SOME FINITE NILPOTENT p-GROUPS

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Consider the following statement:

For every positive integer n and every prime p there is a finite p-group of nilpotency class (precisely) c all of whose $(n-1)$-generator subgroups are nilpotent of class at most n.

This statement is obviously true for $c \leqq n$. It is a consequence of Theorem 2 of Macdonald-Neumann [5] that the statement is false for $c \geqq n+2$. In this note we complete the discussion by showing the statement is true for $c = n+1$. This also answers the question raised in the last paragraph of the first section of [5].

Specifically we show:

THEOREM. *For every positive integer n and every prime p there is a finite p-group of nilpotency class $n+1$ all of whose $(n-1)$-generator subgroups are nilpotent of class at most n. Moreover for $p > n+1$ there is a group of exponent p of the required kind.*

Some partial results are known:

$n = 1, 2$	– trivial;
$n = 3$	– Example 4.1 of [2];
$p = 2$	– an easy consequence of **34.54** of Hanna Neumann's book [6];
$p = 5, n = 4$	– § 5 of Lazard [4].

PROOF. For $p > n+1$ the result is an easy consequence of Higman's theory of varieties of prime exponent and small class [3]. We rely heavily on his exposition and follow his notation.

Let t be the subfunctor of L_{n+1} generated by the irreducible subfunctors which are equivalent to partition functors $[\lambda]$ where (λ) ranges over partitions of $n+1$ into at most $n-1$ parts, and let $\mathfrak{X}$ be the subvariety of $\mathfrak{B}_{p,n+1}$ corresponding to t. It follows from the multiplicity formula ([3] p. 170) that L_{n+1} contains (precisely) one irreducible subfunctor equivalent to $[2, 1^{n-1}]$ where $(2, 1^{n-1})$ is the partition of $n+1$ into n parts. Hence the free group F_p of rank n of $\mathfrak{X}$ is nilpotent of class precisely $n+1$. On the other hand the free group of rank $n-1$ of $\mathfrak{X}$ has class at most n (by the remarks in the last paragraph on p. 168 of [3]). Hence every $(n-1)$-generator subgroup of F_p has class at most n, and therefore F_p has all the properties required.

The result for $p \leqq n+1$ follows by fairly standard arguments. The direct product of the F_p taken over all $p > n+1$ has class $n+1$, the last term of its lower central series contains an element of infinite order and all its $(n-1)$-generator subgroups have class at most n. It follows that there is a finitely generated torsion-free group of class $n+1$ all of whose $(n-1)$-generator subgroups have class at most n. The existence of groups of the required kind follows from the result of Gruenberg that a finitely generated torsion-free nilpotent group is residually a finite p-group for every prime p (Theorem 2.1 of [1]).

REMARK. This is not our original proof which depended on using the theory of basic commutators as developed by Ward [7] to construct a suitable torsion-free group.

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