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RESEARCH ARTICLE

Stochastic comparisons of largest claim and aggregate claim amounts

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Abstract
In this paper, we establish some stochastic comparison results for largest claim amounts of two sets of independent and also for interdependent portfolios under the setup of the proportional odds model. We also establish stochastic comparison results for aggregate claim amounts of two sets of independent portfolios. Further, stochastic comparisons for largest claim amounts from two sets of independent multiple-outlier claims have also been studied. The results we obtained apply to the whole family of extended distributions, also known as the Marshall–Olkin family of distributions. We have given many numerical examples to illustrate the results obtained.

1. Introduction

An insurance policy is an agreement between the insurer and the insured. Consequently, there are two thought processes in any insurance policy, namely, one from insurer side and the other one from insured side. An insured always looks into the plan (that contains the annual premium amount, total time period, whether it is the individual or the group insurance policy, etc.) and the key benefits (namely, sum insured amount, withdrawal facility, tax saving facility, etc.) of a policy before having it. On the other hand, the insurer comes up with a policy whose existence and upgradation (as and when necessary) depend on different key factors, namely, number of claims in a given time frame, size of the portfolio, aggregate claim amount, largest claim amount, smallest claim amount, etc. Thus, numerous researchers have shown their keen interest in studying useful characteristics of these key factors.

Assume that $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli random variables (r.v.'s), independent of r.v.'s X_i's, with $\mathbb{E}(I_{p_i}) = p_i$, $i = 1, \ldots, n$. Let $X_i^* = X_i I_{p_i}$, $i = 1, \ldots, n$, and denote $X_{n:n}^* = \max(X_1^*, \ldots, X_n^*)$. In actuarial science, it represents the largest claim amount in a portfolio of risks [3, 7, 28], where X_i's represent random claims that can be made by a policy in an insurance period and I_{p_i}'s indicate the occurrence of these claims ($I_{p_i} = 1$ if the ith policy makes random claim X_i and $I_{p_i} = 0$ if there is no claim). Here $\sum_{i=1}^n X_i I_{p_i}$ represents the aggregate claim amount for this portfolio of risks. Similarly, denote $Y_{n:n}^* = \max(Y_1^*, \ldots, Y_n^*)$, which represents the largest claim amount in an another portfolio of risks, where $Y_i^* = Y_i I_{q_i}$, $i = 1, \ldots, n$. In this case, $\sum_{i=1}^n Y_i I_{q_i}$ represents the aggregate claim amount. As discussed, the smallest, the largest, and the aggregate claim amounts have important roles in determining the annual premium and the coverage of a policy. It is also important for an actuary to be able to compare different portfolios of risks according to these important information. In this prospect, stochastic comparisons of maximum, minimum, and aggregate claim amounts arising from two sets of portfolios have great importance in actuarial science on both theoretical and practical grounds [2–4, 6, 7, 27, 28, 30].

The proportional odds (PO) model [8, 15, 18, 23] is an important model in reliability theory and survival analysis. Let X and Y be two r.v.'s with cumulative distribution functions (CDFs) F, G and survival functions $\bar{F}$, $\bar{G}$, respectively. If X represents a lifetime r.v., then the odds function $\rho_X(t)$, defined by $\rho_X(t) = \bar{F}(t)/F(t)$, represents the odds on surviving beyond time t. Similarly, if X represents a random claim amount, then $\rho_X(x)$ represents the odds that claim amount be more than a specific quantity x. Two r.v.'s X and Y are said to satisfy the PO model if

$$\rho_Y(x) = \alpha\, \rho_X(x) \tag{1}$$

for all admissible x, where proportionality constant α is known as odds ratio. If X and Y represent random claims corresponding to two policies, and we consider the event "claim amount be more than a specific quantity", then $\alpha > (<)1$ indicates that the event is more (less) likely to occur for the first policy than the second one. From Eq. (1), we have that the survival functions of X and Y are related as

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}, \tag{2}$$

where $\bar{\alpha} = 1 - \alpha$. We will say that Y is following the PO model with baseline survival function $\bar{F}$ and odds ratio α denoted as $Y \sim PO(\bar{F}, \alpha)$. Let us denote the hazard rate functions of X and Y as $r_X(\cdot) = f(\cdot)/\bar{F}(\cdot)$, $r_Y(\cdot) = g(\cdot)/\bar{G}(\cdot)$, respectively, where $f(\cdot)$ and $g(\cdot)$ denote the respective probability density functions of X and Y. Then, from Eq. (2), we have the hazard ratio of the two r.v.'s X and Y as $r_Y(t)/r_X(t) = 1/(1 - \bar{\alpha}\bar{F}(t))$, which converges to unity as t tends to ∞. This is in contrast to the proportional hazard rate model, where the hazard ratio of the two r.v.'s remains constant with time. The convergence property of the hazard ratio makes the PO model reasonable in many practical applications as discussed in [8, 12, 15, 19, 24]. Marshall and Olkin [20] discussed the model in Eq. (2) with $0 < \alpha < \infty$ (tilt parameter) as a method of generating flexible new family of distributions, known as the Marshall–Olkin family of distributions (or extended distributions) [13, 20], from an existing family of distributions. Many researchers studied this family of distributions considering the baseline distributions as some known distributions like exponential, Weibull, gamma, Pareto, Lomax, and linear failure-rate (see [5] and references therein). Note that we have studied it in general setup, that is, without considering any specific baseline distributions.

In this paper, we investigate stochastic comparisons of the largest claim amounts from two sets of heterogeneous portfolios in the sense of some stochastic orderings under the setup of the PO model. We also investigate stochastic comparisons of aggregate claim amounts. It is worth noting that our results are not limited to be applied in actuarial science. For instance, our proposed results can be used to compare the lifetimes of two parallel systems whose components are subject to random shocks instantaneously. Suppose that the r.v. X_i denotes the lifetime of the ith component of a parallel system which may receive a random shock defined by the Bernoulli r.v. I_{p_i}, where $I_{p_i} = 1$ if the shock does not occur with $p_i = P\left(I_{p_i} = 1\right)$ and 0 if the shock occurs. Then $X^*_{n:n}$ represents the lifetime of a parallel system whose components are subject to random shocks instantaneously [1, 11, 16].

The rest of the paper is organized as follows. Section 2 describes some preliminary concepts. Section 3 presents some stochastic comparison results for largest claim amounts of two sets of independent and also for interdependent portfolios under the setup of the PO model. Section 4 presents star ordering result for two sets of independent multiple-outlier claims. Section 5 presents comparison results on aggregate claim amounts under two sets of independent portfolios. Finally, the concluding remarks are given in Section 6.

2. Preliminaries

We first give a brief overview of some preliminaries, namely, stochastic orders and majorization orders.

Let $U \subseteq \mathbb{R}$ denote a subset of the real line. Further, let, for any vector $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \mathbb{R}^n, \vartheta_{(1)} \leq \vartheta_{(2)} \leq \ldots \leq \vartheta_{(n)}$ denote the increasing arrangement of components in $\boldsymbol{\vartheta}$.

Definition 2.1. **[21]** *Let $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n) \in U^n$. The vector $\boldsymbol{\vartheta}$ is said to*

(i) majorize the vector $\boldsymbol{\eta}$ (denoted by $\boldsymbol{\vartheta} \overset{m}{\geq} \boldsymbol{\eta}$) if

$$\sum_{i=1}^{j} \vartheta_{(i)} \leq \sum_{i=1}^{j} \eta_{(i)}, \ \forall j = 1,\ 2,\ \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} \vartheta_{(i)} = \sum_{i=1}^{n} \eta_{(i)}.$$

(ii) weakly supermajorize $\boldsymbol{\eta}$ (denoted by $\boldsymbol{\vartheta} \overset{w}{\geq} \boldsymbol{\eta}$) if

$$\sum_{i=1}^{j} \vartheta_{(i)} \leq \sum_{i=1}^{j} \eta_{(i)}, \ \forall j = 1, 2, \ldots, n.$$

(iii) weakly submajorize $\boldsymbol{\eta}$ (denoted by $\boldsymbol{\vartheta} \ \geq_w \ \boldsymbol{\eta}$) if

$$\sum_{i=j}^{n} \vartheta_{(i)} \geq \sum_{i=j}^{n} \eta_{(i)}, \ \forall j = 1, 2, \ldots, n.$$

It is to be noted that $\boldsymbol{\vartheta} \overset{w}{\geq} \boldsymbol{\eta}$ $(\boldsymbol{\vartheta} \geq_w \boldsymbol{\eta}) \implies \boldsymbol{\vartheta} \overset{m}{\geq} \boldsymbol{\eta}$, but the reverse is not true. Later, we give some definitions related to multivariate majorization (Marshall et al. [21], Chap. 15).

Definition 2.2. *Let A and B be two $m \times n$ matrices. Further, let $a_1^R, \ldots, a_m^R$ and $b_1^R, \ldots, b_m^R$ are the rows of A and B, respectively. Then A is said to be*

(i) row majorize B (denoted by $A >^{row} B$) if $a_i^R \overset{m}{\geq} b_i^R$, $i = 1, \ldots, m$.

(ii) row weakly supermajorize (submajorize) B (denoted by $A >^{w} (>_w) B$) if $a_i^R \overset{w}{\geq} (\geq_w)\, b_i^R$, $i = 1, \ldots, m$.

Next we give the definitions of some stochastic orders (see [26]).

Definition 2.3. *Let X_1 and X_2 be two absolutely continuous nonnegative r.v.'s with the CDFs $F_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$, the survival functions $\bar{F}_{X_1}(\cdot)$ and $\bar{F}_{X_2}(\cdot)$, the probability density function (p.d.f.'s) $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$, and the reversed hazard rate functions $\tilde{r}_{X_1}(\cdot) = f_{X_1}(\cdot)/F_{X_1}(\cdot)$ and $\tilde{r}_{X_2}(\cdot) = f_{X_2}(\cdot)/F_{X_2}(\cdot)$, respectively. Further, let $F_{X_1}^{-1}(\cdot)$ and $F_{X_2}^{-1}(\cdot)$ be the right continuous inverses of $F_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$, respectively. Then X_1 is said to be smaller than X_2 in the*

(i) usual stochastic order, denoted by $X_1 \leq_{st} X_2$, if $\bar{F}_{X_1}(t) \leq \bar{F}_{X_2}(t)\ \forall t \geq 0$;

(ii) reversed hazard rate order, denoted by $X_1 \leq_{rh} X_2$, if $F_{X_2}(t)/F_{X_1}(t)$ is increasing in $t \geq 0$, or equivalently if $\tilde{r}_{X_1}(t) \leq \tilde{r}_{X_2}(t)\ \forall t \geq 0$;

(iii) star order, denoted by $X_1 \leq_\star X_2$, if $F_{X_2}^{-1}(F_{X_1}(t))/t$ is increasing in $t > 0$ or equivalently $X_1 \leq_\star X_2$ iff $F_{X_2}^{-1}(t)/F_{X_1}^{-1}(t)$ is increasing in $t \in (0, 1)$. □

In what follows, we introduce a notation. Let

$$\mathcal{U}_n = \left\{(\boldsymbol{\vartheta},\boldsymbol{\eta}) = \begin{pmatrix} \vartheta_1 & \cdots & \vartheta_n \\ \eta_1 & \cdots & \eta_n \end{pmatrix} : \vartheta_i > 0, \eta_i > 0, \quad \text{and} \right.$$
$$\left. (\vartheta_i - \vartheta_j)(\eta_i - \eta_j) \geq 0, \ \forall\, i,j = 1,\ldots,n \right\}.$$

Notation. *Let us denote the following notations:*

(i) $\mathcal{D} = \{(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) : \vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n \geq 0\}$
(ii) $\mathcal{E} = \{(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) : 0 \leq \vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_n\}$

Lemma 2.4. **[17, 21]** *Let $\varphi : \mathcal{E} \to \mathbb{R}$ is continuously differentiable on the interior of $\mathcal{E}$. Then, for $\boldsymbol{\vartheta}, \boldsymbol{\eta} \in \mathcal{E}$,*

$$\boldsymbol{\vartheta} \overset{m}{\geq} \boldsymbol{\eta} \implies \varphi(\boldsymbol{\vartheta}) \geq (\text{resp.} \ \leq)\ \varphi(\boldsymbol{\eta})$$

iff $\varphi_{(k)}(\mathbf{z})$ is increasing (respectively, decreasing) in $k = 1, \ldots, n$, where $\varphi_{(k)} = \partial\varphi(\mathbf{z})/\partial z_k$ denotes the partial derivative of φ with respect to its kth argument.

Lemma 2.5. **[21]** *Let $\varphi : S \to \mathbb{R}$ be a function, $S \subseteq \mathbb{R}^n$. Then, for $\boldsymbol{\vartheta}, \boldsymbol{\eta} \in S$,*

$$\boldsymbol{\vartheta} \geq_w \boldsymbol{\eta} \implies \varphi(\boldsymbol{\vartheta}) \geq (\text{resp.} \ \leq)\ \varphi(\boldsymbol{\eta})$$

iff φ is increasing (respectively, decreasing) and Schur-convex (respectively, Schur-concave) on S. Similarly,

$$\boldsymbol{\vartheta} \overset{w}{\geq} \boldsymbol{\eta} \implies \varphi(\boldsymbol{\vartheta}) \geq (\text{resp.} \ \leq)\ \varphi(\boldsymbol{\eta})$$

iff φ is decreasing (respectively, increasing) and Schur-convex (respectively, Schur-concave) on S. □

Lemma 2.6. **[14, 21]** *Let $\varphi : \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ be a continuous function and continuously differentiable on the interior of $\mathcal{D}(\mathcal{E})$. Then*

$$\varphi(\boldsymbol{\vartheta}) \geq \varphi(\boldsymbol{\eta}) \text{ whenever } \boldsymbol{\vartheta} \geq_w \boldsymbol{\eta} \text{ on } \mathcal{D}\ (\mathcal{E})$$

iff $\varphi_{(k)}(z)$ is a nonnegative decreasing (increasing) function in k for all z in the interior of $\mathcal{D}(\mathcal{E})$. Similarly,

$$\varphi(\boldsymbol{\vartheta}) \geq \varphi(\boldsymbol{\eta}) \text{ whenever } \boldsymbol{\vartheta} \overset{w}{\geq} \boldsymbol{\eta} \text{ on } \mathcal{D}\ (\mathcal{E})$$

iff $\varphi_{(k)}(z)$ is a nonpositive decreasing (increasing) function of k for all z in the interior of $\mathcal{D}(\mathcal{E})$.

We conclude this section by giving the following useful definition (for details, see [22]).

Definition 2.7. [22] *Let C_1 and C_2 be two copulas. Then C_1 is said to be less positively lower orthant dependent (PLOD) than C_2, denoted by $C_1 \prec C_2$, if $C_1(\mathbf{v}) \leq C_2(\mathbf{v})$, for all $\mathbf{v} \in [0,1]^n$.*

3. Stochastic comparisons of largest claim amounts

In this section, we derive some stochastic comparison results for largest claim amounts of two different portfolios of risks. Unless otherwise specified, we assume that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are two sets of independent r.v.'s.

Assume that I_{p_i}, $i = 1, \ldots, n$, are independent Bernoulli r.v.'s, independent of X_i's, with $E(I_{p_i}) = p_i$. Denote multivariate Bernoulli random vector $\boldsymbol{I} = \left(I_{p_1}, \ldots, I_{p_n}\right)$. Let $X_i^* = X_i I_{p_i}$, $i = 1, \ldots, n$, and denote $X_{n:n}^* = \max(X_1^*, \ldots, X_n^*)$. Then $X_{n:n}^*$ corresponds to the largest claim amount in a portfolio of risks, where X_i's represent random claim amount that can be made by a policy in an insurance period, and I_{p_i}'s indicate the occurrence of these claims. Further, suppose odds function of each X_i in X is proportional to that of a baseline r.v. with proportionality constant (odds ratio) $\alpha_i > 0$, that is, $X_i \sim \mathrm{PO}(\bar{F}, \alpha_i)$, $i = 1, \ldots, n$, where $\bar{F}$ denotes the survival function of the baseline r.v. Let us denote $X_{n:n}^\circ = \max(X_1^\circ, \ldots, X_n^\circ)$, where $X_i^\circ = X_i I_{q_i}$ and I_{q_i}, $i = 1, \ldots, n$, are independent Bernoulli r.v.'s, independent of X_i's, with $E(I_{q_i}) = q_i$.

Similarly suppose $Y_i \sim \mathrm{PO}(\bar{F}, \beta_i)$, $\beta_i > 0$, $i = 1, \ldots, n$. Denote $Y_{n:n}^* = \max(Y_1^*, \ldots, Y_n^*)$, where $Y_i^* = Y_i I_{p_i}$, and $Y_{n:n}^\circ = \max(Y_1^\circ, \ldots, Y_n^\circ)$, where $Y_i^\circ = Y_i I_{q_i}$, $i = 1, \ldots, n$.

Theorem 3.1 established that more heterogeneity among the odds of claim in terms of the weakly supermajorization order results in less largest claim amount in the sense of the usual stochastic orders when both portfolios have common occurrence of claim $\mathbf{p}$. By the symbol $a \overset{sign}{=} b$, we mean that a and b have the same sign.

Theorem 3.1. *Let $\kappa : [0,1] \to R_+$ be a differentiable and strictly decreasing function. Then, for $(\kappa(\boldsymbol{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ and $(\kappa(\boldsymbol{p}), \boldsymbol{\beta}) \in \mathcal{U}_n$,*

$$\boldsymbol{\alpha} \overset{w}{\geq} \boldsymbol{\beta} \implies X_{n:n}^* \leq_{st} Y_{n:n}^*.$$

Proof. We have $F_{X_{n:n}^*}(x) = \prod_{i=1}^n (1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x))$, where $\bar{F}_{X_i}(x) = \frac{\alpha_i \bar{F}(x)}{1 - \bar{\alpha}_i \bar{F}(x)}$, and $u_i = \kappa(p_i)$, $i = 1, 2, \ldots, n$. Note that $\bar{F}_{X_i}$ is increasing and concave in α_i. Now,

$$\begin{aligned}
\frac{\partial F_{X_{n:n}^*}(x)}{\partial \alpha_i} = \frac{-\kappa^{-1}(u_i)\frac{\partial \bar{F}_{X_i}}{\partial \alpha_i}}{1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x)} F_{X_{n:n}^*}(x) &= -\frac{\kappa^{-1}(u_i)\frac{\bar{F}(x)F(x)}{(1-\bar{\alpha}_i\bar{F}(x))^2}}{1 - \kappa^{-1}(u_i)\frac{\alpha_i\bar{F}(x)}{1-\bar{\alpha}_i\bar{F}(x)}} F_{X_{n:n}^*}(x) \\
&= -\frac{\kappa^{-1}(u_i)\bar{F}(x)F(x)}{(1-\bar{\alpha}_i\bar{F}(x))^2 - \kappa^{-1}(u_i)\alpha_i(1-\bar{\alpha}_i\bar{F}(x))\bar{F}(x)} F_{X_{n:n}^*}(x) \\
&= -g(z_i, \alpha_i)\bar{F}(x)F(x)F_{X_{n:n}^*}(x) \ (say),
\end{aligned}$$

where $z_i = \kappa^{-1}(u_i)$. Again,

$$\frac{\partial g}{\partial \alpha_i} \overset{\text{sign}}{=} -\kappa^{-1}(u_i)\bar{F}(x)\left[(2 - \kappa^{-1}(u_i))(1 - \bar{F}(x)) + 2(1 - \kappa^{-1}(u_i))\alpha_i\bar{F}(x)\right] \leq 0.$$

So $g(z_i, \alpha_i)$ is decreasing in α_i. Further,

$$\frac{\partial g}{\partial z_i} \overset{\text{sign}}{=} (1 - \bar{\alpha}_i\bar{F}(x))^2 \geq 0,$$

so $g(z_i, \alpha_i)$ is increasing in $z_i = \kappa^{-1}(u_i)$, and so it is decreasing in u_i as $z_i = \kappa^{-1}(u_i)$ is decreasing in u_i. Without loss of generality, we assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ so that $(\boldsymbol{\kappa}(\mathbf{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ implies $\kappa(p_1) \geq \kappa(p_2) \geq \cdots \geq \kappa(p_n)$. Now for any pair i,j such that $1 \leq i < j \leq n$, we have $\alpha_i \geq \alpha_j$ and

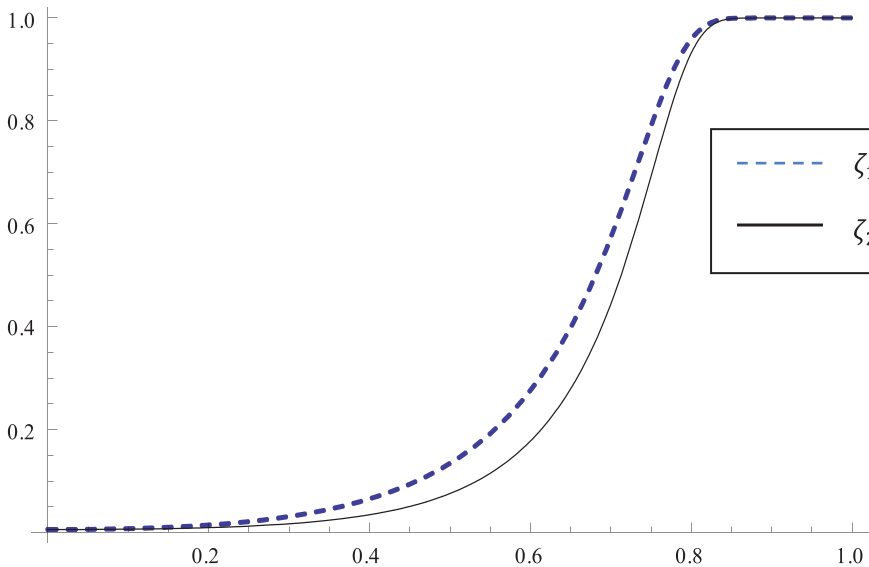

Figure 1. *Plots of $\zeta_1(t)$ and $\zeta_2(t)$, $t \in [0, 1]$.*

$u_i \geq u_j$. Thus, we have

$$g(z_i, \alpha_i) \leq g(z_j, \alpha_i) \leq g(z_j, \alpha_j) \implies \frac{\partial F_{X^*_{n:n}}(x)}{\partial \alpha_j} \leq \frac{\partial F_{X^*_{n:n}}(x)}{\partial \alpha_i} \leq 0. \tag{3}$$

So from Lemma 2.6, we get

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \implies X^*_{n:n} \leq_{st} Y^*_{n:n}.$$

The following example demonstrates Theorem 3.1.

Example 3.2. Suppose that $\{X_1, X_2, X_3, X_4\}$ and $\{Y_1, Y_2, Y_3, Y_4\}$ are two sets of independent nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}(x), \alpha_i)$ and $Y_i \sim \mathrm{PO}(\bar{F}(x), \beta_i)$, $i = 1, 2, 3, 4$, where $\bar{F}(x) = \mathrm{e}^{-(0.5x)^2}$, $x > 0$. Further, suppose that $\{I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4}\}$ is a set of Bernoulli r.v.'s, independent of X_i's and Y_i's. Set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.2, 0.6, 1.5, 2.8)$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.5, 0.8, 2.5, 4.8)$, $(p_1, p_2, p_3, p_4) = (0.95, 0.65, 0.5, 0.35)$, and $\kappa(p) = 1/p^2$, satisfying all the conditions of Theorem 3.1. We consider the transformation $x = t/(1-t)$ so that, for $t \in [0, 1)$, we have $x \in [0, \infty)$. Then denote the distribution functions of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \zeta_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \zeta_2(t)$. Figure 1 shows that $\zeta_1(t) \geq \zeta_2(t)$, for all $t \in [0, 1)$, and hence, $X^*_{n:n} \leq_{st} Y^*_{n:n}$ □

Next we provide a counterexample to show that the stochastic ordering result in Theorem 3.1 may not hold if we relax the weakly supermajorize condition under a strictly decreasing function.

Counterexample 3.3. *In Example 3.2, let us take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.2, 0.9, 1.5, 4.5)$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.35, 0.4, 2.9, 3.8)$ so that $\boldsymbol{\alpha} \overset{w}{\not\succeq} \boldsymbol{\beta}$. In Figure 2, we have plotted $\zeta_1(t) - \zeta_2(t)$ for all $t \in [0, 1)$, from which it is clear that stochastic ordering result of Theorem 3.1 does not hold.*

Theorem 3.4 establishes that largest claim amounts of two portfolios of risks might be increased in terms of the usual stochastic order with the increased heterogeneity among the probabilities of occurrence of claims when both the portfolio of risks have common odds of claim vector $\boldsymbol{\alpha}$.

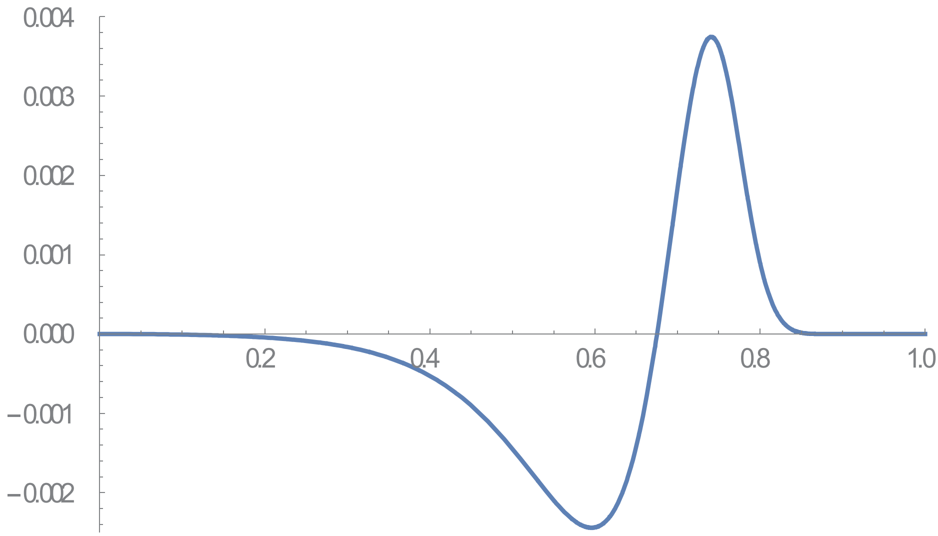

Figure 2. *Plot of* $\zeta_1(t) - \zeta_2(t)$, $t \in [0,1]$.

Theorem 3.4. *Let* $\kappa : [0,1] \to R_+$ *be a differentiable and strictly increasing concave function. Then, for* $(\boldsymbol{\kappa}(\boldsymbol{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ *and* $(\boldsymbol{\kappa}(\boldsymbol{q}), \boldsymbol{\alpha}) \in \mathcal{U}_n$,

$$(\kappa(p_1), \kappa(p_2), \ldots, \kappa(p_n)) \geq_w (\kappa(q_1), \kappa(q_2), \ldots, \kappa(q_n)) \implies X^*_{n:n} \geq_{st} X^\circ_{n:n}.$$

Proof. Here $F_{X^*_{n:n}}(x) = \prod_{i=1}^n (1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x))$, where $\bar{F}_{X_i}(x) = \frac{\alpha_i \bar{F}(x)}{1-\bar{\alpha}_i \bar{F}(x)}$. It is to be noted that $\bar{F}_{X_i}(x)$ is increasing in α_i. Also, κ^{-1} is strictly increasing and convex. Let $\phi(\mathbf{u}) = -F_{X^*_{n:n}}(x)$. We have

$$\frac{\partial \phi(\mathbf{u})}{\partial u_i} = \frac{\bar{F}_{X_i}(x)\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}}{1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x)} F_{X^*_{n:n}}(x) \geq 0$$

according to $\kappa^{-1}(.)$ is increasing.

Let $\ell(\tau_i, u_i) = \frac{\tau_i \frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}}{1-\kappa^{-1}(u_i)\tau_i}$, where $\tau_i = \bar{F}_{X_i}(x)$. Then

$$\frac{\partial \ell}{\partial u_i} \overset{\text{sign}}{=} \tau_i(1 - \tau_i \kappa^{-1}(u_i)) \frac{\mathrm{d}^2\kappa^{-1}(u_i)}{\mathrm{d}u_i^2} + \tau_i^2 \left(\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i} \right)^2 \geq 0,$$

which holds as $\kappa^{-1}(u_i)$ is convex. So, $\ell(\tau_i, u_i)$ is increasing in u_i. Further,

$$\frac{\partial \ell}{\partial \tau_i} \overset{\text{sign}}{=} (1 - \tau_i \kappa^{-1}(u_i)) \frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i} - \tau_i \frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}(-\kappa^{-1}(u_i)) = \frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i} \geq 0,$$

since $\kappa^{-1}(u_i)$ is increasing in u_i. Then $\ell(\tau_i, u_i)$ is increasing in τ_i (i.e., in $\bar{F}_{X_i}(x)$), so that it is increasing in α_i as $\bar{F}_{X_i}(x)$ is increasing in α_i.

Without loss of generality, we assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, so that $(\boldsymbol{\kappa}(\mathbf{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ implies $\kappa(p_1) \geq \kappa(p_2) \geq \cdots \geq \kappa(p_n)$. Now for any pair i, j with $1 \leq i < j \leq n$, we have $\alpha_i \geq \alpha_j$ and $u_i \geq u_j$.

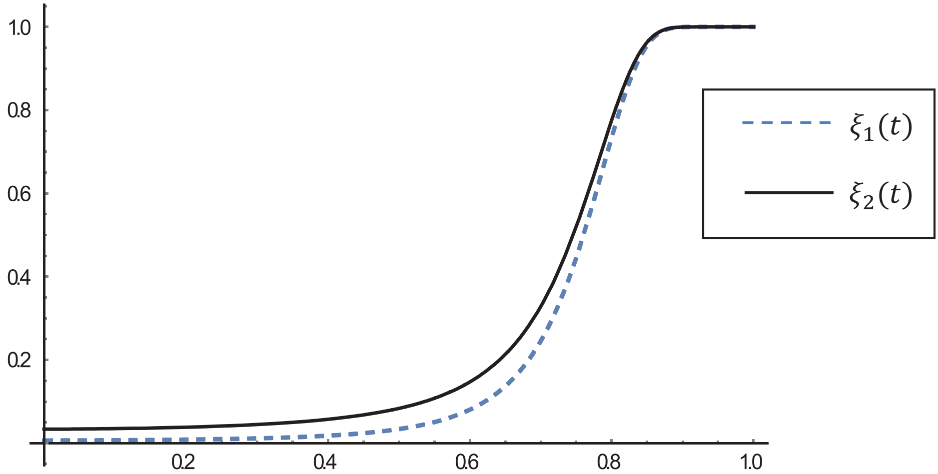

Figure 3. *Plots of $\xi_1(t)$ and $\xi_2(t)$, $t \in [0, 1]$.*

Then, if $\kappa^{-1}(u_i)$ is increasing and convex in u_i, we have

$$\ell(\tau_i, u_i) \geq \ell(\tau_j, u_i) \geq \ell(\tau_j, u_j),$$

$$\text{i.e. } \frac{\partial \phi(\mathbf{u})}{\partial u_i} \geq \frac{\partial \phi(\mathbf{u})}{\partial u_j} \geq 0. \tag{4}$$

So from Lemma 2.6, we have

$$(\kappa(p_1), \kappa(p_2), \ldots, \kappa(p_n)) \succeq_w (\kappa(q_1), \kappa(q_2), \ldots, \kappa(q_n)) \implies X^*_{n:n} \geq_{\text{st}} X^\circ_{n:n}.$$

We illustrate Theorem 3.4 with the following example.

Example 3.5. Suppose that $\{X_1, X_2, X_3, X_4\}$ is a set of independent nonnegative r.v.'s with $X_i \sim \text{PO}(\bar{F}(x), \alpha_i), i = 1, 2, 3, 4$, where $\bar{F}(x) = \mathrm{e}^{-(x/2)^{1.5}}$, $x > 0$. Set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.9, 1.36, 2.55, 3.5)$, $(p_1, p_2, p_3, p_4) = (0.35, 0.5, 0.8, 0.9)$, $(q_1, q_2, q_3, q_4) = (0.2, 0.4, 0.65, 0.8)$, and $\kappa(p) = p/(1 + p)$, satisfying all the conditions of Theorem 3.4. We consider the transformation $x = t/(1 - t)$ so that, for $t \in [0, 1)$, we have $x \in [0, \infty)$. After this substitution, let us denote the respective distribution functions of $X^*_{n:n}$ and $X^\circ_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \xi_1(t)$ and $F_{X^\circ_{n:n}}(t/(1-t)) = \xi_2(t)$. From Figure 3, it is clear that $\xi_1(t) \leq \xi_2(t)$, $\forall t \in [0, 1)$, and hence, $X^*_{n:n} \geq_{st} X^\circ_{n:n}$ □

Next we provide a counterexample to show that the stochastic ordering result in Theorem 3.4 may not hold if we relax the weakly submajorize condition under an increasing concave function.

Counterexample 3.6. *In Example 3.5, let us take $(p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.85, 0.95)$ and $(q_1, q_2, q_3, q_4) = (0.5, 0.55, 0.75, 0.8)$ so that $(\kappa(p_1), \kappa(p_2), \kappa(p_3), \kappa(p_4)) \not\succeq_w (\kappa(q_1), \kappa(q_2), \kappa(q_3), \kappa(q_4))$. In Figure 4, we have plotted $\xi_1(t) - \xi_2(t)$ $\forall t \in [0, 1)$, from which it is clear that none of these distribution functions dominate each other.*

Cai and Wei [10] proposed some multivariate dependence notions based on stochastic arrangement increasing (SAI) notion, including weakly SAI through left tail probability (LWSAI), to model multivariate dependent risks. Since then it has been applied in finance and actuarial science to model dependent

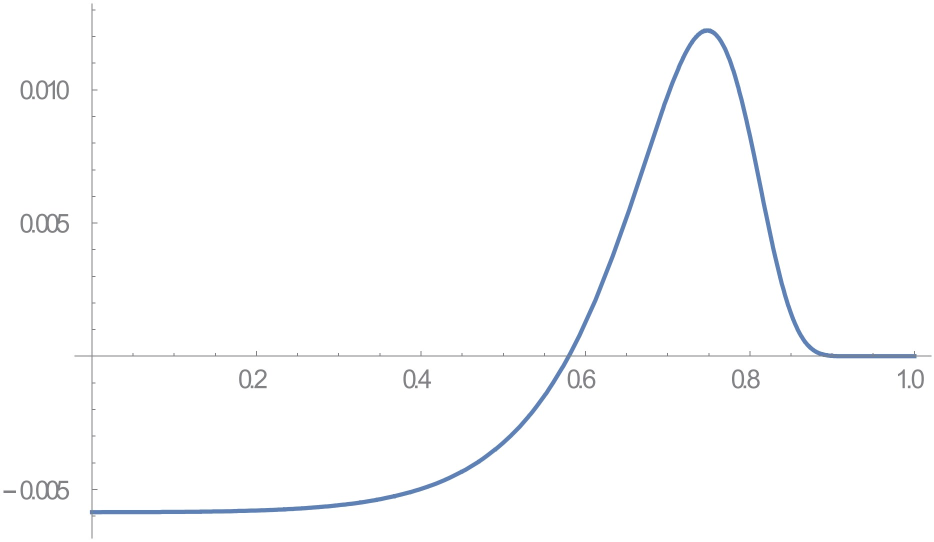

Figure 4. *Plot of* $\xi_1(t) - \xi_2(t)$, $t \in [0, 1]$

stochastic returns and risks [10, 29, 30]. For a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, one of the ways to define or describe its dependence notion is to characterize the expectations of the transformations of the random vector [9]. For any (i, j) with $1 \le i < j \le n$, denote

$$\mathcal{G}^{i,j}_{\text{LWSAI}}(n) = \left\{ g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\pi_{ij}(\boldsymbol{x})) \text{ is decreasing in } x_i \le x_j \right\},$$

where $\pi_{i,j}$ is the special permutation of transposition defined as $\pi_{i,j}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$. A random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ or its distribution is said to be the LWSAI [10] if $\mathbb{E}[g(\boldsymbol{X})] \ge \mathbb{E}[g(\tau_{i,j}(\boldsymbol{X}))]$for any $g(\boldsymbol{x}) \in \mathcal{G}^{i,j}_{LWSAI}(n)$ and any $1 \le i < j \le n$.

Next, we present a stochastic ordering result when the occurrence probabilities are interdependent in terms of LWSAI. Let us denote $S_k = \{\chi | \chi_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, n,\ \chi_1 + \cdots + \chi_n = k\}, k = 0, \ldots, n$, and $S^{i,j}_k(\eta_i, \eta_j) = \{\chi \in S_k | \chi_i = \eta_i, \chi_j = \eta_j,\ \eta_i, \eta_j \in \{0, 1\}\}$, for any $1 \le i \ne j \le n$, $k = 1, \ldots, n-1$. Then $S_k = \bigcup_{\eta_i, \eta_j \in \{0,1\}} S^{i,j}_k(\eta_i, \eta_j)$. Also denote $p(\chi) = \mathbb{P}(\boldsymbol{I} = \chi) = \mathbb{P}\left(I_{p_1} = \chi_1, \ldots, I_{p_n} = \chi_n\right)$.

Lemma 3.7. *(Cai and Wei* [10]*; Balakrishnan et al.* [3]*) A multivariate Bernoulli random vector* $\boldsymbol{I}$ *is LWSAI iff* $p(\chi) \ge p(\pi_{ij}(\chi))$ *for all* $\chi \in S^{i,j}_k(0, 1)$, $1 \le i < j \le n$, *and* $k = 1, \ldots, n-1$.

Theorem 3.8. *Suppose that* $\boldsymbol{I} = \left(I_{p_1}, \ldots, I_{p_n}\right)$ *is LWSAI. If*

$$\boldsymbol{\alpha} \overset{m}{\ge} \boldsymbol{\beta} \text{ such that } \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \text{ and } \beta_1 \le \beta_2 \le \cdots \le \beta_n,$$

then $X^*_{n:n} \le_{st} Y^*_{n:n}$.

Proof. From Theorem 4.1 of Kundu et al. [18], it follows that $\boldsymbol{\alpha} \overset{m}{\ge} \boldsymbol{\beta} \implies X_{n:n} \le_{st} Y_{n:n}$, that is, $F_{X_{n:n}}(t) \ge F_{Y_{n:n}}(t)$ for all t, where $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$. We desire to show that $F_{X^*_{n:n}}(t) \ge F_{Y^*_{n:n}}(t)$ for all $t \in \Re_+$. By the nature of majorization order, it suffices to prove the result when $(\alpha_i, \alpha_j) \overset{m}{\ge} (\beta_i, \beta_j)$ for some pair $1 \le i < j \le n$ and $\alpha_r = \beta_r$ for all $r \ne i, j$. Now, we have

$$F_{X^*_{n:n}}(t) = \mathbb{P}\left(\max\{I_{p_1}X_1, \ldots, I_{p_n}X_n\} \le t\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \sum_{\chi \in S_k} \mathbb{P}\left(\max\{I_{p_1}X_1, \ldots, I_{p_n}X_n\} \le t | \boldsymbol{I} = \chi\right) p(\chi) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(\max\{X_1, \ldots, X_n\} \le t) + \sum_{k=1}^{n-1} \sum_{\chi \in S_k} p(\chi)\mathbb{P}(\max\{\chi_1 X_1, \ldots, \chi_n X_n\} \le t) \\
&= p(\mathbf{0}) + p(\mathbf{1})F_{X_{n:n}}(t) + \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\chi) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right. \\
&\quad + \sum_{\chi \in S_k^{i,j}(0,1)} p(\chi) F_{X_j}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) + \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi)) F_{X_i}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \\
&\quad \left. + \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi) F_{X_i}(t) F_{X_j}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_{Y^*_{n:n}}(t) &= p(\mathbf{0}) + p(\mathbf{1})F_{Y_{n:n}}(t) + \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\chi) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) \right. \\
&\quad + \sum_{\chi \in S_k^{i,j}(0,1)} p(\chi) F_{Y_j}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) + \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi)) F_{Y_i}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) \\
&\quad \left. + \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi) F_{Y_i}(t) F_{Y_j}(t) \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) \right\}.
\end{aligned}
$$

Upon using the condition that $\mathbb{P}(\chi_r X_r \le t) = \mathbb{P}(\chi_r Y_r \le t)$ for all $r \ne i,j$, we have

$$
\begin{aligned}
F_{X^\star_{n:n}}(t) - F_{Y^*_{n:n}}(t) &= p(\mathbf{1})[F_{X_{n:n}}(t) - F_{Y_{n:n}}(t)] \\
&\quad \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,1)} p(\chi)[F_{X_j}(t) - F_{Y_j}(t)] \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) + \right. \\
&\quad \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi))[F_{X_i}(t) - F_{Y_i}(t)] \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) + \\
&\quad \left. \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi)[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)] \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right\} \\
&\ge \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi))[F_{X_j}(t) - F_{Y_j}(t)] \prod_{r \ne i,j}^{n} \mathbb{P}(\chi_r X_r \le t) + \right.
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi))[F_{X_i}(t) - F_{Y_i}(t)] \prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) + \\
& \left. \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi)[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)] \prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) \right\} \\
= & \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi))[F_{X_i}(t) + F_{X_j}(t) - F_{Y_i}(t) - F_{Y_j}(t)] \prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) + \right. \\
& \left. \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi)[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)] \prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) \right\} \\
\geq & \; 0,
\end{aligned}
$$

where the first inequality follows from the fact that $F_{X_{n:n}}(t) \geq F_{Y_{n:n}}(t)$, Lemma 3.7, and the fact that $F_{X_i}(t) = \frac{F(t)}{1-\bar{\alpha}_i \bar{F}(t)}$ is decreasing in α_i. For the last inequality, we have the following explanation. Since $F_{X_i}(t)$ is convex in α_i, it follows that $F_{X_i}(t) + F_{X_j}(t) \geq F_{Y_i}(t) + F_{Y_j}(t)$. Let

$$\phi(\alpha_i, \alpha_j) = F_{X_i}(t)F_{X_j}(t) = \frac{F^2(t)}{(1 - \bar{\alpha}_i \bar{F}(t))(1 - \bar{\alpha}_j \bar{F}(t))}.$$

Then, for $1 \leq i < j \leq n$,

$$\frac{\partial \phi}{\partial \alpha_i} - \frac{\partial \phi}{\partial \alpha_j} = \frac{(\alpha_i - \alpha_j)\bar{F}(x)}{(1 - \bar{\alpha}_i \bar{F}(t))^2 (1 - \bar{\alpha}_j \bar{F}(t))^2} \leq 0.$$

So, from Lemma 2.4, we get that $(\alpha_i, \alpha_j) \overset{m}{\geq} (\beta_i, \beta_j) \Rightarrow \phi(\alpha_i, \alpha_j) \geq \phi(\beta_i, \beta_j)$, and thus the proof is completed. □

Remark 3.9. Here it is to be noted that in Theorem 3.11 of Balakrishnan et al. [7], they derived similar results under the assumption that the survival function $\bar{F}(x; \alpha)$ is decreasing and convex in $\alpha > 0$, which is not satisfied by the PO model. Here our established results in Theorem 3.8 can be generalized for any semi-parametric model for which $\bar{F}(x; \alpha)$ is increasing and concave in $\alpha > 0$.

We illustrate Theorem 3.8 with the following example.

Example 3.10. Suppose that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}(x), \alpha_i), i = 1, 2$, and $Y_i \sim \mathrm{PO}(\bar{F}(x), \beta_i), i = 1, 2$, where $\bar{F}(x) = \mathrm{e}^{-(0.08x)^{0.08}}$, $x > 0$. Set $(\alpha_1, \alpha_2) = (0.55, 1.45)$, $(\beta_1, \beta_2) = (0.65, 1.35)$, $p(0,0) = P(I_{p_1} = 0, I_{p_2} = 0) = 0.14$, $p(0,1) = 0.47$, $p(1,0) = 0.25$, and$p(1,1) = 0.14$. Then $I = \{I_{p_1}, I_{p_2}\}$ is LWSAI. We consider the transformation $x = t/(1-t)$ so that for $t \in [0,1)$, we have $x \in [0, \infty)$. After this substitution, let us denote the respective distribution functions of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \varphi_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \varphi_2(t)$. Figure 5 shows that $\varphi_1(t) \geq \varphi_2(t)$ for all $t \in [0,1)$. Hence, $X^*_{n:n} \leq_{st} Y^*_{n:n}$. □

Theorems 3.11–3.13 compare the largest claim amounts of two interdependent heterogeneous portfolios of risks where the joint distribution functions are modeled using copulas.

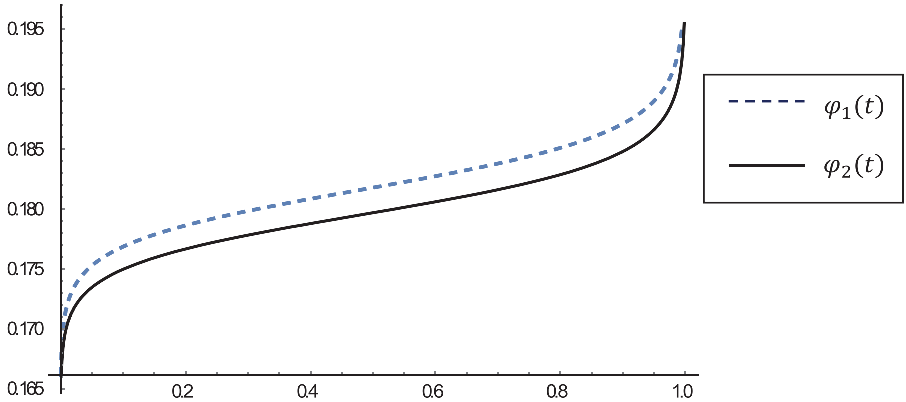

Figure 5. *Plots of $\varphi_1(t)$ and $\varphi_2(t)$, $t \in [0,1]$.*

Theorem 3.11. *Let $X_1, X_2, \dots, X_n$ $(Y_1, Y_2, \dots, Y_n)$ be nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}, \alpha_i)$ $(Y_i \sim \mathrm{PO}(\bar{F}, \beta_i))$, $i = 1, 2, \dots, n$, and let the associated copula be C. Further, suppose that $\{I_{p_1}, I_{p_2}, \dots, I_{p_n}\}$ is a set of independent Bernoulli r.v.'s, independent of X_i's (Y_i's). Then*

$$\alpha_i \le \beta_i, \forall i = 1, 2, \dots, n \implies X^*_{n:n} \le_{st} Y^*_{n:n}.$$

Proof. The distribution function of $X^*_{n:n}$ can be written as

$$\begin{aligned}
G_{X^*_{n:n}}(t) &= \mathbb{P}(X^*_1 \le t, X^*_2 \le t, \dots, X^*_n \le t)\\
&= \mathbb{P}(I_{p_1}X_1 \le t, \dots, I_{p_n}X_n \le t)\\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} \mathbb{P}\left(I_{p_1}X_1 \le t, \dots, I_{p_n}X_n \le t | \boldsymbol{I} = \boldsymbol{\chi}\right) p(\boldsymbol{\chi})\\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} p(\boldsymbol{\chi}) \mathbb{P}\left(\chi_1 X_1 \le t, \dots, \chi_n X_n \le t\right)\\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} p(\boldsymbol{\chi}) C\left([F_{X_1})]^{\chi_1}, \dots, [F_{X_n}]^{\chi_n}\right).
\end{aligned}$$

Since $F_{X_i}(x) = \frac{F(x)}{1-\bar{\alpha}_i \bar{F}(x)}$ is decreasing in α_i and the copula is component-wise increasing, we have that $G_{X^*_{n:n}}(x)$ is decreasing in α_i, for $i = 1, 2, \dots, n$. Hence, the desired result follows. □

Theorem 3.12. *Let $X_1, X_2, \dots, X_n$ be nonnegative r.v.'s with $X_i \sim PO(\bar{F}, \alpha_i)$, $i = 1, 2, \dots, n$, and let the associated copula be C (C'). Further, suppose that $\{I_{p_1}, I_{p_2}, \dots, I_{p_n}\}$ is a set of independent Bernoulli r.v.'s, independent of X_i's. Then*

$$C' \prec C \implies X^*_{n:n} \le_{st} X^{*'}_{n:n},$$

where the r.v.'s $X^*_{n:n}$ and $X^{*'}_{n:n}$ represent the largest claim amount with the associated copula C (C').

Proof. The proof follows from Theorem 3.11 and the fact "C' is less PLOD than C". □

For the next theorem, proof follows from Theorems 3.11 and 3.12 and, hence, omitted.

Theorem 3.13. *Let $X_1, X_2, \dots, X_n$ $(Y_1, Y_2, \dots, Y_n)$ be nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}, \alpha_i)$ $(Y_i \sim \mathrm{PO}(\bar{F}, \beta_i))$, $i = 1, 2, \dots, n$, and let the associated copula be C (C'). Further, suppose that*

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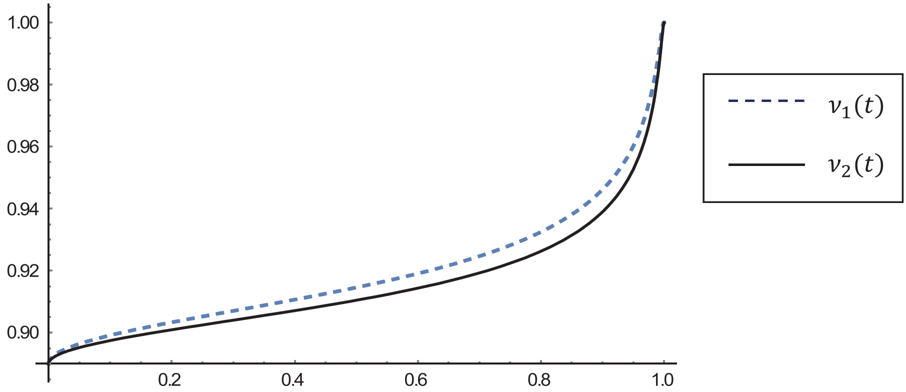

Figure 6. *Plots of $\nu_1(t)$ and $\nu_2(t)$, $t \in [0,1]$.*

$\{I_{p_1}, I_{p_2}, \ldots, I_{p_n}\}$ is a set of independent Bernoulli r.v.'s, independent of X_i's (Y_i's). Then

$$\alpha_i \le \beta_i,\ \forall i = 1,2,\ldots,n,\ \text{and}\ \alpha_i \le \beta_i,\ \forall i = 1,2,\ldots,n,\ \text{and}\ C' \prec C \implies X^*_{n:n} \le_{st} Y^*_{n:n}.$$

The following example demonstrates the result given in the above theorem.

Example 3.14. Suppose $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent nonnegative r.v.'s with $X_i \sim \text{PO}(\bar{F}(x), \alpha_i)$, $i = 1,2$, and $Y_i \sim \text{PO}(\bar{F}(x), \beta_i)$, $i = 1,2$, where $\bar{F}(x) = \mathrm{e}^{-(0.05x)^{0.5}}$, $x > 0$. Set $(\alpha_1, \alpha_2) = (0.5, 1.25)$, $(\beta_1, \beta_2) = (0.75, 1.55)$, $p(0,0) = 0.89$, $p(0,1) = 0.06$, $p(1,0) = 0.04$, $p(1,1) = 0.01$. Here we take $C(x_1, x_2) = \mathrm{e}^{-\{(\log(x_1))^{\theta_1} + (\log(x_2))^{\theta_1}\}^{1/\theta_1}}$ and $C'(x_1, x_2) = \mathrm{e}^{-\{(\log(x_1))^{\theta_2} + (\log(x_2))^{\theta_2}\}^{1/\theta_2}}$, where $\theta_1 = 2$ and $\theta_2 = 5$. We consider the transformation $x = t/(1-t)$, so that for $t \in [0,1)$, we have $x \in [0, \infty)$. After this substitution, we denote the distribution functions of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \nu_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \nu_2(t)$, respectively. Figure 6 shows that $\nu_1(t) \ge \nu_2(t)$ for all $t \in [0,1)$. Hence, $X^*_{n:n} \le_{st} Y^*_{n:n}$. □

The following theorem compares the largest claim amounts of two sets of heterogeneous portfolios of risks in terms of the reversed hazard rate order. Here we assume that the odds ratios are the same but the probabilities of occurrences of claims are different.

Theorem 3.15. *Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s with $X_i \sim \text{PO}(\bar{F}, \alpha)$, $i = 1, \ldots, n$, and let I_{p_i} (I_{q_i}), $i = 1, \ldots, n$, be independent Bernoulli r.v.'s, independent of X_i's. Further, let $X^*_i = X_i I_{p_i}$ and $X^\circ_i = X_i I_{q_i}$, $i = 1, \ldots, n$. Let $\kappa : [0,1] \to R_+$ be a differentiable function. Then*

(i) $(\kappa(p_1), \kappa(p_2), \ldots, \kappa(p_n)) \overset{w}{\succeq} (\kappa(q_1), \kappa(q_2), \ldots, \kappa(q_n)) \implies X^*_{n:n} \ge_{rh} X^\circ_{n:n}$,
if $\kappa(x)$ is strictly decreasing and convex in x;

(ii) $(\kappa(p_1), \kappa(p_2), \ldots, \kappa(p_n)) \succeq_w (\kappa(q_1), \kappa(q_2), \ldots, \kappa(q_n)) \implies X^*_{n:n} \ge_{rh} X^\circ_{n:n}$,
if $\kappa(x)$ is strictly increasing and concave in x.

Proof. We have $F_{X^*_{n:n}}(t) = \prod_{i=1}^n (1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t))$, where $\kappa(p_i) = u_i$, $i = 1, \ldots, n$. Since $X_i \sim \text{PO}(\bar{F}, \alpha)$ for $i = 1, \ldots, n$. We have $\bar{F}_\alpha(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}$.

Now $f_{X^*_{n:n}}(t) = \sum_{i=1}^n \left(\frac{\kappa^{-1}(u_i) f_\alpha(t)}{1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t)} \right) F_{X^*_{n:n}}(t)$ and therefore

$$\tilde{r}_{X^*_{n:n}}(t) = \sum_{i=1}^n \frac{\kappa^{-1}(u_i) f_\alpha(t)}{1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t)}.$$

So, we have

$$\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_i} = \frac{\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i} f_\alpha(t)}{(1-\kappa^{-1}(u_i)\bar{F}_\alpha(t))} + \frac{\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i} \kappa^{-1}(u_i) f_\alpha(t)\bar{F}_\alpha(t)}{(1-\kappa^{-1}(u_i)\bar{F}_\alpha(t))^2}$$

and

$$\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_j} = \frac{\frac{\mathrm{d}\kappa^{-1}(u_j)}{\mathrm{d}u_j} f_\alpha(t)}{(1-\kappa^{-1}(u_j)\bar{F}_\alpha(t))} + \frac{\frac{\mathrm{d}\kappa^{-1}(u_j)}{\mathrm{d}u_j} \kappa^{-1}(u_j) f_\alpha(t)\bar{F}_\alpha(t)}{(1-\kappa^{-1}(u_j)\bar{F}_\alpha(t))^2}.$$

Now, consider the following two cases.

Case I: Let κ be strictly decreasing and convex. Then κ^{-1} is strictly decreasing and convex. Consequently, $\tilde{r}_{X^*_{n:n}}(t)$ is decreasing in u_i. Further,

$$(u_i-u_j)\left(\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_i} - \frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_j}\right) \overset{\text{sign}}{=} (u_i-u_j)\left[\left(\frac{\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\frac{\mathrm{d}\kappa^{-1}(u_j)}{\mathrm{d}u_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right) + \left(\frac{\kappa^{-1}(u_i)\frac{\mathrm{d}\kappa^{-1}(u_i)}{|rmdu_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\kappa^{-1}(u_j)\frac{|rmd\kappa^{-1}(u_j)}{\mathrm{d}u_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right)\right] \geq 0,$$

which follows from the fact that κ^{-1} is decreasing and convex. Consequently, we have that $\tilde{r}_{X^*_{n:n}}(t)$ is decreasing and Schur-convex in u_i. So, from Lemma 2.5, the first part of the theorem follows.

Case II: Let κ be strictly increasing and concave. Then κ^{-1} is strictly increasing and convex. Consequently, $\tilde{r}_{X^*_{n:n}}(t)$ is increasing in u_i. Further,

$$(u_i-u_j)\left(\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_i} - \frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_j}\right) \overset{\text{sign}}{=} (u_i-u_j)\left[\left(\frac{\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\frac{\mathrm{d}\kappa^{-1}(u_j)}{\mathrm{d}u_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right) + \left(\frac{\kappa^{-1}(u_i)\frac{\mathrm{d}\kappa^{-1}(u_i)}{\mathrm{d}u_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\kappa^{-1}(u_j)\frac{\mathrm{d}\kappa^{-1}(u_j)}{\mathrm{d}u_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right)\right] \geq 0,$$

which follows from the fact that κ^{-1} is increasing and convex. Consequently, we have that $\tilde{r}_{X^*_{n:n}}(t)$ is increasing and Schur-convex in u_i. So, from Lemma 2.5, the second part of the theorem follows. □

We illustrate Theorem 3.15 with the following example.

Example 3.16. Suppose that $\{X_1, X_2, X_3, X_4\}$ is a set of independent nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}(x), \alpha)$, $i = 1, 2, 3, 4$, where $\bar{F}(x) = \mathrm{e}^{-(0.5x)^{1.5}}$, $x > 0$, and $\alpha = 0.75$. Further, suppose that $\{I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4}\}$ and $\{I_{q_1}, I_{q_2}, I_{q_3}, I_{q_4}\}$ are two sets of Bernoulli r.v.'s, independent of X_i's, $i = 1, 2, 3, 4$. Set $(p_1, p_2, p_3, p_4) = (0.35, 0.65, 0.85, 0.96)$, $(q_1, q_2, q_3, q_4) = (0.15, 0.35, 0.55, 0.82)$. Let $\kappa(x) = \log(1+x)$, which is strictly increasing and concave. We consider the transformation $x = t/(1-t)$ so that for $t \in [0, 1)$, we have $x \in [0, \infty)$. After this substitution, let us denote the respective reverse hazard functions by $\tilde{r}_{X^*_{n:n}}(t/(1-t) = \tilde{r}_1(t)$ and $\tilde{r}_{X^\circ_{n:n}}(t/(1-t)) = \tilde{r}_2(t)$. From Figure 7, we see that $\tilde{r}_1(t) \geq \tilde{r}_2(t)$ for all $t \in [0, 1)$. Hence, $X^*_{n:n} \geq_{rh} X^\circ_{n:n}$. □

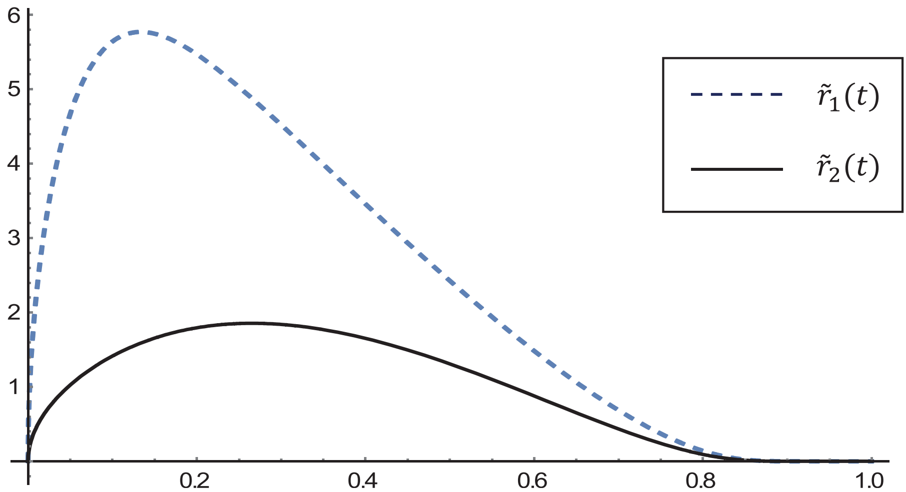

Figure 7. *Plots of* $\tilde{r}_1(t)$ *and* $\tilde{r}_2(t)$, $t \in [0, 1]$.

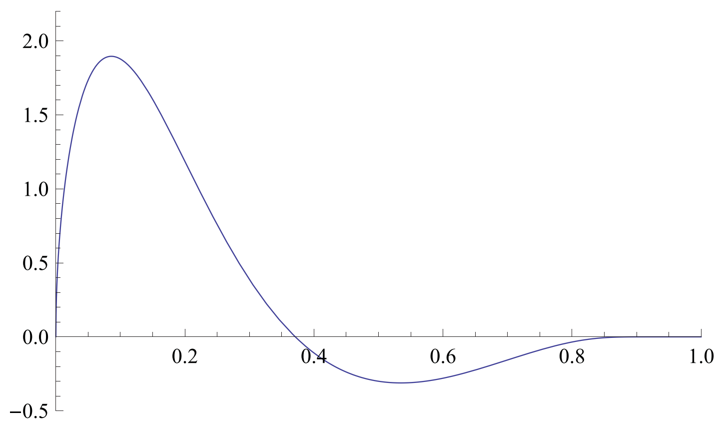

Figure 8. *Plot of* $\tilde{r}_1(t) - \tilde{r}_2(t)$, $t \in [0, 1]$.

Next, we provide a counterexample to show that the ordering result in Theorem 3.15 may not hold if we relax the stated majorization conditions.

Counterexample 3.17. *In Example 3.16, let us take* $(p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.85, 0.95)$ *and* $(q_1, q_2, q_3, q_4) = (0.5, 0.65, 0.8, 0.85)$ *so that* $(\kappa(p_1), \kappa(p_2), \kappa(p_3), \kappa(p_4)) \not\succeq_w (\kappa(q_1), \kappa(q_2), \kappa(q_3), \kappa(q_4))$. *In Figure 8, we have plotted* $\tilde{r}_1(t) - \tilde{r}_2(t)$ *for all* $t \in [0, 1)$, *which shows that the hazard rate ordering result of Theorem 3.15(ii) does not hold in this case.*

4. Star order for multiple-outlier claim

Let X_i, $i = 1, 2, \ldots, r$, have a common distribution F, and X_j, $j = r + 1, r + 2, \ldots, n$, have a common distribution G, where $r = 1, 2, \ldots, n-1$. This type of model is known as outlier model, where F is called the original distribution, whereas the G is called the outlier distribution. For $r = 1, 2, \ldots, n - 2$, it is called multiple-outlier model. In actuarial practice, even for a portfolio of risks consisting of similar kind of insureds, it may happen that some insureds have different (higher/lower) probabilities of occurrence of claims, claim sizes or odds of claims than the rest. Then this phenomena falls in the multiple-outlier claims model.

Star order is one of the most important transform order used to compare the skewness of probability distributions. Since in general the insurance claims follow positively skewed and heavy-tailed distributions, it is therefore of interest to establish sufficient conditions for star order between them to analyze the effects of the heterogeneity among occurrence probabilities and claim severity parameters (e.g., the odds ratio in our considered model) on the skewness of their distributions.

In Theorem 4.2, we derive stochastic comparisons on the largest claim amounts in case of multiple-outlier claims model with respect to star order.

The following lemma is derived from Saunders and Moran [25], which will be used in proving Theorem 4.2.

Lemma 4.1. *Let $\{G_\lambda|\lambda \in \mathbb{R}_+\}$ be a class of distribution function such that G_λ is supported on some interval $\mathbf{I} \subseteq \mathbb{R}_+$. Then, $G_\lambda \geq_\star G_{\lambda^\star}$ for $\lambda \leq \lambda^\star$ iff $\frac{\frac{\partial G_\lambda(x)}{\partial \lambda}}{x g_\lambda(x)}$ is increasing in x, where the density g_λ of G_λ does not vanishe on any subinterval of* $\mathbf{I}$.

Theorem 4.2. *Let $X_i \sim \mathrm{PO}(\bar{F}, \alpha_1)(Y_i \sim \mathrm{PO}(\bar{F}, \beta_1))$ for $i = 1, 2, \ldots, n_1$, and let $X_j \sim \mathrm{PO}(\bar{F}, \alpha_2)(Y_j \sim \mathrm{PO}(\bar{F}, \beta_2))$ for $j = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\ (= n)$. Assume that X_i's are independent and that the Y_j's are independent. Further, let I_{p_i}, $i = 1, 2, \ldots, n_1$, be independent Bernoulli r.v.'s such that $\mathbb{E}[I_{p_i}] = p_1$, and let I_{p_j}, $j = n_1 + 1, \ldots, n$, be another set of independent Bernoulli r.v.'s such that $\mathbb{E}[I_{p_j}] = p_2$. Then, for $n_1 p_1 \geq n_2 p_2$, $p_1 \geq p_2$, $\alpha_1 \leq \alpha_2$, and $\beta_1 \leq \beta_2$,*

$$\frac{\alpha_1}{\alpha_2} \leq \frac{\beta_1}{\beta_2} \implies X^*_{n:n} \geq_\star X^*_{n:n}.$$

Proof. Consider the following two cases.

Case I: Let $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = c$ (say). Further, let $\alpha_1 = \alpha \leq \alpha_2$ and $\beta_1 = \beta \leq \beta_2$, so that $\alpha \in [0, c/2]$. Then the distribution function of $X^*_{n:n}$ is given by

$$F_{n,\alpha}(x) = \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2}. \tag{5}$$

Here $\bar{F}_\alpha(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}$ and $\bar{F}_{c-\alpha}(x) = \frac{(c-\alpha)\bar{F}(x)}{1 - \overline{(c-\alpha)}\bar{F}(x)}$.

The density function corresponding to Eq. (5) is

$$\begin{aligned} f_{n,\alpha}(x) &= \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1 - 1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2 - 1} f(x) \\ &\quad \times \left[\frac{n_1 p_1 \alpha (1 - p_2 \bar{F}_{c-\alpha}(x))}{(1 - \bar{\alpha}\bar{F}(x))^2} + \frac{n_2 p_2 (c-\alpha)(1 - p_1 \bar{F}_\alpha(x))}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2}\right]. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial F_{n,\alpha}(x)}{\partial \alpha} &= F(x)\bar{F}(x) \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1 - 1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2 - 1} \\ &\quad \times \left[\frac{-n_1 p_1 (1 - p_2 \bar{F}_{c-\alpha}(x))}{(1 - \bar{\alpha}\bar{F}(x))^2} + \frac{n_2 p_2 (1 - p_1 \bar{F}_\alpha(x))}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2}\right]. \end{aligned}$$

Let $\Lambda_1(x) = \frac{1 - p_2 \bar{F}_{c-\alpha}(x)}{(1 - \bar{\alpha}\bar{F}(x))^2}$ and $\Lambda_2(x) = \frac{1 - p_1 \bar{F}_\alpha(x)}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2}$. Then, by using Lemma 4.1, it suffices to prove that

$$\frac{\frac{\partial F_{n,\alpha}(x)}{\partial \alpha}}{x f_{n,\alpha}(x)} = \frac{F(x)\bar{F}(x)}{x f(x)} \left[\frac{-n_1 p_1 \Lambda_1(x) + n_2 p_2 \Lambda_2(x)}{n_1 p_1 \alpha \Lambda_1(x) + n_2 p_2 (c-\alpha)\Lambda_2(x)}\right]$$

$$= \frac{F(x)\bar{F}(x)}{xf(x)} \times \Lambda(x)$$

is increasing in $x \in \mathbb{R}_+$, for $\alpha \in [0, c/2]$, where

$$\begin{aligned}
\Lambda(x) &= \left[\frac{n_1p_1\alpha\Lambda_1(x) + n_2p_2(c-\alpha)\Lambda_2(x)}{-n_1p_1\Lambda_1(x) + n_2p_2\Lambda_2(x)}\right]^{-1} \\
&= \left[\frac{cn_2p_2\Lambda_2(x)}{n_2p_2\Lambda_2(x) - n_1p_1\Lambda_1(x)} - \alpha\right]^{-1} \\
&= \left[c\left(1 - \frac{n_1p_1}{n_2p_2}\frac{\Lambda_1(x)}{\Lambda_2(x)}\right)^{-1} - \alpha\right]^{-1}.
\end{aligned}$$

Further, let

$$\begin{aligned}
\Lambda_3(x) &= \frac{\Lambda_1(x)}{\Lambda_2(x)} \\
&= \frac{\left(\frac{1-p_2\bar{F}_{c-\alpha}(x)}{(1-\bar{\alpha}\bar{F}(x))^2}\right)}{\left(\frac{1-p_1\bar{F}_\lambda(x)}{(1-\overline{(c-\alpha)}\bar{F}(x))^2}\right)} \\
&= \left(\frac{[1-\overline{(c-\alpha)}\bar{F}(x) - p_2(c-\alpha)\bar{F}(x)]}{(1-\bar{\alpha}\bar{F}(x))}\right)\left(\frac{(1-\bar{\alpha}\bar{F}(x))(1-\overline{(c-\alpha)}\bar{F}(x))^2}{[1-\bar{\alpha}\bar{F}(x) - p_1\alpha\bar{F}(x)]}\right) \\
&= \left(\frac{[F(x) + (1-p_2)(c-\alpha)\bar{F}(x)]}{[F(x) + (1-p_1)\alpha\bar{F}(x)}]\right)\left(\frac{1-\overline{(c-\alpha)}\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right) \\
&= \Delta_1(x) \times \Delta_2(x).
\end{aligned}$$

It is clear that $\Delta_1(x) \geq 0$ and $\Delta_2(x) \geq 0$ $\forall x \in \mathbb{R}_+$. Now we have

$$\begin{aligned}
\Delta_1'(x) &\overset{\text{sign}}{=} [1-(1-p_2)(c-\alpha)][F(x) + \alpha(1-p_1)\bar{F}(x)] \\
&\quad - [1-\alpha(1-p_1)][F(x) + (1-p_2)(c-\alpha)\bar{F}(x)] \\
&\overset{\text{sign}}{=} [\alpha(1-p_1) - (1-p_2)(c-\alpha)](\bar{F}(x) + F(x)) \\
&\overset{\text{sign}}{=} [\alpha(1-p_1) - (1-p_2)(c-\alpha)] \\
&\leq 0,
\end{aligned}$$

which holds as $\alpha \leq c - \alpha$ and $p_1 \geq p_2$. Further,

$$\begin{aligned}
\Delta_2'(x) &\overset{\text{sign}}{=} (\overline{c-\alpha})(1-\bar{\alpha}\bar{F}(x))(1 - c \bar{-} \alpha\bar{F}(x)) \\
&\overset{\text{sign}}{=} (c \bar{-} \alpha) - \bar{\alpha} \leq 0.
\end{aligned}$$

Hence, ultimately we have $\Lambda_3'(x) \leq 0$. Consequently, $\Lambda_3(x)$ is nonnegative and decreasing in x.

$$\begin{aligned}
\text{Now} \qquad & \alpha \leq (c-\alpha) \\
& \implies \frac{1-\overline{(c-\alpha)}\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} \geq 1
\end{aligned}$$

$$\text{and} \qquad (c-\alpha) \geq \alpha, p_1 \geq p_2$$

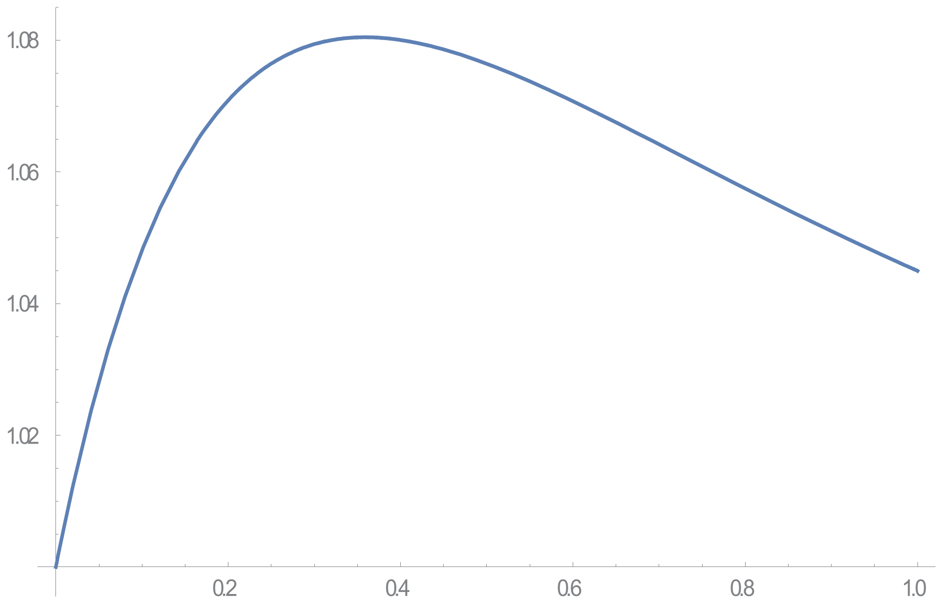

Figure 9. *Plots of derivative of* $F^{-1}_{X^*_{n:n}}(t)/F^{-1}_{Y^*_{n:n}}(t)$ *with respect to* t *for* $t \in (0, 1)$.

$$\implies F(x) + (1 - p_2)(c - \alpha)\bar{F}(x) \geq F(x) + (1 - p_1)\alpha\bar{F}(x),$$

and hence, $\Lambda_3(x) \geq 1$.

Now if $n_1 \geq n_2$, then $n_1p_1 \geq n_2p_2$. On combining all of these results, we have

$$\frac{n_1p_1}{n_2p_2}\Lambda_3(x) \geq 1$$
$$\implies \left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right) \leq 0.$$

Hence, $\left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right)$ is increasing in x, which implies $\left[\left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right)^{-1} - \alpha\right]^{-1}$ is increasing in x. So ultimately we have $\Lambda(x)$ is increasing in x. This completes the proof.

Case II: Let $\alpha_1 + \alpha_2 \neq \beta_1 + \beta_2$. In this case, there exists some $\kappa > 0$ such that $\alpha_1 + \alpha_2 = \kappa(\beta_1 + \beta_2)$. Now, let $Z_{n:n}$ be the largest claim amount from $I_1Z_1, \ldots, I_{n_1}Z_{n_1}, I_{n_1+1}Z_{n_1+1}, \ldots, I_nZ_n$, where $Z_1, \ldots, Z_{n_1}$ have the distribution $F_{\kappa\mu_1}$ and $Z_{n_1+1}, \ldots, Z_n$ have the distribution $F_{\kappa\mu_2}$. Finally, on using the result of Case I and the scale invariant property of the star order, the desired result follows.

Example 4.3. Suppose that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent nonnegative r.v.'s with $X_i \sim \mathrm{PO}(\bar{F}(x), \alpha_i)$, $i = 1, 2$, and $Y_i \sim \mathrm{PO}(\bar{F}(x), \beta_i)$, $i = 1, 2$, where $\bar{F}(x) = \mathrm{e}^{-x}$, $x > 0$. Set $(\alpha_1, \alpha_2) = (0.5, 1.9)$, $(\beta_1, \beta_2) = (0.8, 1.2)$, $(p_1, p_2) = (1/4, 1/8)$, and $(n_1, n_2) = (3, 2)$. Then all the conditions of Theorem 4.2 are satisfied. In Figure 9, we have plotted the derivative of $F^{-1}_{X^*_{n:n}}(t)/F^{-1}_{Y^*_{n:n}}(t)$ with respect to t from which it is clear that $F^{-1}_{X^*_{n:n}}(t)/F^{-1}_{Y^*_{n:n}}(t)$ is increasing for $t \in (0, 1)$. Hence, $X^*_{n:n} \geq_\star Y^*_{n:n}$. □

5. Aggregate claim amount

The aggregate claim of a portfolio is the sum of all amounts payable during the reference period. Our next theorem derives sufficient conditions that the aggregate claim amount increases on reducing the heterogeneity in the sense of majorization among the concerned parameters of a considered semi-parametric family of distributions when they are in ascending order. Here we assume that occurrence probabilities are arranged according to LWSAI.

Theorem 5.1. *Suppose that $\boldsymbol{I} = (I_{p_1}, \ldots, I_{p_n})$ is LWSAI. Let $X_{\alpha_i} \sim \bar{F}(x; \alpha_i)$ $(X_{\beta_i} \sim \bar{F}(x; \beta_i)), i = 1, \ldots, n$, be independent r.v.'s. Suppose that the following conditions hold:*

(i) $\bar{F}(x; \alpha)$ is increasing and concave in $\alpha > 0$; and
(ii) the survival function of $X_{\mu_1} + X_{\mu_2}$ is Schur-concave in (μ_1, μ_2), $\mu_1, \mu_2 > 0$.

If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, then $\sum_{i=1}^{n} I_{p_i} X_{\alpha_i} \leq_{st} \sum_{i=1}^{n} I_{p_i} X_{\beta_i}$.

Proof. Let $A(\boldsymbol{I}, \alpha) = \sum_{i=1}^{n} I_i X_{\alpha_i}$ and $A(\boldsymbol{I}, \boldsymbol{\beta}) = \sum_{i=1}^{n} I_i X_{\beta_i}$. We have to prove that $F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t) \geq F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t)$ $\forall t \in \Re_+$. By the nature of majorization order, it suffices to prove it when $(\alpha_i, \alpha_j) \overset{m}{\succeq} (\beta_i, \beta_j)$ for some pair $1 \leq i < j \leq n$, and $\alpha_r = \beta_r$ for all $r \neq i, j$. Note that

$$
\begin{aligned}
F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t) &= \mathbb{P}\left(\sum_{i=1}^{n} I_{p_i} X_{\alpha_i} \leq t\right) \\
&= \sum_{k=0}^{n} \sum_{\chi \in S_k} \mathbb{P}\left(\sum_{i=1}^{n} I_{p_i} X_{\alpha_i} \leq t \mid \boldsymbol{I} = \chi\right) p(\chi) \\
&= p(\mathbf{0}) + p(\mathbf{1}) \mathbb{P}\left(\sum_{i=1}^{n} X_{\alpha_i} \leq t\right) + \sum_{k=1}^{n} \sum_{\chi \in S_k} p(\chi) \mathbb{P}\left(\sum_{i=1}^{n} \chi_i X_{\alpha_i} \leq t\right) \\
&= p(\mathbf{0}) + p(\mathbf{1}) \mathbb{P}\left(\sum_{i=1}^{n} X_{\alpha_i} \leq t\right) + \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\chi) \mathbb{P}\left(\sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t\right) + \right. \\
&\quad \sum_{\chi \in S_k^{i,j}(0,1)} p(\chi) \mathbb{P}\left(X_{\alpha_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t\right) + \\
&\quad \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi)) \mathbb{P}\left(X_{\alpha_i} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t\right) + \\
&\quad \left. \sum_{\chi \in S_k^{i,j}(1,1)} p(\chi) \mathbb{P}\left(X_{\alpha_i} + X_{\alpha_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t\right) \right\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t) &= \mathbb{P}\left(\sum_{i=1}^{n} I_i X_{\beta_i} \leq t\right) \\
&= p(\mathbf{0}) + p(\mathbf{1}) \mathbb{P}\left(\sum_{i=1}^{n} X_{\beta_i} \leq t\right) + \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\chi) \mathbb{P}\left(\sum_{r \neq i,j}^{n} \chi_r X_{\beta_r} \leq t\right) + \right.
\end{aligned}
$$

$$\sum_{\chi\in S_k^{i,j}(0,1)} p(\chi)\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_r}\leq t\right)+$$
$$\sum_{\chi\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\chi))\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_r}\leq t\right)+$$
$$\left.\sum_{\chi\in S_k^{i,j}(1,1)} p(\chi)\mathbb{P}\left(X_{\beta_i}+X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_r}\leq t\right)\right\}.$$

Under assumption (*ii*), it holds that

$$\mathbb{P}\left(\sum_{i=1}^{n}X_{\alpha_i}\leq t\right)\geq\mathbb{P}\left(\sum_{i=1}^{n}X_{\beta_i}\leq t\right)\tag{6}$$

and, for any $\chi\in S_k^{i,j}(1,1), k=1,\ldots,n-1$,

$$\mathbb{P}\left(X_{\alpha_i}+X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_i}\leq t\right)\geq\mathbb{P}\left(X_{\beta_i}+X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_i}\leq t\right).\tag{7}$$

Then combining above two, we have

$$\begin{aligned}
&F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t)-F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t)\\
&=p(\mathbf{1})\left[\mathbb{P}\left(\sum_{i=1}^{n}X_{\alpha_i}\leq t\right)-\mathbb{P}\left(\sum_{i=1}^{n}X_{\beta_i}\leq t\right)\right]\\
&+\sum_{k=1}^{n-1}\left\{\sum_{\chi\in S_k^{i,j}(0,1)}p(\chi)\left[\mathbb{P}\left(X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)-\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)\right]\right.\\
&+\sum_{\chi\in S_k^{i,j}(0,1)}p(\tau_{i,j}\chi)\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)\right]\\
&\left.+\sum_{\chi\in S_k^{i,j}(1,1)}p(\chi)\left[\mathbb{P}\left(X_{\alpha_i}+X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_i}\leq t\right)\geq\mathbb{P}\left(X_{\beta_i}+X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_i}\leq t\right)\right]\right\}\\
&\geq\sum_{k=1}^{n-1}\left\{\sum_{\chi\in S_k^{i,j}(0,1)}p(\chi)\left[\mathbb{P}\left(X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)-\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)\right]\right.\\
&\left.+\sum_{\chi\in S_k^{i,j}(0,1)}p(\tau_{i,j}\chi)\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)\right]\right\}\\
&\geq\sum_{k=1}^{n-1}\left\{\sum_{\chi\in S_k^{i,j}(0,1)}p(\tau_{i,j}\chi)\left[\mathbb{P}\left(X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)-\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\leq t\right)\right]\right.
\end{aligned}$$

$$+\sum_{\chi\in S_k^{i,j}(0,1)} p(\tau_{i,j}\chi)\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\Bigg\}$$

$$=\sum_{k=1}^{n-1}\Bigg\{\sum_{\chi\in S_k^{i,j}(0,1)} p(\tau_{i,j}\chi)\int\cdots\int_{\mathbb{R}_+^{n-2}}\left[\mathbb{P}\left(X_{\alpha_i}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)+\mathbb{P}\left(X_{\alpha_j}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right.$$

$$\left.-\,\mathbb{P}\left(X_{\beta_i}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)-\mathbb{P}\left(X_{\beta_j}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right]\prod_{r\neq i,j}^{n} g_{X_{\alpha_r}}(x_{\alpha_r})\,\mathrm{d}x_{\alpha_r}\Bigg\}$$

$$=\sum_{k=1}^{n-1}\Bigg\{\sum_{\chi\in S_k^{i,j}(0,1)} p(\tau_{i,j}\chi)\int\cdots\int_{\mathbb{R}_+^{n-2}}\left[\bar{F}_{X_{\beta_i}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)+\bar{F}_{X_{\beta_j}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right.$$

$$\left.-\bar{F}_{X_{\alpha_i}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)-\bar{F}_{X_{\alpha_j}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right]\prod_{r\neq i,j}^{n} g_{X_{\alpha_r}}(x_{\alpha_r})\,\mathrm{d}x_{\alpha_r}\Bigg\}$$

$$\ge 0,$$

where $g_{X_{\alpha_r}}(x)$ is the density function of X_{α_r}. The first inequality follows from Eqs. (6) and (7), the second inequality from Lemma 3.7, and finally the last inequality is due to the fact that $\bar{F}_{X_\alpha}$ is concave in $\alpha\in\mathbb{R}_+$ as per assumption (i).

Remark 5.2. Theorem 5.1 hold true for the PO model, that is, for $X_{\alpha_i}\sim \mathrm{PO}(\bar{F},\alpha_i)$ $(X_{\beta_i}\sim PO(\bar{F},\beta_i))$, $i=1,\ldots,n$, with $\bar{F}(t)=\mathrm{e}^{-\lambda t}$, $\lambda>0$. This family of distribution is known as Marshall–Olkin extended exponential distribution. Note that $\bar{F}_{X_\alpha}(t)=\alpha\bar{F}(t)/1-\bar{\alpha}\bar{F}(t)$ is increasing and concave in α. The condition (ii), that is, the survival function of $X_{\mu_1}+X_{\mu_2}$ is Schur-concave in (μ_1,μ_2) follows from the Corrollary F.12.a. (p. 235) of [21] with the fact that both the survival function $\bar{F}_{X_\mu}(t)=\mu e^{-\lambda t}/1-\bar{\mu}e^{-\lambda t}$ and p.d.f. $f_{X_\mu}(t)=\mu\cdot\lambda e^{-\lambda t}/(1-\bar{\mu}e^{-\lambda t})^2$ are concave in μ.

It is to be noted that exponentiated Weibull distribution having the distribution function $F(t;\alpha,\beta))=\left(1-\mathrm{e}^{-t^\beta}\right)^\alpha$, $\alpha,\beta>0$, also satisfies both the conditions (i) and (ii) of Theorem 5.1 with respect to the parameter α.

Remark 5.3. It is worth to be mention that in Theorem 4.7 of Zhang et al. [30], they compared aggregate claim amounts of two sets of heterogeneous portfolios under the assumption that the survival function $\bar{F}(x;\alpha)$ is decreasing and convex in $\alpha>0$.

The following example illustrates the result given in Theorem 5.1.

Example 5.4. Suppose that $\{X_1,X_2\}$ and $\{Y_1,Y_2\}$ are two sets of independent nonnegative r.v.'s with $X_i\sim \mathrm{PO}(\bar{F}(x),\alpha_i), i=1,2$, and $Y_i\sim\mathrm{PO}(\bar{F}(x),\beta_i), i=1,2$, where $\bar{F}(x)=\mathrm{e}^{-0.3x}$, $x>0$. Set $(\alpha_1,\alpha_2)=(0.4,2.6)$, $(\beta_1,\beta_2)=(0.8,2.2)$, $p(0,0)=P(I_{p_1}=0,I_{p_2}=0)=0.15$, $p(0,1)=0.46$, $p(1,0)=0.34$, and $p(1,1)=0.05$. Then, $I=\{I_{p_1},I_{p_2}\}$ is LWSAI. We consider the transformation $x=t/(1-t)$ so that, for $t\in[0,1)$, we have $x\in[0,\infty)$. After this substitution, we denote the distribution functions of $\sum_{i=1}^2 I_{p_i}X_{\alpha_i}$ and $\sum_{i=1}^2 I_{p_i}X_{\beta_i}$ by $F_{A(I,\alpha)}$ and $F_{A(I,\beta)}$, respectively. $F_{A(I,\alpha)}(t/(1-t))=\psi_1(t)$ and $F_{A(I,\beta)}(t/(1-t))=\psi_2(t)$, respectively. From Figure 10, it is clear that $\psi_1(t)\ge\psi_2(t)$ for all $t\in[0,1)$. Hence, $\sum_{i=1}^2 I_{p_i}X_{\alpha_i}\le_{st}\sum_{i=1}^2 I_{p_i}X_{\beta_i}$.

□

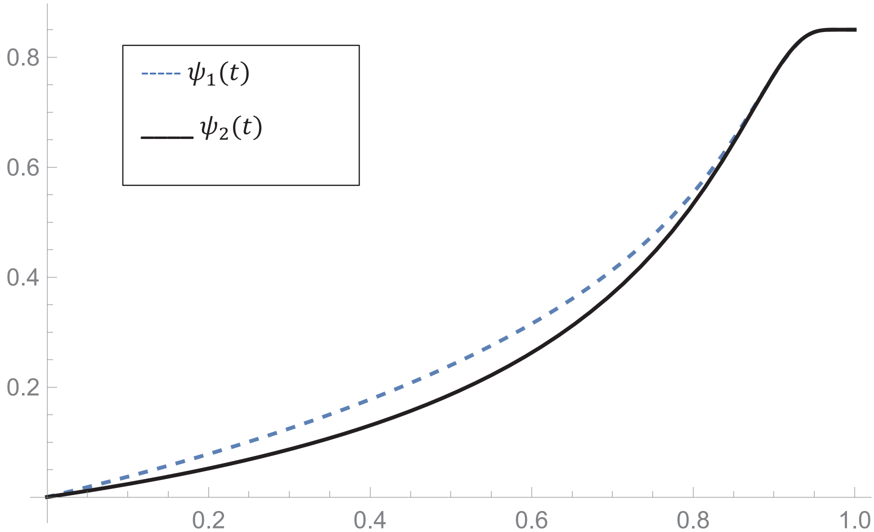

Figure 10. *Plots of $\psi_1(t)$ and $\psi_2(t)$, $t \in [0, 1]$.*

6. Concluding remarks

In this paper, we study some stochastic comparison results for the largest and the aggregate claim amounts under the setup of PO model. We derive the results with respect to the usual stochastic order, the reversed hazard rate order, and the star order. Further, some numerical examples are given to illustrate the derived results.

As discussed, the PO model is one of the important semi-parametric models. It has a large number of applications in different disciplines including financial engineering and actuarial science. It is worth mentioning that the results obtained by us apply to the whole family of extended distributions, for example, extended–exponential, Weibull, gamma, Pareto, Lomax, and linear failure-rate distribution, which have been studied by many researchers in different areas of applications (see [5] and references therein). Further, the largest claim and the aggregate claim amounts contain useful information in determining the key factors in a given insurance policy. For instance, the largest claim amount is referred to as the probable maximum loss, which helps to determine the amount of funds required to pay future claims on policies. Thus, the proposed study may be helpful to the actuaries in determining which portfolio of risks is better (in some stochastic sense) among a list of portfolios with respect to the largest and aggregate claim amounts. Apart from this, our theoretical results can be used to compare the lifetimes of two parallel systems whose components are subject to random shocks instantaneously.

Even though we discussed a lot of results, there are still many open problems that may be explored. Here the results are mostly developed for the usual stochastic order. Thus, the study of the same problem, as in here, for other stochastic orders (namely, likelihood ratio order, convex order, etc.) can be considered as a potential problem.

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