this section we shall introduce a selfadjoint linear operator on T_xS. Its trace is equal to the trace of Q_ξ given by (1.3).

Let $\phi: M^m \to N^n$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N. The Levi-Civita connections of M and N are denoted by ∇ and $\overline{\nabla}$ respectively. For any $X, Y \in C(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h is the second fundamental form of the immersion ϕ. If $V(N, M)$ is the normal bundle of M in N, for a smooth section $\nu \in C(V(N, M))$ we have

$$\overline{\nabla}_X \nu = -A_\nu X + \nabla^{\perp}_X \nu,$$

where A_ν is the so-called the shape operator determined by ν. We know that

$$\langle A_\nu X, Y\rangle = \langle h(X, Y), \nu\rangle. \tag{2.1}$$

For a given integer $p \in (0, m)$ let V be a p-dimensional subspace in $T_x M$. Define a map $B_\nu \colon V \to V$ associated with A_ν by

$$B_\nu X = \text{ orthogonal projection of } A_\nu X \text{ onto } V,$$

where $X \in V$. If $\{e_i\}$ is an orthonormal basis of V, we have

$$B_\nu X = \sum_i \langle A_\nu X, e_i\rangle e_i. \tag{2.2}$$

It may be seen that B_ν is a selfadjoint linear operator on V because A_ν is selfadjoint linear. Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_x(N, M)$ and $A_\lambda = A_{\nu_\lambda}$. Define a selfadjoint linear map $Q^A \colon V \to V$ associated with the immersion ϕ by

$$Q^A X = \sum_\lambda \left[2\left(\sum_i \langle A_\lambda^2 X, e_i\rangle e_i - B_\lambda^2 X\right) - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda) B_\lambda X\right], \tag{2.3}$$

where $X \in V$ and $\{e_i\}$ is an orthonormal basis of V. Q^A is independent of the choice of orthonormal bases of $V_x(N, M)$ and V. And its trace is

$$\begin{aligned} \operatorname{tr} Q^A &= \sum_i \langle Q^A e_i, e_i\rangle \\ &= \sum_\lambda \left[2\left(\sum_i \langle A_\lambda^2 e_i, e_i\rangle - \operatorname{tr} B_\lambda^2\right) - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda)\operatorname{tr} B_\lambda\right]. \end{aligned} \tag{2.4}$$

Let $\{e_\alpha\}$ be an orthonormal basis of $V^\perp$ which is the orthogonal complement of V in $T_x M$. Then $\{e_i, e_\alpha\}$ is an orthonormal basis of $T_x M$ and

$$\sum_i \langle A_\lambda^2 e_i, e_i\rangle = \sum_{i,j} \langle A_\lambda e_i, e_j\rangle^2 + \sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha\rangle^2,$$

$$\operatorname{tr} B_\lambda^2 = \sum_i \langle B_\lambda^2 e_i, e_i\rangle = \sum_{i,j} \langle A_\lambda e_i, e_j\rangle^2.$$

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Hence (2.4) becomes

$$\operatorname{tr} Q^A = \sum_\lambda \left[2\sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha\rangle^2 - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda)\operatorname{tr} B_\lambda \right]. \tag{2.5}$$

Now assume that $\psi\colon N^n \to R^l$ is an isometric immersion of the Riemannian manifold N in the Euclidean space R^l. Let D be the Levi-Civita connection on R^l. Associated with the isometric immersion $x = \psi \circ \phi\colon M^m \to R^l$, the shape operator A'_ν determined by $\nu \in C(V(R^l, M))$ is given by

$$A'_\nu Y = -(D_Y \nu)^T, \tag{2.6}$$

where $Y \in C(TM)$. In particular, if $\nu \in C(V(N, M))$,

$$\begin{aligned} A'_\nu Y &= -(D_Y\nu)^T = -[\overline{\nabla}_Y \nu + \overline{h}(\nu, T)]^T \\ &= -\left(-A_\nu Y + \nabla^\perp_Y \nu\right)^T = A_\nu Y, \end{aligned} \tag{2.7}$$

and if $\nu \in C(V(R^l, N))$,

$$A'_\nu Y = \left(\overline{A}_\nu Y\right)^T. \tag{2.8}$$

For a given vector $v \in R^l$, we define two vector fields v^T and $v^\perp$ on M by

$$\begin{aligned} v^T(x) &= \text{ orthogonal projection of } v \text{ onto } T_x M, \\ v^\perp(x) &= \text{ orthogonal projection of } v \text{ onto } V_x(R^l, M). \end{aligned} \tag{2.9}$$

To any unit, simple p-vector $\xi \in \wedge^p T_x M$, we shall calculate the quadratic form $Q_\xi(v^T)$ given by (1.3). Using (2.6), we have

$$\begin{aligned} a^{v^T}(Y) &= \nabla_Y v^T = \left(D_Y v - D_Y v^\perp\right)^T = A'_{v^\perp} Y, \\ \nabla^\perp_Y v^\perp &= \left(D_Y v - D_Y v^T\right)^\perp = -h'\left(v^T, Y\right). \end{aligned} \tag{2.10}$$

These imply

$$\nabla_{v^T, Y}\, v^T = \left(\nabla^*_{v^T} A'\right)_{v^\perp} Y - A_{h'\left(v^T, v^T\right)} Y, \tag{2.11}$$

where $\nabla^*_{v^T} A'$ is the derivative with respect to the connection of Van der Waerden-Bortolotti ([**1**, p.65]). Putting (2.10) and (2.11) into (1.3), we obtain

$$\begin{aligned} Q_\xi\left(V^T\right) = &\left[\sum_j \langle A'_{v^\perp} e_j, e_j\rangle\right]^2 + 2\sum_{j,\alpha}\langle A'_{v^\perp} e_j, e_\alpha\rangle^2 \\ &+ \sum_j \langle \left(\nabla^*_{v^T} A'\right)_{v^\perp} e_j, e_j\rangle - \sum_j \langle A'_{h'\left(v^T, v^T\right)} e_j, e_j\rangle. \end{aligned} \tag{2.12}$$

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Let (S, ξ) be an oriented, p-rectifiable set. With a point $x \in S$ associate a tangent p-space $V = T_xS \subset T_xM$. Choose an orthonormal basis $\{e_i, e_\alpha\}$ of T_xM such that $\{e_i\}$ is a basis of V and $\xi = e_1 \wedge \ldots \wedge e_p$. Suppose that $\{\nu_\sigma\}$ is an orthonormal basis of $V_x(R^l, M)$ associated with the immersion $\psi \circ \phi \colon M \to R^l$, $A'_\sigma = A'_{\nu_\sigma}$, and $Q^{A'}$ is the selfadjoint linear operator on V defined by (2.3). Consider Q_ξ as a quadratic form defined on the set

(2.13) $$\theta = \{v^T \in C(TM); v \in R^l,\ v^T \text{ is defined by (2.9)}\}.$$

There is the following relation between the quadratic form Q_ξ and the operator $Q^{A'}$.

PROPOSITION 1. *$\operatorname{tr} Q_\xi = \operatorname{tr} Q^{A'}$, where*

(2.14) $$\operatorname{tr} Q^{A'} = \sum_\sigma \Big[2\sum_{i,\alpha}\langle A'_\sigma e_i, e_\alpha\rangle^2 - (\operatorname{tr} A'_\sigma - \operatorname{tr} B'_\sigma)\operatorname{tr} B'_\sigma\Big].$$

PROOF: Observing that at the given point $x \in M$, $\{e_i, e_\alpha, \nu_\sigma\}$ is an orthonormal basis of R^l and $(\nabla^*_{v^T} A')_{v^\perp} = 0$ as $v^T = 0$ or $v^\perp = 0$, from (2.12) we have

$$\begin{aligned}
\operatorname{tr} Q_\xi &= \sum_i Q_\xi(e_i) + \sum_\alpha Q_\xi(e_\alpha) + \sum_\sigma Q_\xi(\nu_\sigma)\\
&= -\sum_{i,j}\langle A'_{h'(e_i,e_i)}e_j, e_j\rangle - \sum_{\alpha,j}\langle A'_{h'(e_\alpha,e_\alpha)}e_j, e_j\rangle\\
&\quad + \sum_\sigma\left\{\Big[\sum_j\langle A'_\sigma e_j, e_j\rangle\Big]^2 + 2\sum_{j,\alpha}\langle A'_\sigma e_j, e_\alpha\rangle^2\right\}\\
&= -\sum_\sigma\Big[\sum_{i,j}\langle A'_\sigma e_i, e_i\rangle\langle A'_\sigma e_j, e_j\rangle + \sum_{\alpha,i}\langle A'_\sigma e_\alpha, e_\alpha\rangle\langle A'_\sigma e_i, e_i\rangle\Big]\\
&\quad + \sum_\sigma\left\{\Big[\sum_j\langle A'_\sigma e_j, e_j\rangle\Big]^2 + 2\sum_{j,\alpha}\langle A'_\sigma e_j, e_\alpha\rangle^2\right\}\\
&= \sum_{\sigma,i,\alpha}[2\langle A'_\sigma e_i, e_\alpha\rangle^2 - \langle A'_\sigma e_\alpha, e_\alpha\rangle\langle A'_\sigma e_i, e_i\rangle].
\end{aligned}$$

Since

$$\operatorname{tr} A'_\sigma = \sum_i\langle A'_\sigma e_i, e_i\rangle + \sum_\alpha\langle A'_\sigma e_\alpha, e_\alpha\rangle$$

and

$$\operatorname{tr} B'_\sigma = \sum_i\langle A'_\sigma e_i, e_i\rangle$$

we obtain $\operatorname{tr} Q_\xi = \operatorname{tr} Q^{A'}$. ∎

From the above proof, expression (2.14) can also be written as

$$\operatorname{tr} Q^{A'} = \sum_{\sigma, i, \alpha} [2\langle A'_\sigma e_i, e_\alpha\rangle^2 - \langle A'_\sigma e_\alpha, e_\alpha\rangle\langle A'_\sigma e_i, e_i\rangle]. \tag{2.15}$$

At a point $x \in M$, we take an orthonormal basis $\{\nu_\lambda, \eta_a\}$ of $V_x(R^l, M)$ so that $\{\nu_\lambda\}$ and $\{\eta_a\}$ are bases of $V_x(N, M)$ and $V_x(R^l, N)$ respectively. From (2.7), (2.8) and (2.15) we obtain

$$\operatorname{tr} Q^{A'} = \operatorname{tr} Q^A + \overline{A}(V), \tag{2.16}$$

where $\operatorname{tr} Q^A$ is given by (2.5) and

$$\overline{A}(V) = \sum_{a, i, \alpha} [2\langle \overline{A}_a e_i, e_\alpha\rangle^2 - \langle \overline{A}_a e_\alpha, e_\alpha\rangle\langle \overline{A}_a e_i, e_i\rangle]. \tag{2.17}$$

Note that $\overline{A}(V) \neq \operatorname{tr} Q^{\overline{A}}$.

THEOREM. *Let $\phi\colon M^m \to N^n$ be an isometric immersion of a compact Riemannian manifold M in a submanifold N of R^l, and p a given integer, $p \in (0, m)$. Suppose that for any $x \in M$ and any p-subspace V of $T_x M$,*

$$\operatorname{tr} Q^A < -\overline{A}(V). \tag{2.18}$$

Then there is no stable p-current in M and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

PROOF: Let θ be the set given by (2.13). If $v^T \in \theta$, v^T is the gradient ∇f of the function $f(x) = \langle v, x\rangle$ on M. To each $\mathcal{S} \in \mathcal{R}_p(M)$ associate a quadratic form $Q_{\mathcal{S}}$ on θ as follows. For $X \in \theta$ let ϕ_t be the flow generated by X and set

$$Q_{\mathcal{S}}(X) = \frac{d^2}{dt^2} M(\phi_{t*}\mathcal{S})\,|_{t=0}\,.$$

From (1.2) we have

$$\operatorname{tr} Q_{\mathcal{S}} = \sum_n n \int_{S_n} \operatorname{tr} Q_{\xi_n}\, d\mathcal{H}^p(x).$$

But from (1.3), Proposition 1 and (2.16), (2.18) implies $\operatorname{tr} Q_{\xi_n} < 0$ for any n. Therefore $\operatorname{tr} Q_{\mathcal{S}} < 0$. This implies that there is no stable p-current in M. By using Federer-Fleming's theorem, we have

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

∎

If N^n in the above theorem is a totally umbilical submanifold immersed in R^l, N^n is of constant curvature $c \geqslant 0$. In this case $\overline{A}(V)$ given by (2.17) becomes

$$\overline{A}(V) = -p(m-p)c.$$

Hence we obtain

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Corollary 1. *Let M^m be a compact submanifold immersed in a totally umbilical submanifold N^n of R^l. If for any $x \in M$ and any p-subspace V of $T_x M$,*

$$\operatorname{tr} Q^A < p(m-p)c,$$

where c is the sectional curvature of N, then there is no stable p-current in M and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Remark 1. In the case $N^n = S^n$ we have $c = 1$, and Corollary 1 becomes Lawson and Simons' theorem. And when $N^n = R^n$, Corollary 1 is due to Xin [4, Theorem 1].

3. Main results

Let $m_1 + m_2 = m$ and

$$M^m = S^{m_1} \times S^{m_2} = \{(x_1, x_2) \in R^{m+2};\, x_\lambda \in R^{m_\lambda+1} \text{ and } \|x_\lambda\| = 1,\ \lambda = 1, 2\}.$$

Then M^m is a submanifold of R^{m+2}. At $x = (x_1, x_2) \in M^m$ we take an orthonormal basis $\{\nu_\lambda\}$ of $V_x(R^{m+2}, M)$ as follows

$$\nu_1 = (x_1, 0), \qquad \nu_2 = (0, x_2).$$

It may be seen that the shape operators A_λ can be denoted by the matrices

$$A_1 = -\begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = -\begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix},$$

where I_λ is the $m_\lambda \times m_\lambda$ identity matrix for each $\lambda = 1, 2$. Hence for any $X \in T_x M$ we have $A_\lambda X = -X_\lambda$, where X_λ is the orthogonal projection of X onto $T_{x_\lambda} S^{m_\lambda}$.

At $x \in M$, we take an orthonormal basis $\{e_i, e_\alpha\}$ of $T_x M$ so that $\{e_i\}$ is an orthonormal basis of the p-subspace V. Denoting the orthogonal projection of e_i (respectively e_α) onto $T_{x_\lambda} S^{m_\lambda}$ by $e_{i\lambda}$ (respectively $e_{\alpha\lambda}$), we have

$$\langle A_\lambda e_i, e_\alpha \rangle = -\langle e_{i\lambda}, e_{\alpha\lambda} \rangle,$$
$$\operatorname{tr} A_\lambda = \sum_i \langle A_\lambda e_i, e_i \rangle + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle = -\sum_i \|e_{i\lambda}\|^2 - \sum_\alpha \|e_{\alpha\lambda}\|^2,$$
$$\operatorname{tr} B_\lambda = \sum_i \langle B_\lambda e_i, e_i \rangle = \sum_i \langle A_\lambda e_i, e_i \rangle = -\sum_i \|e_{i\lambda}\|^2.$$

Substituting these into (2.5) we obtain

$$\operatorname{tr} Q^A = \sum_{i,\alpha} \Big[2\big(\langle e_{i1}, e_{\alpha 1} \rangle^2 + \langle e_{i2}, e_{\alpha 2} \rangle^2\big) - \Big(\|e_{\alpha 1}\|^2 \|e_{i1}\|^2 + \|e_{\alpha 2}\|^2 \|e_{i2}\|^2\Big) \Big]. \tag{3.1}$$

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Since $e_i = e_{i1} + e_{i2}$ and $e_\alpha = e_{\alpha 1} + e_{\alpha 2}$, we have

$$\|e_{i1}\|^2 + \|e_{i2}\|^2 = 1, \qquad \|e_{\alpha 1}\|^2 + \|e_{\alpha 2}\|^2 = 1,$$
$$\langle e_{i1}, e_{\alpha 1}\rangle + \langle e_{i2}, e_{\alpha 2}\rangle = 0.$$

So (3.1) becomes

$$\operatorname{tr} Q^A = \sum_{i,\alpha} \Big[4\langle e_{i1}, e_{\alpha 1}\rangle^2 + \|e_{i1}\|^2 + \|e_{\alpha 1}\|^2 - 2\,\|e_{i1}\|^2\,\|e_{\alpha 1}\|^2\Big] - p(m-p). \tag{3.2}$$

LEMMA. *For each pair of fixed indices i, α, let*

$$f_{i\alpha} = 4\langle e_{i1}, e_{\alpha 1}\rangle^2 + \|e_{i1}\|^2 + \|e_{\alpha 1}\|^2 - 2\,\|e_{i1}\|^2\,\|e_{\alpha 1}\|^2. \tag{3.3}$$

Then $f_{i\alpha} \leqslant 1$ and equality holds if and only if $e_i \in T_{x_1}S^{m_1}$ and $e_\alpha \in T_{x_2}S^{m_2}$, or $e_\alpha \in T_{x_1}S^{m_1}$ and $e_i \in T_{x_2}S^{m_2}$.

PROOF: Let $e_i^s (s = 1, 2, \dots, m_1)$ (respectively e_α^s) be the components of e_{i1} (respectively e_{α_1}) with respect to an orthonormal basis of $T_{x_1}S^{m_1}$. Then (3.3) becomes

$$f_{i\alpha} = 4\Big(\sum_s e_i^s e_\alpha^s\Big)^2 + \sum_s (e_i^s)^2 + \sum_s (e_\alpha^s)^2 - 2\sum_{s,t}(e_i^s)^2(e_\alpha^t)^2, \tag{3.4}$$

where

$$0 \leqslant \sum_s (e_i^s)^2 \leqslant 1, \qquad 0 \leqslant \sum_s (e_\alpha^s)^2 \leqslant 1. \tag{3.5}$$

In order to seek the maximum of $f_{i\alpha}$ under the condition (3.5), partially differentiating (3.4) with respect to each variable and equating to zero, we obtain

$$4\Big(\sum_t e_i^t e_\alpha^t\Big) e_\alpha^s + e_i^s - 2\sum_t \big(e_\alpha^t\big)^2 e_i^s = 0,$$
$$4\Big(\sum_t e_i^t e_\alpha^t\Big) e_i^s + e_\alpha^s - 2\sum_t \big(e_i^t\big)^2 e_\alpha^s = 0.$$

These equations can be expressed by

$$u\, e_i^s = 4w\, e_\alpha^s, \tag{3.6}$$
$$v\, e_\alpha^s = 4w\, e_i^s, \tag{3.7}$$

where $u = 2\sum_s (e^s_\alpha)^2 - 1$, $v = 2\sum_s (e^s_i)^2 - 1$, $w = \sum_s e^s_i e^s_\alpha$. From (3.6) we obtain

$$\frac{1}{2}(1+v)u = 4w^2, \tag{3.8}$$

$$uw = 2(1+u)w. \tag{3.9}$$

Similarly, from (3.7) we have

$$\frac{1}{2}(1+u)v = 4w^2, \tag{3.10}$$

$$vw = 2(1+v)w. \tag{3.11}$$

(3.8) and (3.10) give $u = v$. If $w \neq 0$, from (3.9) we have $u = 2(1+u)$. So $u = -2$ and thus $\sum_s (e^s_\alpha)^2 = -1/2$; this is impossible. Therefore w must be zero. And from (3.8) we have $(1+v)u = 0$, that is,

(i) $1 + v = 0$; this gives $e^s_i = 0$, and $e^s_\alpha = 0$ from $u = v$; or
(ii) $u = v = w = 0$; this implies $\sum_s (e^s_i)^2 = \sum_s (e^s_\alpha)^2 = 1/2$ and $\sum_s e^s_i e^s_\alpha = 0$.

From (3.4),

($\star$) (i) implies $f_{i\alpha} = 0$, and (ii) implies $f_{i\alpha} = 1/2$.

Besides, if $\sum_s (e^s_i)^2 = 1$, that is, $\|e_{i1}\|^2 = 1$, then $e_{i2} = 0$ and thus

$$\sum_s e^s_i e^s_\alpha = \langle e_{i1}, e_{\alpha 1}\rangle = -\langle e_{i2}, e_{\alpha 2}\rangle = 0.$$

From (3.4) we have

$$f_{i\alpha} = 1 - \sum_s (e^s_\alpha)^2 \leqslant 1,$$

equality holds if and only if $\sum_s (e^s_\alpha)^2 = 0$. Combining (*), we see that under the condition (3.5), $f_{i\alpha} \leqslant 1$. Clearly equality holds if and only if $e_i \in T_{x_1}S^{m_1}$ and $e_\alpha \in T_{x_2}S^{m_2}$, or $e_\alpha \in T_{x_1}S^{m_1}$ and $e_i \in T_{x_2}S^{m_2}$. $\square$

From this lemma, (3.2) gives

$$\mathrm{tr}Q^A = \sum_{i,\alpha} f_{i\alpha} - p(m-p) \leqslant 0.$$

It is easy to check that equality holds if and only if $\{e_i\} \subset T_{x_1}S^{m_1}$ and $\{e_\alpha\} \subset T_{x_2}S^{m_2}$, or $\{e_\alpha\} \subset T_{x_1}S^{m_1}$ and $\{e_i\} \subset T_{x_2}S^{m_2}$. These imply the p-subspace $V = T_{x_1}S^{m_1}$ and $V^\perp = T_{x_2}S^{m_2}$, or $V = T_{x_2}S^{m_2}$ and $V^\perp = T_{x_1}S^{m_1}$. Hence we have

Proposition 2. *For the isometric immersion* $S^{m_1} \times S^{m_2} \to R^{m+2}$ $(m = m_1 + m_2)$, $\operatorname{tr} Q^A \leqslant 0$. *Furthermore, if* $p \overline{\in} \{m_1, m_2\}$, $\operatorname{tr} Q^A < 0$.

Proof of Theorem 1: In the Theorem of Section 2 we take $M^m = S^{m_1} \times S^{m_2}$ and $N^n = R^{m+2}$; then $\overline{A}(V) = 0$ from (2.17). Combining Proposition 2 and the Theorem in Section 2, we obtain Theorem 1. □

Proof of Theorem 2: Let $\{e_i, e_\alpha\}$ be an orthonormal basis of $T_x M$ so that $\{e_i\}$ is a basis of the p-subspace V. Note that the shape operators of $S^{n_1} \times S^{n_2} \to R^{n_1+n_2+2}$ are $\overline{A}_a$ $(a = 1, 2)$, $\overline{A}_a X = -X_a$ where $X \in T_x M$ and X_a is the orthogonal projection of X onto $T_{x_a} S^{n_a}$. So $\langle \overline{A}_a e_i, e_\alpha \rangle = -\langle e_{ia}, e_{\alpha a} \rangle$, $\langle \overline{A}_a e_i, e_i \rangle = -\|e_{ia}\|^2$, and $\langle \overline{A}_a e_\alpha, e_\alpha \rangle = -\|e_{\alpha a}\|^2$. Thus (2.17) becomes

$$\overline{A}(V) = \sum_{i,\alpha} \left[2\left(\langle e_{i1}, e_{\alpha 1} \rangle^2 + \langle e_{i2}, e_{\alpha 2} \rangle^2 \right) - \left(\|e_{\alpha 1}\|^2 \|e_{i1}\|^2 + \|e_{\alpha 2}\|^2 \|e_{i2}\|^2 \right) \right].$$

So from the Lemma we have $\overline{A}(V) = \sum_{i,\alpha} f_{i\alpha} - p(m-p) \leqslant 0$. Combining this with the Theorem in Section 2 we obtain Theorem 2. □

Corollary 2. *Let* M^m *be a compact submanifold isometrically immersed in* $S^{n_1} \times S^{n_2}$. *If for any point* $x \in M$ *and any* p*-subspace* V *of* $T_x M (0 < p < m)$ *the selfadjoint linear operator* Q^A *on* V *is negative definite, then there is no stable* p*-current in* M.

Remark 2. Theorems and corollaries in this paper are true if one replaces the integers by any finitely generated abelian coefficient group because the Federer-Fleming theorem remains true in the latter case. Besides, one can easily generalise these theorems and corollaries to arbitrary varifolds on M from [**3**, p.436, Remark 4].

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