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# THE TOPOLOGICAL STABILITY OF DIFFEOMORPHISMS

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## § 1. Introduction

The present paper is concerned with the stability of diffeomorphisms of $C^\infty$ closed manifolds. Let $M$ be a $C^\infty$ closed manifold and $\mathrm{Diff}^r(M)$ be the space of $C^r$ diffeomorphisms of $M$ endowed with the $C^r$ topology (in this paper we deal with only the case $r = 0$ or 1). Let us define

$$\mathscr{F}(M) = \left\{ f \in \mathrm{Diff}^1(M) \,\middle|\, \begin{array}{l} \text{there exists a } C^1 \text{ neighborhood } \mathscr{U}(f) \text{ of} \\ f \text{ such that all periodic points of every} \\ g \in \mathscr{U}(f) \text{ are hyperbolic} \end{array} \right\}.$$

Then every $C^1$ structurally stable and $\Omega$-stable diffeomorphism belongs to $\mathscr{F}(M)$ (see [3]). In light of this result Mañé solved in [5] the $C^1$ Structural Stability Conjecture by Palis and Smale. After that Palis [9] obtained, in proving that every diffeomorphism belonging to $\mathscr{F}(M)$ is approximated by Axiom $A$ diffeomorphisms with no cycle, the $C^1$ $\Omega$-Stability Conjecture. Recently Aoki [2] proved that every diffeomorphism belonging to $\mathscr{F}(M)$ is Axiom $A$ diffeomorphisms with no cycle (a conjecture by Palis and Mañé). For the topological stability Walters [14] proved that every Anosov diffeomorphism is topologically stable. In [7] Nitecki showed that every Axiom $A$ diffeomorphism having strong transversality is topologically stable, and that every Axiom $A$ diffeomorphism having no cycle is $\Omega$-topologically stable.

Thus it will be natural to ask whether topologically stable diffeomorphisms belonging to $\mathrm{Diff}^1(M)$ satisfy Axiom $A$ and strong transversality.

Let $f \in \mathrm{Diff}^1(M)$. Then $f: M \to M$ is topologically stable if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in \mathrm{Diff}^0(M)$ with $d(f, g) < \delta$ there exists a continuous map $h: M \to M$ satisfying $h \circ g = f \circ h$ and $d(h, \mathrm{id}) < \varepsilon$ (where id is the identity). Note that if $\varepsilon$ is sufficiently small then the above continuous map $h$ is surjective since $h$ is homotopic to id. We denote by $\Omega(f)$ the set of nonwandering points of $f$. A diffeo-

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morphism $f$ is $\Omega$-topologically stable if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in \mathrm{Diff}^0(M)$ with $d(f, g) < \delta$ such that there exists a continuous map $h : \Omega(g) \to \Omega(f)$ $(h(\Omega(g)) \subset \Omega(f))$ satisfying $h \circ g = f \circ h$ on $\Omega(g)$ and $d(h(x), x) < \varepsilon$ for all $x \in \Omega(g)$.

A sequence $\{x_i \mid i \in (a, b)\}$ $(-\infty \leq a < b \leq \infty)$ of points is called a $\delta$-pseudo orbit for $f$ if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. Given $\varepsilon > 0$ a pseudo orbit $\{x_i\}$ is said to be $\varepsilon$-traced by a point $x \in M$ if $d(f^i(x), x_i) < \varepsilon$ for $i \in (a, b)$. We say that $f$ has the pseudo orbit tracing property (abbrev. POTP) if for $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo orbit for $f$ can be $\varepsilon$-traced by some point of $M$.

For compact spaces the notions stated above are independent of the compatible metric used. It is known that if $f: M \to M$ is topologically stable then $f$ has POTP and all the periodic points of $f$ are dense in $\Omega(f)$ (see [6], [15]), and that if $f: M \to M$ has POTP then so is $f_{|\Omega(f)} : \Omega(f) \to \Omega(f)$ (see [1]).

To mention precisely our aim let us define the subsets of $\mathrm{Diff}^1(M)$ as

$$\mathrm{AxS}(M) = \{f \mid f \text{ satisfies Axiom } A \text{ and strong transversality}\},$$
$$\mathrm{AxN}(M) = \{f \mid f \text{ satisfies Axiom } A \text{ and no cycle}\},$$
$$\mathrm{POTP}(M) = \mathrm{int}\{f \mid f \text{ has POTP}\},$$
$$\Omega\text{-POTP}(M) = \mathrm{int}\{f \mid f_{|\Omega(f)} \text{ has POTP}\},$$
$$\mathrm{TS}(M) = \mathrm{int}\{f \mid f \text{ is topologically stable}\},$$
$$\Omega\text{-TS}(M) = \mathrm{int}\{f \mid f \text{ is } \Omega\text{-topologically stable}\}.$$

Here int $E$ denotes the interior of $E$. Among these sets exist the following

$$\mathrm{POTP}(M) \subset \Omega\text{-POTP}(M) \text{ ([1])}, \qquad \mathrm{TS}(M) \subset \Omega\text{-TS}(M),$$
$$\mathrm{TS}(M) \subset \mathrm{POTP}(M) \text{ ([6] or [15])}, \qquad \mathrm{AxS}(M) \subset \mathrm{TS}(M) \text{ ([7] or [12])},$$
$$\mathrm{AxN}(M) \subset \Omega\text{-TS}(M) \text{ ([7])}, \qquad \mathrm{AxN}(M) = \mathscr{F}(M) \text{ ([2])}.$$

For the question mentioned above we shall show the following

THEOREM 1. *Under the above notations, the following holds.*

(1) $\Omega\text{-TS}(M) = \mathscr{F}(M)$,

(2) $\mathrm{TS}(M) = \mathrm{AxS}(M)$.

By Theorem 1 the following is concluded.

$$\Omega\text{-TS}(M) = \mathrm{AxN}(M) = \mathscr{F}(M) \subset \mathrm{TS}(M) = \mathrm{AxS}(M).$$

We have the following theorem as an easy conclusion of Theorem 1.

THEOREM 2. *Let* $f \in \mathrm{POTP}(M)$. *If* $\dim W^s(x, f) = 0$ *or* $\dim M$ *or* $\dim M - 1$ *for* $x \in M$, *then* $f$ *belongs to* $\mathrm{AxS}(M)$.

The proof of Theorem 2 will be given in § 5.

The conclusions of Theorem 1 will be obtained in proving the following three propositions.

PROPOSITION 1. $\Omega$-$\mathrm{POTP}(M) \subset \mathscr{F}(M)$.

The proof will be based on the techniques of the proof of Theorem 1 of Franks [3].

If we establish Proposition 1, then we have that $\Omega$-$\mathrm{POTP}(M) = \mathscr{F}(M)$ by the fact mentioned above.

PROPOSITION 2. $\Omega$-$\mathrm{TS}(M) \subset \mathscr{F}(M)$.

For the proof we need the methods in [6] or [15], in which it is proved that topological stability implies POTP, and the facts used in the proof of Proposition 1.

Since Proposition 2 shows $\Omega$-$\mathrm{TS}(M) = \mathscr{F}(M)$, (1) of Theorem 1 is concluded.

PROPOSITION 3. $\mathrm{TS}(M) \subset \mathrm{AxS}(M)$.

A result that Axiom A diffeomorphisms satisfying structural stability have strong tansversality was proved in Robinson [11]. However every diffeomorphism dealt with in Proposition 3 is Axiom $A$ and topologically stable. Thus it does not follow from Robinson's result that the diffeomorphism satisfies strong transversality.

Proposition 3 ensures that $\mathrm{TS}(M) = \mathrm{AxS}(M)$ and therefore (2) of Theorem 1 is concluded.

## § 2. Proof of Proposition 1

Let $P(f)$ denote the set of periodic points of $f \in \Omega$-$\mathrm{POTP}(M)$. If $p \in P(f)$ with the prime period $k$, then $T_pM$ splits into the direct sum $T_pM = E^u(p) \oplus E^s(p) \oplus E^c(p)$ where $E^u(p)$, $E^s(p)$ and $E^c(p)$ are $D_pf^k$-invariant subspaces corresponding to the absolute values of the eigenvalues of $D_pf^k$ with greater than one, less than one and equal to one.

To obtain Proposition 1 it suffices to prove that each $p \in P(f)$ is hyperbolic: i.e. $E^c(p) = \{0\}$. On the contrary suppose that $p \in P(f)$ is non-hyperbolic and let $k > 0$ be the prime period of $p$. Then, for every $\varepsilon > 0$ there exists a linear automorphism $\mathcal{O} : T_pM \to T_pM$ such that

$$(2.1)\quad \begin{cases} \text{(i)} & \|\mathcal{O}\| \le \varepsilon\,, \\ \text{(ii)} & \mathcal{O}(E^{\sigma}(p)) = E^{\sigma}(p) \quad \text{for } \sigma = s,\ u,\ c\,, \\ \text{(iii)} & \text{all eigenvalues of } \mathcal{O}\circ D_pf^k | E^c(p) \text{ are of a root of unity.} \end{cases}$$

Making use of the following Franks's lemma, we can find $\delta_0 > 0$ and a diffeomorphism $g \in \Omega$-POTP$(M)$ such that

$$(2.2)\quad \begin{cases} \text{(i)} & B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(f^j(p)) = \emptyset \quad \text{for } 0 \le i \ne j \le k-1\,, \\ \text{(ii)} & g(x) = f(x) \quad \text{for } x \in \{p, f(p), \cdots, f^{k-1}(p)\} \cup \{M - \bigcup_{i=0}^{k-1} B_{4\delta_0}(f^i(p))\}\,, \\ \text{(iii)} & g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)}f \circ \exp^{-1}_{f^i(p)}(x) \\ & \qquad\qquad \text{for } x \in B_{\delta_0}(f^i(p))(0 \le i \le k-2)\,, \\ \text{(iv)} & g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{k-1}(p)}f \circ \exp^{-1}_{f^{k-1}(p)} \quad \text{for } x \in B_{\delta_0}(f^{k-1}(p))\,. \end{cases}$$

FRANKS'S LEMMA. *For $f \in \mathrm{Diff}^1(M)$ let $F$ be a finite set of distinct points in $M$. If $\varepsilon > 0$ is sufficiently small and $G_x : T_xM \to T_{f(x)}M$ is an isomorphism such that $\|G_x - D_xf\| < \varepsilon/10$ $(x \in F)$, then there exist $\delta > 0$ and a diffeomorphism $g : M \to M$, $\varepsilon$ close to $f$ in the $C^1$ topology, such that $B_{4\delta}(x) \cap B_{4\delta}(y) = \emptyset$ for $x, y \in F$ with $x \ne y$ and $g(z) = \exp_{f(x)} \circ G_x \circ \exp_x^{-1}(z)$ if $z \in B_\delta(x)$ and $g(z) = f(z)$ if $z \notin B_{4\delta}(x)$ $(x \in F)$.*

Define $G = \mathcal{O} \circ D_pf^k$. Then there exists $m > 0$ such that $G^m_{|E^c(p)}$ is the identity by (2.1), and $\delta_1 > 0$ such that

$$(2.3)\qquad g^{mk}|_{\exp_p T_pM(\delta_1)} = \exp_p \circ G^m \circ \exp_p^{-1} \quad (\text{by (2.2)})$$

where $T_pM(\delta_1) = \{v \in T_pM \,|\, \|v\| \le \delta_1\}$. Put $E^c(p, \delta_1) = E^c(p) \cap T_pM(\delta_1)$, then it is clear that

$$(2.4)\qquad g^{mk}|_{\exp_p E^c(p,\delta_1)} = \mathrm{id}|_{\exp_p E^c(p,\delta_1)}\,.$$

Since $g \in \Omega$-POTP$(M)$, we see that $g^{mk}_{|\Omega(g)}$ has POTP. Then, for $0 < \varepsilon < \delta_1/4$ there exists $0 < \delta < \varepsilon$ such that every $\delta$-pseudo orbit is $\varepsilon$-traced by some point in $\Omega(g)$. Now take and fix $y \in \exp_p E^c(p, \delta_1)$ with $d(p, y) = \frac{3}{4}\delta_1$. From (2.4) we can construct a cyclic $\delta$-pseudo orbit $\{x_i\}$ of $g^{mk}$ satisfying

$$(2.5)\qquad \begin{cases} \text{(i)} & \{x_i\} \subset \exp_p E^c(p, \delta_1)\,, \\ \text{(ii)} & x_0 = p \text{ and } x_s = y \text{ for some } s > 0\,, \\ \text{(iii)} & B_\varepsilon(x_i) \subset \exp_p T_pM(\delta_1) \quad \text{for } i \in \mathbf{Z}\,. \end{cases}$$

For the pseudo orbit $\{x_i\}$ there is $z \in \Omega(g)$ such that $d(g^{mki}(z), x_i) < \varepsilon$ for $i \in \mathbf{Z}$ as explained above. By (2.5) (iii) we have $\exp_p^{-1} \circ g^{mki}(z) \in T_pM(\delta_1)$ and letting $u = \exp_p^{-1} z$, $\|G^{mi}(u)\| = \|\exp_p^{-1} \circ g^{mki}(z)\| \le \delta_1$ for $i \in \mathbf{Z}$. Thus

$u \in E^c(p)$ and so $z \in \exp_p E^c(p, \delta_1)$. From (2.4) we have that $d(p, z) \geq d(p, x_s) - d(x_s, z) = d(p, y) - d(x_s, g^{mks}(z)) \geq \frac{3}{4}\delta_1 - \varepsilon > \frac{1}{2}\delta_1 > \varepsilon$, which shows a contradiction.

## § 3. Proof of Proposition 2

Let $f \in \Omega\text{-}TS(M)$. For $\varepsilon > 0$ there exists $\delta > 0$ such that for $g \in \mathrm{Diff}^0(M)$, $d(f(x), g(x_s)) \leq \delta$ $(x \in M)$ implies that there exists a continuous map $h : \Omega(g) \to \Omega(f)$ satisfying $h \circ g = f \circ h$ and $d(h(x), x) \leq \varepsilon$ for $x \in \Omega(g)$. Note that $g$ does not belong to $\Omega$-TS$(M)$.

The proof is divided into two the cases $\dim M = 1$ and $\dim M \geq 2$. For the case $\dim M = 1$ we know that the set of all Morse-Smale diffeomorphisms is open dense in $\mathrm{Diff}^1(M)$. Choose a Morse-Smale diffeomorphism as the diffeomorphism $g$. Then, it is easily checked that $h(P(g)) = P(f)$. Thus $\sharp P(f) \leq \sharp P(g) < \infty$, which implies that $f_{|\bar{P}(f)}$ has POTP. Therefore $f \in \mathscr{F}(M)$ by the same proof as Proposition 1.

For the case $\dim M \geq 2$, we prove directly that $f \in \mathscr{F}(M)$. To do this it suffices to show that every $x \in P(f)$ is hyperbolic. Suppose that $p \in P(f)$ is non-hyperbolic and let $k > 0$ be a prime period of $p$. As in the proof of Proposition 1, for $\varepsilon > 0$ take a linear automorphism $\mathcal{O} : T_pM \to T_pM$ satisfying (2.1) and after that take $\delta_0 > 0$ and $g \in \Omega$-TS$(M)$ satisfying (2.2). Moreover let $m > 0$ be a minimal integer such that $G^m_{|E^c(p)}$ is the identity map on $E^c(p)$. We put $I_0 = E^c(p)$ when $m = 1$. If $m \geq 2$ then we take $v \in E^c(p)$ with $\|v\| = 1$ such that, letting $I_0 = \{tv \mid t \geq 0\}$ and $I_l = G^l(I_0)$ $(0 \leq l \leq m - 1)$

$$(3.1) \qquad \begin{cases} (\mathrm{i}) & I_l \cap I_{l'} = \{0\} \quad (0 \leq l \neq l' \leq m - 1), \\ (\mathrm{ii}) & G^m(I_l) = I_l \quad (0 \leq l \leq m - 1). \end{cases}$$

Let $\delta_1 > 0$ be as in (2.3) and take $y \in \exp_p(I_0 \cap T_pM(\delta_1))$ with $d(p, y) = \frac{3}{4}\delta_1$. Since $g^{mk}_{|\exp_p E^c(p,\delta_1)}$ is the identity on $\exp_p E^c(p, \delta_1)$ by (2.4), for $0 < \varepsilon < \frac{1}{4}\delta_1$ and every $\delta > 0$ we can find a finite sequence $\{x_i\}_{i=0}^{2s}$ of $M$ such that

$$(3.2) \qquad \begin{cases} (\mathrm{i}) & \{x_i\}_{i=0}^{2s} \subset \exp_p\{(I_0 \cap T_pM(\delta_1)) - \{0\}\}, \\ (\mathrm{ii}) & d(x_0, p) < \varepsilon, \\ (\mathrm{iii}) & x_{2s} = x_0 \quad \text{and} \quad x_s = y, \\ (\mathrm{iv}) & x_i \neq x_j \quad \text{for } 0 \leq i \neq j \leq 2s - 1, \\ (\mathrm{v}) & d(x_i, x_{i+1}) < \delta \quad \text{for } 0 \leq i \leq 2s - 1, \\ (\mathrm{vi}) & B_\varepsilon(x_i) \in \exp_p T_pM(\delta_1) \quad \text{for } 0 \leq i \leq 2s. \end{cases}$$

Now define $p_{mki} = x_i$ and $q_{mki} = x_{i+1}$ and $p_{mki+j} = q_{mki+j} = g^j(x_{i+1})$ for $0 \leq i \leq 2s - 1$ and $1 \leq j \leq mk - 1$. Then we have that $d(p_n, q_n) < \delta$, $p_n \neq p_{n'}$ and $q_n \neq q_{n'}$ for $0 \leq n \neq n' \leq 2smk - 1$. Thus, by Lemma 13 of Nitecki and Shub [8] we have that there exists $\varphi \in \mathrm{Diff}^1(M)$ such that $d(\varphi(x), x) < 2\pi\delta$ for $x \in M$ and $\varphi(p_n) = q_n$ for $0 \leq n \leq 2smk - 1$. Define $\tilde{g} = g \circ \varphi$. Since $\delta$ is arbitrary, we can take $\tilde{g}$ such that $\tilde{g}$ is small $C^0$ near to $g$. Thus there exists a continuous map $h : \Omega(\tilde{g}) \to \Omega(g)$ satisfying $h \circ \tilde{g} = g \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in \Omega(\tilde{g})$. Moreover $\tilde{g}^{2smk}(x_0) = x_0$. Thus $d(p, h(x_0)) \leq d(p, x_0) + d(x_0, h(x_0)) \leq 2\varepsilon$ and

$$\begin{aligned} d(y, g^{smk}(h(x_0))) &= d(y, h(\tilde{g}^{smk}(x_0))) \\ &= d(y, h(x_s)) \leq d(y, x_s) + d(x_s, h(x_s)) \leq \varepsilon . \end{aligned}$$

Therefore we have $\frac{3}{4}\delta_1 = d(p, y) \leq d(p, h(x_0)) + d(g^{smk}(h(x_0)), y) \leq 3\varepsilon < \frac{3}{4}\delta_1$, since $g^{smk}(h(x_0)) = h(x_0)$. We arrived at a contradiction.

## § 4. Proof of Proposition 3

Since $TS(M) \subset \mathscr{F}(M)$ by Propositions 1 and 2, it is clear that $f \in TS(M)$ satisfies Axiom $A$ and no cycle. Thus $f$ is $\Omega$-stable. On the other hand, since $f$ is topologically stable, for $\varepsilon > 0$ small enough we can find a small neighborhood $\mathscr{U}(f)$ of $f$ in $\mathrm{Diff}^1(M)$ such that for $g \in \mathscr{U}(f)$ there exists a continuous surjection $h : M \to M$ such that $h \circ g(x) = f \circ h(x)$ and $d(h(x), x) < \varepsilon$ for all $x \in M$ and moreover $h_{|\Omega(g)} : \Omega(g) \to \Omega(f)$ is bijective.

Thus we have that for $x \in M$

$$h^{-1} W^{\sigma}(h(x), f) = W^{\sigma}(x, g) \quad (\sigma = s, u) \tag{4.1}$$

where

$$\begin{aligned} W^s(x, g) &= \{y \in M \mid d(g^n(x), g^n(y)) \to 0 \text{ as } n \to \infty\} , \\ W^u(x, g) &= \{y \in M \mid d(g^{-n}(x), g^{-n}(y)) \to 0 \text{ as } n \to \infty\} . \end{aligned}$$

Indeed, (4.1) is checked as follows. Since $f \in \mathscr{F}(M)$ and $\mathscr{U}(f)$ is a sufficiently small neighborhood, we can take it as $\mathscr{U}(f) \subset \mathscr{F}(M)$. Thus

$$M = \bigcup_{x \in \Omega(g)} W^{\sigma}(x, g) \qquad \text{for } g \in \mathscr{U}(f) \ (\sigma = s, u) . \tag{4.2}$$

Since $hW^{\sigma}(x, g) \subset W^{\sigma}(h(x), f)$ for $x \in M$, we have $W^{\sigma}(x, g) \subset h^{-1} \circ hW^{\sigma}(x, g) \subset h^{-1}W^{\sigma}(h(x), f)$. To obtain (4.1) suppose that $W^{\sigma}(x, g) \neq h^{-1}W^{\sigma}(h(x), f)$. Then $y \notin W^{\sigma}(x, g)$ and $h(y) \in W^{\sigma}(h(x), f)$ for some $y \in M$. By (4.2) there exist $x', y' \in \Omega(g)$ such that $W^{\sigma}(x', g) = W^{\sigma}(x, g)$ and $W^{\sigma}(y', g) = W^{\sigma}(y, g)$. Then we have

$$h(y) \in W^\sigma(h(x), f) \cap W^\sigma(h(y), f) = W^\sigma(h(x'), f) \cap W^\sigma(h(y'), f)$$

and so $W^\sigma(h(x'), f) = W^\sigma(h(y'), f)$. For $\sigma = s$ we have $d(h \circ g^n(x'), h \circ g^n(y')) = d(f^n \circ h(x'), f^n \circ h(y')) \to 0$ as $n \to \infty$. Since $h_{|\Omega(g)}$ is a homeomorphism, it follows $d(g^n(x'), g^n(y')) \to 0$ as $n \to \infty$ and hence $y' \in W^s(x', g) = W^s(x, g)$. Therefore $y \in W^s(x, g)$ which is a contradiction. Similarly we can derive a contradiction for $\sigma = u$.

Next we check that for $x \in M$

$$\dim W^s(x, f) + \dim W^u(x, f) \geq \dim M. \tag{4.3}$$

Since $h_{|\Omega(g)}$ is bijective, for $p$, $q \in P(f)$ with $W^s(p, f) \cap W^u(q, f) \neq \varnothing$ there exist $p'$, $q' \in P(g)$ satisfying $h(p') = p$ and $h(q') = q$. From (4.1) we have

$$\begin{aligned} W^s(p', g) \cap W^u(q', g) &= h^{-1}[W^s(h(p'), f) \cap W^u(h(q'), f)] \\ &= h^{-1}[W^s(p, f) \cap W^u(q, f)] \neq \varnothing . \end{aligned}$$

Use here the fact that the set of all Kupka-Smale diffeomorphisms is residual in $\mathrm{Diff}^1(M)$. Then we can take a Kupka-Smale diffeomorphism as the diffeomorphism $g$. Thus $\dim W^s(p', g) + \dim W^u(q', g) \geq \dim M$. Since $g$ is $C^1$ near to $f$, we have that $\dim W^\sigma(x, g) = \dim W^\sigma(h(x), f)$ for $x \in \Omega(g)$ ($\sigma = s, u$). Therefore (4.3) was obtained for this case.

Since $f$ satisfies Axiom $A$, there exists $\varepsilon > 0$ such that $\bigcap_{n \in Z} f^n(U_\varepsilon(\Lambda_i)) = \Lambda_i$ for each basic set $\Lambda_i$ of $\Omega(f)$. Since topological stability derives POTP, for the number $\varepsilon > 0$ let $\delta > 0$ be a number satisfying properties in the definition of POTP. Since $M = \bigcup_{y \in \Omega(f)} W^\sigma(y, f)$ for $\sigma = s$, $u$, for $x \in M$ there exist $y_i \in \Lambda_i$ and $y_j \in \Lambda_j$ such that $x \in W^s(y_i, f) \cap W^u(y_j, f)$. Take $m > 0$ so large that $d(f^m(x), f^m(y_i)) < \delta$ and $d(f^{-m}(x), f^{-m}(y_j)) < \delta$. Since $\Lambda_k \cap P(f)$ is dense in $\Lambda_k$ for each basic set $\Lambda_k$, we can choose periodic points $p_i \in \Lambda_i$ and $p_j \in \Lambda_j$ satisfying $d(f^m(x), p_i) \leq \delta$ and $d(f^{-m}(x), p_j) \leq \delta$. Then a $\delta$-pseudo orbit $\mathcal{O} = \{\cdots, f^{-2}(p_j), f^{-1}(p_j), f^{-m}(x), \cdots, x, \cdots, f^{m-1}(x), p_i, f(p_i), \cdots\}$ is $\varepsilon$-traced by a point $z$ in $M$. Obviously $z \in W^s(f^{-m}(p_i), f) \cap W^u(f^m(p_j), f)$, and hence $\dim W^s(f^{-m}(p_i), f) + \dim W^u(f^m(p_j), f) \geq \dim M$ as above. Therefore we have

$$\begin{aligned} \dim W^s(x, f) + \dim W^u(x, f) &= \dim W^s(p_i, f) + \dim W^u(p_j, f) \\ &= \dim W^s(f^{-m}(p_i), f) + \dim W^u(f^m(p_j), f) \\ &\geq \dim M. \end{aligned}$$

We now are ready to prove Proposition 3.

For $x \in M - \Omega(f)$ it suffices to prove that $W^s(x,f)$ and $W^u(x,f)$ meet transversally. Since $M = \bigcup_{y \in \Omega(f)} W^\sigma(y,f)$ for $\sigma = s, u$, there exist $y_1, y_2 \in \Omega(f)$ such that

$$W^s(x,f) = W^s(y_1,f) \quad \text{and} \quad W^u(x,f) = W^u(y_2,f) .$$

We know (cf. see [4]) that there is $\varepsilon_1 > 0$ with $B_{\varepsilon_1}(x) \cap B_{\varepsilon_1}(\Omega(f)) = \varnothing$ such that for $0 < \varepsilon < \varepsilon_1$ and $y \in \Omega(f)$

$$(4.4) \quad \begin{cases} \text{(i)} & W^\sigma_\varepsilon(y,f) \text{ is a } C^1\text{-disk for } \sigma = s, u , \\ \text{(ii)} & W^s(y,f) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(f^n(y),f)) , \\ \text{(iii)} & W^u(y,f) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(f^{-n}(y),f)) . \end{cases}$$

Thus, for $0 < \varepsilon_2 < \varepsilon_1$ there exist $n_1, n_2 > 0$ satisfying

$$(4.5) \quad \begin{cases} \text{(i)} & f^{n_1}(x) \in \operatorname{int} W^s_{\varepsilon_2}(f^{n_1}(y_1),f) , \\ \text{(ii)} & f^{-n_2}(x) \in \operatorname{int} W^u_{\varepsilon_2}(f^{-n_2}(y_2),f) \end{cases}$$

where $\operatorname{int} W^\sigma_{\varepsilon_2}(y,f)$ denotes the interior of $W^\sigma_{\varepsilon_2}(y,f)$ in $W^\sigma_{\varepsilon_1}(y,f)$, and $\delta_0 > 0$ satisfying

$$(4.6) \quad \begin{cases} \text{(i)} & B_{\delta_0}(f^n(x)) \cap B_{\delta_0}(f^m(x)) = \varnothing \quad \text{for} \quad -n_2 \leq n \neq m \leq -n_1 , \\ \text{(ii)} & f^{-1}[B_{\delta_0}(f^{-n_2}(x))] \cap B_{\delta_0}(f^n(x)) = \varnothing \quad \text{for} \quad -n_2 \leq n \leq n_1 , \\ \text{(iii)} & f^m[B_{\delta_0}(f^n(x))] \cap B_{\delta_0}(f^n(x)) = \varnothing \quad \text{for} \quad -n_2 \leq n \leq n_1 \text{ and } m \neq 0 . \end{cases}$$

Denote by $C^\sigma_\delta(y,f)$ the connected component of $y$ in $B_\delta(y) \cap W^\sigma(y,f)$ for $\sigma = s, u$. From (4.4) and (4.5) it follows that there is $0 < \delta_1 < \delta_0$ such that for $0 < \delta \leq \delta_1$

$$\begin{aligned} \operatorname{int} W^s_{\varepsilon_2}(f^{n_1}(y_1),f) \cap B_\delta(f^{n_1}(x)) &= W^s_{\varepsilon_1}(f^{n_1}(y_1),f) \cap B_\delta(f^{n_1}(x)) \\ &= C^s_\delta(f^{n_1}(x),f) , \\ \operatorname{int} W^u_{\varepsilon_2}(f^{-n_2}(y_2),f) \cap B_\delta(f^{-n_2}(x)) &= W^u_{\varepsilon_1}(f^{-n_2}(y_2),f) \cap B_\delta(f^{-n_2}(x)) \\ &= C^u_\delta(f^{-n_2}(x),f) . \end{aligned}$$

Let $\mathscr{U}(f)$ be a small neighborhood of $f$ in TS($M$). Given a sufficiently small $0 < \delta_2 < \delta_1$ we can construct diffeomorphisms $\varphi_i$ $(i = 1, 2)$, $C^1$ near to the identity, such that

$$\begin{cases} \varphi_1(f^{n_1}(x)) = f^{n_1}(x) , \\ \varphi_1(C^s_{\delta_1}(f^{n_1}(x),f) \cap B_{\delta_2}(f^{n_1}(x)) = \exp_{f^{n_1}(x)}(Df^{n_1}E_1)(\delta_2) , \\ \varphi_1 = \text{id} \quad \text{on } M - B_{\delta_1}(f^{n_1}(x)) , \\ f \circ \varphi_1^{-1} \in \mathscr{U}(f) \end{cases}$$

where $E_1$ denotes the tangent space at $x$ of $W^s(x,f)$, and

$$\begin{cases}\varphi_2(f^{-n_2}(x)) = f^{-n_2}(x)\,,\\ \varphi_2(C^u_{\delta_2}(f^{-n_2}(x),f)\cap B_{\delta_2}(f^{-n_2}(x)) = \exp_{f^{-n_2}(x)}(Df^{-n_2}E_2)(\delta_2)\,.\\ \varphi_2 = \mathrm{id} \quad \text{on } M - B_{\delta_1}(f^{-n_2}(x))\,,\\ \varphi_2\circ f \in \mathscr{U}(f)\end{cases}$$

where $E_2 = T_xW^u(x,f)$. In general $\varphi_1$ and $\varphi_2$ can be constructed as follows. For $y\in\Omega(f)$ let $F_1 = T_yW^s(y,f)$ and write $F_2 = F_1^{\perp}$. Since there exists a $C^1$ map $\gamma: F_1(\delta)\to F_2$ such that $\mathrm{graph}(\gamma) = \exp_y^{-1}(C^s_\delta(y,f))$, we can define a $C^1$ embedding $Q: T_yM(\delta)\to T_yM$ satisfying $Q(z) = Q(z_1,z_2) = (z_1, z_2+\gamma(z_1))$ for $z=(z_1,z_2)\in(F_1\oplus F_2)\cap T_yM(\delta)$. Clearly $D_0Q = \mathrm{id}$ and so $Q$ is $C^1$ near to $\mathrm{id}_{|T_yM(\delta)}$ when $\delta$ is small enough. As usual define a $C^\infty$ bump function $\alpha:\mathbf{R}\to[0,1]$ such that $\alpha(t)=0$ if $|t|\le 1$, $\alpha(t)=1$ if $|t|\ge 2$ and $|\alpha'(t)|<2$. Then, for a sufficiently small $\delta'$ with $0<2\delta'<\delta$ we set

$$\varphi(z) = \begin{cases} z \text{ if } z\notin B_\delta(y)\\ \exp_y\{k\cdot\exp_y^{-1}z + (1-k)Q^{-1}(\exp_y^{-1}z)\} \text{ if } z\in B_\delta(y)\\ \qquad\qquad \text{where } k = \alpha\left(\dfrac{\|\exp_y^{-1}z\|}{\delta'}\right).\end{cases}$$

Then $\varphi: M\to M$ is a diffeomorphism $C^1$ near to id such that $\varphi(y)=y$ and $\varphi(C^s_{\delta'}(y,f)) = F_1(\delta')$.

As the finite set $F$ and the isomorphism $G_x$ of Franks's lemma (mentioned above), we set $F=\{f^{-n_2}(x), f^{-n_2+1}(x),\cdots, f^{n_1-1}(x)\}$ and $G_{f^n(x)} = D_{f^n(x)}f$ $(-n_2\le n\le n_1-1)$. Then we see that for $0<\delta_2<\delta_1$ small enough there is $g_3\in\mathscr{U}(f)$ satisfying

$$\begin{cases} g_3(f^n(x)) = f^{n+1}(x) \quad \text{for } -n_2\le n\le n_1-1\,,\\ g_3 = f \quad \text{on } M - \bigcup_{n=-n_2}^{n_1-1} B_{\delta_1}(f^n(x))\,,\\ g_3 = \exp_{f^{n+1}(x)}\circ D_{f^n(x)}f\circ\exp^{-1}_{f^n(x)}\\ \qquad\qquad \text{on } B_{\delta_2}(f^n(x)) \text{ for } -n_2\le n\le n_1-1\,.\end{cases}$$

Thus by (4.6) we can define a diffeomorphism $g$ belonging to $\mathscr{U}(f)$ by

$$g(y) = \begin{cases} f\circ\varphi_1^{-1}(y) & \text{if } y\in B_{\delta_1}(f^{n_1}(x))\\ \varphi_2\circ f(y) & \text{if } y\in f^{-1}(B_{\delta_1}(f^{-n_2}(x)))\\ g_3(y) & \text{otherwise}\,.\end{cases}$$

Then it is easily checked that for $\delta_3>0$ small enough

$$(4.7)\quad \begin{cases} (\text{i}) & g^{-n_1}(\operatorname{int} W^s_{\varepsilon_2}(g^{n_1}(y_1), g)) \cap B_{\delta_3}(x) = g^{-n_1}(W^s_{\varepsilon_1}(g^{n_1}(y_1), g)) \cap B_{\delta_3}(x) \\ & \qquad = \exp_x(E_1(\delta_3)), \\ (\text{ii}) & g^{n_2}(\operatorname{int} W^u_{\varepsilon_2}(g^{-n_2}(y_2), g)) \cap B_{\delta_3}(x) = g^{n_2}(W^u_{\varepsilon_1}(g^{-n_2}(y_2), g)) \cap B_{\delta_3}(x) \\ & \qquad = \exp_x(E_2(\delta_3)). \end{cases}$$

To obtain the conclusion suppose that $W^s(x, f)$ is not transversal to $W^u(x, f)$. Then $E = E_1 \cap E_2$ is non trivial ($E \neq \{0\}$) by (4.3). Since $g \in \mathscr{U}(f)$ ($\subset \mathrm{TS}(M)$), take $\varepsilon > 0$ smaller than $\min\{\varepsilon_1 - \varepsilon_2, \delta_3/4\}$, and let $\delta > 0$ be a number satisfying the definition of the topological stability i.e. for $\tilde{g} \in \mathrm{Diff}^0(M)$ with $d(g, \tilde{g}) \leq \delta$, there exists a continuous surjection $h: M \to M$ satisfying $h \circ \tilde{g} = g \circ h$ and $d(h(y), y) \leq \varepsilon$ for all $y \in M$. For $\delta' > 0$ sufficiently small we can find a homeomorphism $\mathcal{O}: T_xM \to T_xM$ with $\mathcal{O}(0) = 0$ such that

(i) $\max\{\|\mathcal{O}^{-1}(u) - u\|: u \in T_xM(\delta_3)\} \leq \delta'$,

(ii) letting $E' = \mathcal{O}(E_1) \cap E_2$,

- (a) $E'$ is a non trivial linear subspace, or $E' = \{0\}$,
- (b) $E' \subset E$,
- (c) $\dim E' < \dim E$.

Put $\tilde{g} = g$ on $M - B_{\delta_3}(x)$ and $\tilde{g} = g \circ \exp_x \circ \mathcal{O}^{-1} \circ \exp_x^{-1}$ on $B_{\delta_3/2}(x)$. Then $\tilde{g} \in \mathrm{Diff}^0(M)$ and $d(g, \tilde{g}) \leq \delta$. Thus we have

$$\tilde{g}^{-n_1}(W^s_{\varepsilon_1}(\tilde{g}^{n_1}(y_1), \tilde{g})) \cap B_{\delta_3/2}(x) = \mathcal{O}(E_1)(\delta_3/2)$$

and by (4.7)

$$(4.8)\quad \tilde{g}^{-n_1}(W^s_{\varepsilon_1}(\tilde{g}^{n_1}(y_1), \tilde{g})) \cap \tilde{g}^{n_2}(W^u_{\varepsilon_1}(\tilde{g}^{-n_2}(y_2), \tilde{g})) \cap B_{\delta_3/2}(x) = \exp_x(E'(\delta_3/2)).$$

Since $\dim E' < \dim E$ by (ii) (c), if $\varepsilon > 0$ is sufficiently small, then we can take $z' \in \exp_x(E(\delta_3))$ satisfying $z' \notin B_\varepsilon(\exp_x E'(\delta_3/2))$ and $d(z', x) \leq \delta_3/4$. Let $h: M \to M$ be a semi-conjugacy found as above for $\tilde{g}$. Then $h(z) = z'$ for some $z \in M$. Since $d(z, z') = d(z, h(z)) \leq \varepsilon$, we have that $z \notin \exp_x(E'(\delta_3/2))$ and $d(z, x) \leq d(z, z') + d(z', x) \leq \varepsilon + \delta_3/4 < \varepsilon_3/2$. Thus, by (4.8).

$$z \notin \tilde{g}^{-n_1}(W^s_{\varepsilon_1}(\tilde{g}^{n_1}(y_1), \tilde{g})) \cap \tilde{g}^{n_2}(W^u_{\varepsilon_1}(\tilde{g}^{-n_2}(y_2), \tilde{g})).$$

If $z$ does not belong to the left hand set of the above relation, then $d(\tilde{g}^n(z), \tilde{g}^n(y_1)) = d(\tilde{g}^n(z), g^n(y_1)) > \varepsilon_1$ for a certain $n$ larger than $n_1$ and by (4.7).

$$h(z) \in \exp_x(E(\delta_3)) \subset \exp_x(E_1(\delta_3)) = g^{-n_1}(\operatorname{int} W^s_{\varepsilon_2}(g^{n_1}(y_1), g)) \cap B_{\delta_3}(x).$$

Thus $d(g^n(y_1), g^n(h(z))) \leq \varepsilon_2$ and so

$$d(h(\tilde{g}^n(z)), \tilde{g}^n(z)) = d(g^n(h(z)), \tilde{g}^n(z))$$
$$\geq d(\tilde{g}^n(z), g^n(y_1)) - d(g^n(y_1), g^n(h(z))) > \varepsilon_1 - \varepsilon_2 > \varepsilon .$$

This is inconsistent with the property of $h$. For the case

$$z \notin \tilde{g}^{n_2}(W^u_{\varepsilon_1}(\tilde{g}^{-n_2}(y_2), \tilde{g}))$$

we obtain a contradiction by the same way. Therefore $W^s(x, f)$ is transversal to $W^u(x, f)$ for all $x \in M$. The proof of Proposition 3 is complete.

## § 5. Proof of Theorem 2

As in the proof of Proposition 3, we can construct $g \in \mathrm{POTP}(M)$ satisfying (4.7). Now assume that $\dim W^s(x, f) = \dim M - 1$ and $W^s(x, f)$ is not transversal to $W^u(x, f)$. Then $T_x W^s(x, f) \supset T_x W^u(x, f)$ and so $E = T_x W^s(x, f) \cap T_x W^u(x, f) = T_x W^u(x, f)$. Take $\delta_3 > 0$ small enough, then there exist $\varepsilon' > 0$ and $0 < \varepsilon < \varepsilon'$ such that $W^u_\varepsilon(x, g) \subset \exp_x(E(\delta_3)) \subset W^s_{\varepsilon'}(x, g)$ and $W^s_{2\varepsilon'}(x, g) \subset W^s(x, g)$. Since $g$ has POTP, there exists $\delta > 0$ such that if $d(y, z) \leq \delta$ $(y, z \in M)$ then $W^s_\varepsilon(y, g) \cap W^u_\varepsilon(z, g) \neq \emptyset$. Thus we have $W^s_\varepsilon(y, g) \cap W^u_\varepsilon(x, g) \neq \emptyset$ for all $y \in B_\delta(x)$, and so $W^s_\varepsilon(y, g) \cap W^s_{\varepsilon'}(x, g) \neq \emptyset$. Therefore $y \in W^s_{\varepsilon+\varepsilon'}(x, g) \subset W^s_{2\varepsilon'}(x, g) \subset W^s(x, g)$, and so $B_\delta(x) \subset W^s(x, g)$. This contradicts $\dim W^s(x, f) = \dim W^s(x, g) = \dim M - 1$.

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