AN EXTREME COVERING OF 4-SPACE BY SPHERES

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1. Introduction

Let Λ be a lattice in n-dimensional Euclidean space E_n. For any lattice there is a unique minimal positive number μ such that if spheres of radius μ are placed at the points of the lattice then the entire space is covered, i.e. every point in E_n lies in at least one of the spheres. The *density* of this covering is defined to be $\theta_n(\Lambda) = J_n \mu^n / d(\Lambda)$, where J_n is the volume of an n-dimensional unit sphere and $d(\Lambda)$ is the determinant of the lattice.

The *density of the thinnest lattice covering* of E_n by spheres is the minimum of $\theta_n(\Lambda)$ over all lattices Λ and will be denoted by θ_n.

This minimum may be interpreted in terms of positive definite quadratic forms. If f is any positive definite n-ary quadratic form with determinant D then the *inhomogeneous minimum* of f, $m(f)$, is defined by

$$m(f) = \max_{\boldsymbol{\lambda} \in E_n} \min_{\boldsymbol{l}} f(\boldsymbol{l}+\boldsymbol{\lambda}),$$

where $\boldsymbol{l}$ ranges over all integral points.

If $\Lambda : \boldsymbol{\xi} = T\boldsymbol{x}$ ($\boldsymbol{x}$ integral) and $f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x}$, where $A = T'T$, i.e. $f(\boldsymbol{x}) = \boldsymbol{\xi}'\boldsymbol{\xi}$, then Λ and f are an *associated lattice and form* [4] and it can be shown that

$$\theta_n(\Lambda) = \frac{J_n\{m(f)\}^{\frac{1}{2}n}}{D^{\frac{1}{2}}}.$$

As $\{m(f)\}^{\frac{1}{2}n}/D^{\frac{1}{2}}$ is a function from the $\frac{1}{2}n(n+1)$-dimensional space of coefficients of f, it will be denoted by $\phi(\boldsymbol{a})$, $\boldsymbol{a} = (a_{11}, a_{22}, \cdots, a_{nn}, a_{12}, a_{13}, \cdots, a_{n-1}, n)$, where

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j \text{ and } a_{ij} = a_{ji}.$$

Thus Λ yields the minimum of $\theta_n(\Lambda)$ if and only if its associated form f yields a minimum of $\phi(\boldsymbol{a})$.

The form f is called *extreme* if $\phi(\boldsymbol{a})$ is a local minimum, i.e. sufficiently small variations in $\boldsymbol{a}$ do not decrease $\phi(\boldsymbol{a})$. This corresponds to a local

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minimum of $\theta_n(\Lambda)$ in the space of lattices. f is called ***absolutely extreme*** if $\phi(\boldsymbol{a})$ is an absolute minimum.

$\phi(\boldsymbol{a})$ is invariant under an equivalence transformation (integral unimodular transformation) and is unaltered by multiplying f by an arbitrary positive constant. Hence the property of being extreme is shared by the class of forms consisting of all forms equivalent to a multiple of some one form of the class.

Bleicher [3] proved that the class of forms represented by

$$n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j$$

is extreme for all values of n.

It has been proved [1], [2], [5] that this form is absolutely extreme for $n = 2, 3, 4$, giving

$$\theta_2 = \frac{2\pi}{3\sqrt{3}} \simeq 1\cdot 21, \quad \theta_3 = \frac{5\sqrt{5}}{24}\pi \simeq 1\cdot 46, \qquad \theta_4 = \frac{2}{5\sqrt{5}}\pi^2 \simeq 1\cdot 77.$$

Barnes [2] has shown that this is the only class of extreme forms for $n = 2, 3$. No other extreme forms have yet been found for any n.

It is the purpose of this paper to show that, when $n = 4$, $\phi(\boldsymbol{a})$ has a minimum at $\boldsymbol{a}_0 = (2, 2, 2, 2, \alpha, -1, -1, -1, -1, 1-\alpha)$, where $\alpha = \frac{1}{2}(5-\sqrt{13})$, i.e. the following quaternary form is extreme:

$$f_0(\boldsymbol{x}) = 2\sum_{i=1}^{4} x_i^2 - 2\sum_{\substack{i=1,2\\ j=3,4}} x_i x_j + 2\alpha x_1 x_2 + 2(1-\alpha)x_3 x_4. \tag{1}$$

2. Associated parallelohedra

The set of points of space which are at least as near to the origin as to any integral point $\boldsymbol{l}$ (with the metric defined by f) form a closed bounded convex parallelohedron Π. Π is thus the intersection of the half spaces $f(\boldsymbol{x}) \leqq f(\boldsymbol{x}-\boldsymbol{l})$, where $\boldsymbol{l}$ runs through all integral points.

In fact, only a finite number of these inequalities, i.e. only a finite number of integral points, is necessary to define Π.

Voronoï ([6], p. 277) established the following criterion to determine which points are required: *a point $\boldsymbol{l}$ $(\neq \boldsymbol{0})$ is necessary to define Π if and only if the minimum of $f(\boldsymbol{x})$ over $\boldsymbol{x} \equiv \boldsymbol{l}$ (mod 2) is attained only at $\boldsymbol{x} = \pm\boldsymbol{l}$.* This means that at most $2(2^n-1)$ inequalities are required.

The planes $f(\boldsymbol{x}) = f(\boldsymbol{x}\pm\boldsymbol{l})$, where $\boldsymbol{l}$ satisfies the required condition, thus form the faces of Π.

We can see from the definition of Π that

$$m(f) = \max_{x \in \Pi} f(x),$$

and as Π is convex,

$$m(f) = \max f(x) \text{ over all vertices of } \Pi.$$

Now Π is defined by $f(x) \leqq f(x \pm l)$,
i.e. $2|l'Ax| \leqq l'Al = f(l)$, where $A = (a_{ij})$,
i.e. $2|l'y| \leqq f(l)$, where $y = Ax$.

Also $f(x) = y'A^{-1}y = F(y)$ where F is the form inverse to f, so that $m(f) = \max F(y)$ over all vertices of Π.

For convenience in the work that follows, in place of $F(y)$ we shall use the form $F^*(y) = y'A^*y$ where $A^* = DA^{-1}$, i.e. A^* is the matrix of cofactors of A.

Thus

$$m(f) = \max \frac{F^*(y)}{D}$$

and

$$\vartheta(a) = \max \frac{\{F^*(y)\}^{\frac{1}{2}n}}{D^{\frac{1}{2}(n+1)}},$$

where the maxima are taken over all the vertices of Π.

3. The form f_0

Voronoï [7] showed that any quaternary positive definite quadratic form is equivalent to a form belonging to one of 3 domains in the 10 dimensional space of the coefficients of f. These domains may be specified by the set of integral l required to define Π.

Any form in the third Voronoï domain can be written as

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 \omega + \lambda_6 (x_1 - x_3)^2 + \lambda_7 (x_1 - x_4)^2$$
$$+ \lambda_8 (x_2 - x_3)^2 + \lambda_9 (x_2 - x_4)^2 + \lambda_{10} (x_1 + x_2 - x_3 - x_4)^2,$$

where $\lambda_i \geqq 0$, for all i, and

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 - 2x_1x_4 - 2x_2x_4.$$

For any form in the interior of this domain (i.e. $\lambda_i > 0$ for all i) the required set of integral l is

(1, 0, 0, 0) (0, 1, 0, 0) (0, 0, 1, 0) (0, 0, 0, 1) (1, −1, 0, 0)
(1, 0, 1, 0) (1, 0, 0, 1) (0, 1, 1, 0) (0, 1, 0, 1) (0, 0, 1, −1)
(1, 1, 1, 0) (1, 1, 0, 1) (1, 0, 1, 1) (0, 1, 1, 1) (1, 1, 1, 1)

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and Π has 30 faces and 120 vertices. (If f lies on the boundary of the domain, Π may have fewer faces and vertices).

The form (1) above can be written as

$$f_0(x) = (1-\alpha)[x_1^2+x_2^2+x_3^2+x_4^2+(x_1-x_3)^2+(x_1-x_4)^2 \\ +(x_2-x_3)^2+(x_2-x_4)^2+(x_1+x_2-x_3-x_4)^2]+(2\alpha-1)\omega.$$

Since $(1-\alpha) > 0$ and $(2\alpha-1) > 0$, $\boldsymbol{a}_0$ is an interior point of the third Voronoï domain.

There are sets of vertices for which $F^*(\boldsymbol{y})$ has the same value regardless of f. These are called *congruent* vertices. If $\boldsymbol{x}$ is a vertex defined by the 4 planes, $f(\boldsymbol{x}) = f(\boldsymbol{x}-\boldsymbol{l}_i)$ $(i = 1, \cdots, 4)$, say $\boldsymbol{x}$ corresponds to the set $(\boldsymbol{0}, \boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3, \boldsymbol{l}_4)$. Then a vertex is congruent to $\boldsymbol{x}$ if it corresponds to the set $\pm(-\boldsymbol{l}, \boldsymbol{l}_1-\boldsymbol{l}, \boldsymbol{l}_2-\boldsymbol{l}, \boldsymbol{l}_3-\boldsymbol{l}, \boldsymbol{l}_4-\boldsymbol{l})$ where $\boldsymbol{l} = \boldsymbol{l}_i$ for some i. Thus there are 12 different sets of congruent vertices so we must find the maximum of $F^*(\boldsymbol{y})$ over these 12 sets. Voronoï called these 12 sets of vertices I, II, III, etc. and we will refer to these throughout.

Let

$$\phi_X(\boldsymbol{a}) = \frac{\{F^*(\boldsymbol{y})\}^2}{D^{\frac{5}{4}}},$$

where $\boldsymbol{y}$ is a vertex of type X, so that

$$\phi(\boldsymbol{a}) = \max_X \phi_X(\boldsymbol{a}).$$

Voronoï also constructed a table ([7], p. 173) indicating which planes intersect at each vertex. The integral $\boldsymbol{l}$ are numbered in the order above and each of the planes $\boldsymbol{l}'\boldsymbol{y} = \pm\frac{1}{2}f(\boldsymbol{l})$ is referred to by the appropriate number (marked with a dash when the negative sign is required).

Now for the form being considered,

$$A = A_0 = \begin{bmatrix} 2 & \alpha & -1 & -1 \\ \alpha & 2 & -1 & -1 \\ -1 & -1 & 2 & 1-\alpha \\ -1 & -1 & 1-\alpha & 2 \end{bmatrix},$$

$$A^* = A_0^* = (2-\alpha)(1+\alpha) \begin{bmatrix} 2 & 1-\alpha & 1 & 1 \\ 1-\alpha & 2 & 1 & 1 \\ 1 & 1 & 2 & \alpha \\ 1 & 1 & \alpha & 2 \end{bmatrix},$$

and $D = D_0 = (2-\alpha)^2(1+\alpha)^2$.

The inequalities defining Π are then

1, 2, 3, 4	$\lvert y_i\rvert \leqq 1;\ i = 1, 2, 3, 4$
5	$\lvert y_1-y_2\rvert \leqq 2-\alpha$
6, 7, 8, 9	$\lvert y_i+y_j\rvert \leqq 1;\ i = 1, 2,\ j = 3, 4$
10	$\lvert y_3-y_4\rvert \leqq 1+\alpha$
11, 12	$\lvert y_1+y_2+y_j\rvert \leqq 1+\alpha;\ i = 3, 4$
13, 14	$\lvert y_i+y_3+y_4\rvert \leqq 2-\alpha;\ i = 1, 2$
15	$\lvert y_1+y_2+y_3+y_4\rvert \leqq 1.$

Making use of the Voronoï table [7 p. 173] we can calculate the value of $F^*(y)$ at all vertices.

For example:

Vertex of type I: defined by inequalities 3, 4, 13, 14 i.e. the vertex is the intersection of planes

$$y_3 = 1,\ y_4 = 1,\ y_1+y_3+y_4 = 2-\alpha,$$
$$y_2+y_3+y_4 = 2-\alpha.$$

These give

$$y_1 = -\alpha,\ y_2 = -\alpha,\ y_3 = 1,\ y_4 = 1 \text{ at vertex,}$$

whence [1]

$$F^*(y) = (2-\alpha)(1+\alpha)[-2\alpha^3+6\alpha^2-6\alpha+4]$$
$$= 4(2-\alpha)(1-\alpha)(1+\alpha)^2.$$

Continuing in this way, we find that for vertices of types I, II, VI:

$$F^*(y) = 4(2-\alpha)(1+\alpha)^2(1-\alpha)$$

and for vertices of other types

$$F^*(y) = 2(2-\alpha)(1+\alpha).$$

Hence

$$\max F^*(y) = 4(2-\alpha)(1+\alpha)^2(1-\alpha) = F_0^*$$
$$[\text{as } \alpha < \cdot 7,\ 4(1-\alpha^2) > 2],$$

thus

$$m(f_0) = \frac{4(1-\alpha)}{(2-\alpha)}$$

and so

$$\phi(\boldsymbol{a}_0) = \frac{16(1-\alpha)^2}{(2-\alpha)^3(1+\alpha)} \doteqdot \cdot 386.$$

[1] Here and in subsequent calculations we simplify by using the relation $\alpha^2-5\alpha+3 = 0$.

For the associated lattice Λ,

$$\theta(\Lambda) = \frac{\pi^2}{2}\,\phi(\boldsymbol{a}_0) \simeq 1{\cdot}93,$$

which is fairly close to the absolute minimum.

4. Proof of extremity

$\boldsymbol{a}_0$ yields a minimum of $\phi(\boldsymbol{a})$ if, for every sufficiently short vector

$$\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{44}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}, \varepsilon_{34}),$$

there is some X such that

$$\phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) \geqq \phi_{\mathrm{I}}(\boldsymbol{a}_0).$$

We show that in fact there is some X such that

$$\phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) > \phi_{\mathrm{I}}(\boldsymbol{a}_0), \tag{2}$$

unless $\boldsymbol{\varepsilon}$ is a multiple of $\boldsymbol{a}_0$. It is obvious that X can only be I, II or VI.

We establish (2) in the following steps:

Step (A). It will be shown that the sum of the *directional derivatives* of $\phi_{\mathrm{I}}(\boldsymbol{a})$, $\phi_{\mathrm{II}}(\boldsymbol{a})$ and $\phi_{\mathrm{VI}}(\boldsymbol{a})$ evaluated at $\boldsymbol{a}_0$ is zero for any given direction.

Step (B). It will be shown that the *sum of the second order terms* in the Taylor series in $\boldsymbol{\varepsilon}$ for $\phi_{\mathrm{I}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})+\phi_{\mathrm{II}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})+\phi_{\mathrm{VI}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})$ is a *positive semi-definite quadratic form*.

Step (C). It will be shown that the above semi-definite form can be zero only when $\boldsymbol{\varepsilon} = k\boldsymbol{a}_0$.

If $\boldsymbol{\varepsilon}$ is in such a direction that one of the directional derivatives of $\phi_{\mathrm{I}}(\boldsymbol{a})$, $\phi_{\mathrm{II}}(\boldsymbol{a})$ or $\phi_{\mathrm{VI}}(\boldsymbol{a})$ is positive then (2) is automatically satisfied. If none is positive and (A) is true then they must all be zero.

If this is the case and (B) and (C) are true, then, unless $\boldsymbol{\varepsilon} = k\boldsymbol{a}_0$ the sum of the second order terms must be positive for one of the three functions and hence (2) is still satisfied for all sufficiently small $\boldsymbol{\varepsilon}$.

If $\boldsymbol{\varepsilon} = k\boldsymbol{a}_0$, the quadratic form is a multiple of the given form and $\phi(\boldsymbol{a}_0)$ is unchanged.

Proof of step (A). It suffices to establish (A) for the 10 independent directions given by the unit vectors, i.e. for $\boldsymbol{\varepsilon} = \varepsilon\,(1, 0, \cdots, 0)$ etc. The calculations are similar in all cases (by the symmetry of the form in x_1, x_2 and in x_3, x_4 we need in fact only perform 5 calculations) and it will suffice to give the details only for $\boldsymbol{\varepsilon} = \varepsilon\,(1, 0, \cdots, 0)$.

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So for $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{\varepsilon}$;

$$A = \begin{bmatrix} 2+\varepsilon & \alpha & -1 & -1 \\ \alpha & 2 & -1 & -1 \\ -1 & -1 & 2 & 1-\alpha \\ -1 & -1 & 1-\alpha & 2 \end{bmatrix}$$

and

$$A^* = A_0^* + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1+\alpha)(3-\alpha)\varepsilon & (1+\alpha)\varepsilon & (1+\alpha)\varepsilon \\ 0 & (1+\alpha)\varepsilon & 3\varepsilon & (2\alpha-1)\varepsilon \\ 0 & (1+\alpha)\varepsilon & (2\alpha-1)\varepsilon & 3\varepsilon \end{bmatrix}.$$

The inequalities defining Π are now:

1	$\lvert y_1 \rvert \leqq 1+\frac{1}{2}\varepsilon$
2	$\lvert y_2 \rvert \leqq 1$
3	$\lvert y_3 \rvert \leqq 1$
4	$\lvert y_4 \rvert \leqq 1$
5	$\lvert y_1 - y_2 \rvert \leqq 2-\alpha+\frac{1}{2}\varepsilon$
6	$\lvert y_1 + y_3 \rvert \leqq 1+\frac{1}{2}\varepsilon$
7	$\lvert y_1 + y_4 \rvert \leqq 1+\frac{1}{2}\varepsilon$
8	$\lvert y_2 + y_3 \rvert \leqq 1$
9	$\lvert y_2 + y_4 \rvert \leqq 1$
10	$\lvert y_3 - y_4 \rvert \leqq 1+\alpha$
11	$\lvert y_1 + y_2 + y_3 \rvert \leqq 1+\alpha+\frac{1}{2}\varepsilon$
12	$\lvert y_1 + y_2 + y_4 \rvert \leqq 1+\alpha+\frac{1}{2}\varepsilon$
13	$\lvert y_1 + y_3 + y_4 \rvert \leqq 2-\alpha+\frac{1}{2}\varepsilon$
14	$\lvert y_2 + y_3 + y_4 \rvert \leqq 2-\alpha$
15	$\lvert y_1 + y_2 + y_3 + y_4 \rvert \leqq 1+\frac{1}{2}\varepsilon.$

Consider the vertex of type I defined by 3, 4, 13, 14.

$$y_1 = -\alpha+\tfrac{1}{2}\varepsilon,\ y_2 = -\alpha,\ y_3 = 1,\ y_4 = 1.$$

By examining the extra terms introduced by the ε, we obtain,

$$\begin{aligned} F^*(y) = {} & F_0^* + 2(2-\alpha)(1+\alpha)(-\alpha)\varepsilon + (1+\alpha)(3-\alpha)\alpha^2\varepsilon \\ & + 3\varepsilon + 3\varepsilon - \alpha(1-\alpha)(2-\alpha)(1+\alpha)\varepsilon \\ & + (2-\alpha)(1+\alpha)\varepsilon + (2-\alpha)(1+\alpha)\varepsilon \\ & - 4\alpha(1+\alpha)\varepsilon + 2(2\alpha-1)\varepsilon + O(\varepsilon^2) \\ = {} & F_0^* + 2(1+\alpha)(2-\alpha)(3\alpha-1)\varepsilon + O(\varepsilon^2). \end{aligned}$$

A similar calculation for a vertex of type II (1, 2, 11, 15) yields

$$F^*(y) = F_0^* + (2-\alpha)(1+\alpha)(2\alpha+1)\varepsilon + O(\varepsilon^2).$$

By the symmetry of the given form in the variables x_3 and x_4, $F^*(y)$ for a vertex of type VI (1, 2, 12, 15) is equal to the above value for type II.

Also $D = D_0 + 2(2-\alpha)(1+\alpha)\varepsilon$, and so

$$\phi_{\mathrm{I}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = \phi(\boldsymbol{a}_0)\,\frac{\left\{1+\dfrac{3\alpha-1}{2(1-\alpha)(1+\alpha)}\,\varepsilon+O(\varepsilon^2)\right\}^2}{\left\{1+\dfrac{2\varepsilon}{(2-\alpha)(1+\alpha)}\right\}^{\frac{5}{2}}}$$

and

$$\phi_{\mathrm{II}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = \phi_{\mathrm{VI}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = \phi(\boldsymbol{a}_0)\,\frac{\left\{1+\dfrac{2\alpha+1}{4(1-\alpha)(1+\alpha)}\,\varepsilon+O(\varepsilon^2)\right\}^2}{\left\{1+\dfrac{2\varepsilon}{(2-\alpha)(1+\alpha)}\right\}^{\frac{5}{2}}}.$$

Thus

$$\begin{aligned}
&\phi_{\mathrm{I}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})+\phi_{\mathrm{II}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})+\phi_{\mathrm{VI}}(\boldsymbol{a}_0+\boldsymbol{\varepsilon})\\
&\quad= 3\phi(\boldsymbol{a}_0)\,\frac{\left\{1+\dfrac{5\alpha}{3(1-\alpha)(1+\alpha)}\,\varepsilon+O(\varepsilon^2)\right\}}{\left\{1+\dfrac{2\varepsilon}{(2-\alpha)(1+\alpha)}\right\}^{\frac{5}{2}}}\\
&\quad= 3\phi(\boldsymbol{a}_0)\left\{1+\left[\frac{5\alpha}{3(1-\alpha)(1+\alpha)}-\frac{5}{(2-\alpha)(1+\alpha)}\right]\varepsilon+O(\varepsilon^2)\right\}\\
&\quad= 3\phi(\boldsymbol{a}_0)\left\{1-\frac{5\alpha^2-25\alpha+15}{3(2-\alpha)(1+\alpha)(1-\alpha)}\,\varepsilon+O(\varepsilon^2)\right\}\\
&\quad= 3\phi(\boldsymbol{a}_0)\{1+O(\varepsilon^2)\},
\end{aligned}$$

since $5\alpha^2-25\alpha+15 = 5(\alpha^2-5\alpha+3) = 0$.

Thus (A) is proven for $\boldsymbol{\varepsilon} = \varepsilon(1, 0, 0, \cdots, 0)$.

Proof of step (B).

The coefficients of ε_{11}^2, ε_{22}^2, $\varepsilon_{11}\varepsilon_{22}$, etc. will be calculated separately. The terms are of two types

(i) *Terms of the form* $K\varepsilon_{ij}^2$.

There are 10 terms of this form for which the calculations are similar (by symmetry we need in fact perform only 5 calculations).

Let $\boldsymbol{\varepsilon} = \varepsilon\,(0, 0, \cdots, 0, 1, 0, \cdots, 0)$. Then for $\boldsymbol{a} = \boldsymbol{a}_0+\boldsymbol{\varepsilon}$, let $F^*(y)$ be equal to

(a) $F_0^*\{1+b\varepsilon+b'\varepsilon^2+O(\varepsilon^3)\}$ at a vertex of type I,

(b) $F_0^*\{1+c\varepsilon+c'\varepsilon^2+O(\varepsilon^3)\}$ at a vertex of type II,

(c) $F_0^*\{1+d\varepsilon+d'\varepsilon^2+O(\varepsilon^3)\}$ at a vertex of type VI;

and let $D = D_0\{1+f\varepsilon+f'\varepsilon^2\}$.

Then

$$\sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)\{1+[\tfrac{2}{3}(b+c+d)-\tfrac{5}{2}f]\varepsilon + [\tfrac{2}{3}(b'+c'+d')+\tfrac{1}{3}(b^2+c^2+d^2) -\tfrac{5}{2}f'+\tfrac{35}{8}f^2-\tfrac{5}{3}f(b+c+d)]\varepsilon^2+O(\varepsilon^3)\}.$$

But from (A) we have $\frac{2}{3}(b+c+d) = \frac{5}{2}f$, so that

$$\sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)\{1+[\tfrac{2}{3}(b'+c'+d') +\tfrac{1}{3}(b^2+c^2+d^2)-\tfrac{5}{2}f'-\tfrac{15}{8}f^2]\varepsilon^2+O(\varepsilon)\}. \tag{3}$$

By varying $\boldsymbol{\varepsilon}$ above we can obtain the term in ε_{ij}^2 for all i, j. It will suffice to give details only for the term in ε_{11}^2.

If $\boldsymbol{\varepsilon} = \varepsilon\,(1, 0, \cdots, 0)$, then from above calculations we have:

$$f = \frac{2}{(2-\alpha)(1+\alpha)}, \quad f' = 0, \quad b = \frac{3\alpha-1}{2(1-\alpha)(1+\alpha)},$$
$$c = d = \frac{2\alpha+1}{4(1-\alpha)(1+\alpha)}.$$

A calculation of the ε^2 terms of $F^*(\boldsymbol{y})$ for the 3 vertices gives

$$b' = c' = d' = \frac{1}{8(1-\alpha)(1+\alpha)}.$$

Substitution of these quantities in (3) gives, on simplification

$$\sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)\left[1+\frac{7(3-\alpha)}{24(1+\alpha)^2}\,\varepsilon+O(\varepsilon^3)\right].$$

Thus the coefficient of ε_{11}^2 and, by the symmetry of the form, also ε_{22}^2, in the required Taylor Series is

$$\phi(\boldsymbol{a}_0)\,\frac{7(3-\alpha)}{8(1+\alpha)^2}.$$

(ii) *Terms of the form* $K\varepsilon_{ij}\varepsilon_{kl}$, $ij \neq kl$.

There are 45 terms of this form for which the calculations are similar (by symmetry we need, in fact, perform only 17 calculations).

Let $\boldsymbol{\varepsilon} = (0, \cdots 0,\ \varepsilon, 0, \cdots 0, \eta, 0, \cdots, 0)$. Then, for $\boldsymbol{a} = \boldsymbol{a}_0+\boldsymbol{\varepsilon}$ let $F^*(\boldsymbol{y})$ be equal to

(a) $F_0^*\{1+b\varepsilon+b'\eta+b''\varepsilon\eta\}$ at a vertex of type I,

(b) $F_0^*\{1+c\varepsilon+c'\eta+c''\varepsilon\eta\}$ at a vertex of type II,

(c) $F_0^*\{1+d\varepsilon+d'\eta+d''\varepsilon\eta\}$ at a vertex of type VI;

and let $D = D_0\{1+f\varepsilon+f'\eta+f''\varepsilon\eta\}$, neglecting in each case terms involving $\varepsilon^r\eta^s$ where $\max(r, s) \geqq 2$.

Then

$$\sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)[1+\{\tfrac{2}{3}(b+c+d)-\tfrac{5}{2}f\}\varepsilon \\ +\{\tfrac{2}{3}(b'+c'+d')-\tfrac{5}{2}f'\}\eta+\{\tfrac{2}{3}(bb'+cc'+dd') \\ +\tfrac{2}{3}(b''+c''+d'')-\tfrac{5}{2}f''+\tfrac{35}{4}ff'-\tfrac{5}{3}f(b'+c'+d') \\ -\tfrac{5}{3}f'(b+c+d)\}\varepsilon\eta]+\cdots.$$

But from (A) we have: $\frac{2}{3}(b+c+d) = \frac{5}{2}f$ and $\frac{2}{3}(b'+c'+d') = \frac{5}{2}f'$, so that

$$(4) \qquad \sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)[1+\{\tfrac{2}{3}(bb'+cc'+dd') \\ +\tfrac{2}{3}(b''+c''+d'')-\tfrac{5}{2}f''-\tfrac{15}{4}ff'\}\varepsilon\eta]+\cdots.$$

By varying $\boldsymbol{\varepsilon}$ we can obtain the term in $\varepsilon_{ij}\varepsilon_{kl}$ for all i, j, k, l. It will suffice to give details only for the terms in $\varepsilon_{11}\varepsilon_{22}$.

If $\boldsymbol{\varepsilon} = (\varepsilon, \eta, 0, \cdots 0)$ then from the earlier calculations given in the proof of (A) and from the symmetry of the form in x_1 and x_2 we already have

$$b = b' = \frac{3\alpha-1}{2(1-\alpha^2)}; \quad c = c' = d = d' = \frac{2\alpha+1}{4(1-\alpha^2)}.$$

Now

$$A = \begin{bmatrix} 2+\varepsilon & \alpha & -1 & -1 \\ \alpha & 2+\eta & -1 & -1 \\ -1 & -1 & 2 & 1-\alpha \\ -1 & -1 & 1-\alpha & 2 \end{bmatrix}$$

and $A^* = A_0^* +$

$$\begin{bmatrix} (3-\alpha)(1+\alpha)\eta & 0 & (1+\alpha)\eta & (1+\alpha)\eta \\ 0 & (3-\alpha)(1+\alpha)\varepsilon & (1+\alpha)\varepsilon & (1+\alpha)\varepsilon \\ (1+\alpha)\eta & (1+\alpha)\varepsilon & 3\varepsilon+3\eta+2\varepsilon\eta & (2\alpha-1)(\varepsilon+\eta) \\ & & & -(1-\alpha)\varepsilon\eta \\ (1+\alpha)\eta & (1+\alpha)\varepsilon & (2\alpha-1)(\varepsilon+\eta) & 3\varepsilon+3\eta+2\varepsilon\eta \\ & & -(1-\alpha)\varepsilon\eta & \end{bmatrix}.$$

The inequalities defining II are now

1 $\quad |y_1| \leqq 1+\frac{1}{2}\varepsilon$

2 $\quad |y_2| \leqq 1+\frac{1}{2}\eta$

3 $\quad |y_3| \leqq 1$

4 $|y_4| \leqq 1$

5 $|y_1-y_2| \leqq 2-\alpha+\frac{1}{2}(\varepsilon+\eta)$

6 $|y_1+y_3| \leqq 1+\frac{1}{2}\varepsilon$

7 $|y_1+y_4| \leqq 1+\frac{1}{2}\varepsilon$

8 $|y_2+y_3| \leqq 1+\frac{1}{2}\eta$

9 $|y_2+y_4| \leqq 1+\frac{1}{2}\eta$

10 $|y_3-y_4| \leqq 1+\alpha$

11 $|y_1+y_2+y_3| \leqq 1+\alpha+\frac{1}{2}(\varepsilon+\eta)$

12 $|y_1+y_2+y_4| \leqq 1+\alpha+\frac{1}{2}(\varepsilon+\eta)$

13 $|y_1+y_3+y_4| \leqq 2-\alpha+\frac{1}{2}\varepsilon$

14 $|y_2+y_3+y_4| \leqq 2-\alpha+\frac{1}{2}\eta$

15 $|y_1+y_2+y_3+y_4| \leqq 1+\frac{1}{2}\varepsilon+\frac{1}{2}\eta$

Consider a vertex of type I (3, 4, 13, 14) where

$$y_1 = -\alpha+\tfrac{1}{2}\varepsilon, \quad y_2 = -\alpha+\tfrac{1}{2}\eta, \quad y_3 = 1, \quad y_4 = 1.$$

Then

$$\begin{aligned} b'' &= \frac{1}{F_0^*}\Big[-2\alpha(3-\alpha)(1+\alpha)+4+\tfrac{1}{2}(1-\alpha)(2-\alpha)(1+\alpha) \\ &\qquad\qquad +2(1+\alpha)+2(1+\alpha)-2(1-\alpha)\Big] \\ &= \frac{26-9\alpha}{8(2-\alpha)(1+\alpha)}, \end{aligned}$$

by choice only of terms involving $\varepsilon\eta$ in the calculation of $F^*(\boldsymbol{y})$.

Similarly

$$c'' = d'' = \frac{26-5\alpha}{8(2-\alpha)(1+\alpha)}.$$

A simple calculation gives

$$f = f' = \frac{2}{(2-\alpha)(1+\alpha)}, \quad f'' = \frac{3-\alpha}{(1+\alpha)^2}.$$

Substituting in (4) gives

$$\sum_{X=\mathrm{I,II,VI}} \phi_X(\boldsymbol{a}_0+\boldsymbol{\varepsilon}) = 3\phi(\boldsymbol{a}_0)\left[1+\frac{7\alpha-6}{6(1+\alpha)^2}\,\varepsilon\eta\right]$$

plus terms involving $\varepsilon^r\eta^s$ where $\max\,(r, s) \geqq 2$.

Thus the coefficient of $\varepsilon_{11}\varepsilon_{22}$ in the Taylor's series is

$$\phi(\boldsymbol{a}_0)\cdot\frac{7\alpha-6}{2(1+\alpha)^2}.$$

$$M = \begin{bmatrix}
14(3-\alpha) & 4(7\alpha-6) & -3\alpha-1 & -3\alpha-1 & 4 & 14(3-\alpha) & 14(3-\alpha) & 2(3\alpha-5) & 2(3\alpha-5) & 4(\alpha+5) \\
4(7\alpha-6) & 14(3-\alpha) & -3\alpha-1 & -3\alpha-1 & 4 & 2(3\alpha-5) & 2(3\alpha-5) & 14(3-\alpha) & 14(3-\alpha) & 4(\alpha+5) \\
-3\alpha-1 & -3\alpha-1 & 14(3-\alpha) & 19\alpha-15 & 2(11-9\alpha) & 14(3-\alpha) & 4(2-3\alpha) & 14(3-\alpha) & 4(2-3\alpha) & 2(19-7\alpha) \\
-3\alpha-1 & -3\alpha-1 & 19\alpha-15 & 14(3-\alpha) & 2(11-9\alpha) & 4(2-3\alpha) & 14(3-\alpha) & 4(2-3\alpha) & 14(3-\alpha) & 2(19-7\alpha) \\
4 & 4 & 2(11-9\alpha) & 2(11-9\alpha) & 28(7-\alpha) & 4(14-5\alpha) & 4(14-5\alpha) & 4(14-5\alpha) & 4(14-5\alpha) & 4(7\alpha-12) \\
14(3-\alpha) & 2(3\alpha-5) & 14(3-\alpha) & 4(2-3\alpha) & 4(14-5\alpha) & 4(32-9\alpha) & 4(15-8\alpha) & 4(6+\alpha) & 4(9-14\alpha) & 8(9-2\alpha) \\
14(3-\alpha) & 2(3\alpha-5) & 4(2-3\alpha) & 14(3-\alpha) & 4(14-5\alpha) & 4(15-8\alpha) & 4(32-9\alpha) & 4(9-14\alpha) & 4(6+\alpha) & 8(9-2\alpha) \\
2(3\alpha-5) & 14(3-\alpha) & 14(3-\alpha) & 4(2-3\alpha) & 4(14-5\alpha) & 4(6+\alpha) & 4(9-14\alpha) & 4(32-9\alpha) & 4(15-8\alpha) & 8(9-2\alpha) \\
2(3\alpha-5) & 14(3-\alpha) & 4(2-3\alpha) & 14(3-\alpha) & 4(14-5\alpha) & 4(9-14\alpha) & 4(6+\alpha) & 4(15-8\alpha) & 4(32-9\alpha) & 8(9-2\alpha) \\
4(\alpha+5) & 4(\alpha+5) & 2(19-7\alpha) & 2(19-7\alpha) & 4(7\alpha-12) & 8(9-2\alpha) & 8(9-2\alpha) & 8(9-2\alpha) & 8(9-2\alpha) & 4(53-6\alpha)
\end{bmatrix}$$

$$M^* = \begin{bmatrix}
14(3-\alpha) & 4(7\alpha-6) & -3\alpha-1 & -3\alpha-1 & 40-68\alpha & -3\alpha-1 & -3\alpha-1 & 19\alpha-15 & 19\alpha-15 & 4(7\alpha-6) \\
4(7\alpha-6) & 14(3-\alpha) & -3\alpha-1 & -3\alpha-1 & 40-68\alpha & 19\alpha-15 & 19\alpha-15 & -3\alpha-1 & -3\alpha-1 & 4(7\alpha-6) \\
-3\alpha-1 & -3\alpha-1 & 14(3-\alpha) & 19\alpha-15 & 40-68\alpha & -3\alpha-1 & 4(7\alpha-6) & -3\alpha-1 & 4(7\alpha-6) & 19\alpha-15 \\
-3\alpha-1 & -3\alpha-1 & 19\alpha-15 & 14(3-\alpha) & 40-68\alpha & 4(7\alpha-6) & -3\alpha-1 & 4(7\alpha-6) & -3\alpha-1 & 19\alpha-15 \\
40-68\alpha & 40-68\alpha & 40-68\alpha & 40-68\alpha & 9(68\alpha-40) & 40-68\alpha & 40-68\alpha & 40-68\alpha & 40-68\alpha & 40-68\alpha \\
-3\alpha-1 & 19\alpha-15 & -3\alpha-1 & 4(7\alpha-6) & 40-68\alpha & 14(3-\alpha) & 19\alpha-15 & 4(7\alpha-6) & -3\alpha-1 & -3\alpha-1 \\
-3\alpha-1 & 19\alpha-15 & 4(7\alpha-6) & -3\alpha-1 & 40-68\alpha & 19\alpha-15 & 14(3-\alpha) & -3\alpha-1 & 4(7\alpha-6) & -3\alpha-1 \\
19\alpha-15 & -3\alpha-1 & -3\alpha-1 & 4(7\alpha-6) & 40-68\alpha & 4(7\alpha-6) & -3\alpha-1 & 14(3-\alpha) & 19\alpha-15 & -3\alpha-1 \\
19\alpha-15 & -3\alpha-1 & 4(7\alpha-6) & -3\alpha-1 & 40-68\alpha & -3\alpha-1 & 4(7\alpha-6) & 19\alpha-15 & 14(3-\alpha) & -3\alpha-1 \\
4(7\alpha-6) & 4(7\alpha-6) & 19\alpha-15 & 19\alpha-15 & 40-68\alpha & -3\alpha-1 & -3\alpha-1 & -3\alpha-1 & -3\alpha-1 & 14(3-\alpha)
\end{bmatrix}$$

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On completing these calculations the required sum of the second order terms for any ε is a multiple of

$$h(\varepsilon) = \varepsilon' M \varepsilon,$$

where M is the upper matrix on page 12.

Note that $M\varepsilon = 0$ if

$$\begin{aligned} \varepsilon &= K(2, 2, 2, 2, \alpha, -1, -1, -1, -1, 1-\alpha) \\ &= K\boldsymbol{a}_0. \end{aligned} \tag{5}$$

Let $M^* = T'MT$ where $T =$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 2(2-\alpha) & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 2(2-\alpha) & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 2(2-\alpha) & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2(2-\alpha) & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & (2-\alpha) & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -(2-\alpha) & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -(2-\alpha) & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -(2-\alpha) & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -(2-\alpha) & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Then M^* is the lower matrix on page 12.

Now the sum of the elements in each row of M^* is zero. Also all elements are negative except those on the diagonal.

Hence

$$h^*(\boldsymbol{x}) = \boldsymbol{x}' M^* \boldsymbol{x} = \sum_{i<j} \rho_{ij}(x_i - x_j)^2; \quad \rho_{ij} > 0 \text{ for all } i, j.$$

Thus $h^*(\boldsymbol{x})$ is a positive semi-definite form and, as M^* is congruent to M, $h(\varepsilon)$ is also positive semi-definite. Thus (B) is proved.

Proof of step (C).

As h^* obviously has only one independent zero then h also has only one. We have already shown (5) that this occurs when $\varepsilon = k\boldsymbol{a}_0$.

5. Discussion

Since no criteria are known for extremeness, the above verification is necessarily direct and cumbersome. Some such criteria must be found for work on this problem to proceed much further.

The presence of quadratic irrationals in the coefficients of the above form shows that any criterion for extremeness cannot be linear, as it is for the dual problem of lattice packing of spheres in n-space.

In 4 dimensions we now have two extreme forms, one in each of the first and third Voronoï domains. It is possible that there is also an extreme form in the second domain, similar to the above.

Bleicher ([3], p. 649) made the conjecture that any extreme form probably gives rise to a primitive parallelohedron with all its vertices lying on a sphere, i.e. such that $\phi_X(\boldsymbol{a})$ has the same value for all X. However, the above form shows this conjecture to be false since we now have an extreme form with $\phi_X(\boldsymbol{a})$ greater for 3 types of vertices than it is for the other 9. The author has verified that there are in fact no forms in the third Voronoï domain which give rise to a primitive parallelohedron with $\phi_X(\boldsymbol{a})$ constant for all vertices.

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