21
The Maxwell radiation field

There are two ways of describing the interaction between matter and the electromagnetic field: the first and most fundamental way is to consider the electromagnetic field to be coupled to every individual microscopic charge in a physical system explicitly. Apart from these charges, the field lives on a background vacuum. The charges are represented by an $(n + 1)$ dimensional current vector J_μ.

In systems with very complex distributions of charge, this approach is too cumbersome, and an alternative view is useful: that of an electromagnetic field in dielectric media. This approach is an effective-field-theory approach in which the average effect of a very complex, on average neutral, distribution of charges is taken into account by introducing an effective speed of light, or equivalently effective permittivities and permeabilities. Any remainder charges which make the system non-neutral can then be handled explicitly by an $(n+1)$ dimensional current vector. Although the second of these two approaches is a popular simplification in many cases, it has only a limited range of validity, whereas the first approach is fundamental. We shall consider these two cases separately.

21.1 Charges in a vacuum

21.1.1 The action

The action for the electromagnetic field in a vacuum is given by

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right\}, \tag{21.1}$$

where the anti-symmetric field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{21.2}$$

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and

$$A^\mu = \begin{pmatrix} c^{-1}\phi \\ \mathbf{A} \end{pmatrix}. \tag{21.3}$$

In $3+1$ dimensions, the field components are given by

$$\begin{aligned} E_i = -\partial_i \phi - \partial_t A_i = c\ F_{i0} \\ \epsilon_{ijk} B_k = F_{ij}, \end{aligned} \tag{21.4}$$

where $i = 1, 2, 3$. The latter equation may be inverted to give

$$B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}. \tag{21.5}$$

Note that the indices on the electric and magnetic field *vectors* in $3+1$ dimensions are always written as subscripts, never as superscripts. In $2+1$ dimensions, the magnetic field is a pseudo-scalar, and one has

$$\begin{aligned} E_i = -\partial_i \phi - \partial_t A_i = c\ F_{i0} \\ B = F_{12}, \end{aligned} \tag{21.6}$$

where $i = 1, 2$. In higher dimensions, the tensor character of $F_{\mu\nu}$ is unavoidable, and E and B cease to lose their separate identities. A further important point is that the derivatives in the action are purely classical – there are no factors of $\hbar$ present here.

A phenomenological source can be added to the Maxwell action, in an ambient vacuum:

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \right\}. \tag{21.7}$$

This describes an electromagnetic field, extended in a vacuum around source charges. What we really mean here is that there is no ambient background matter present: we allow positive and negative charges to exist freely in a vacuum, but there is no overall neutral, polarizable matter present. The case of polarization in the ambient medium is dealt with later.

21.1.2 Field equations and continuity

The variation of the action leads to

$$\begin{aligned} \delta S &= \int (\mathrm{d}x) \left\{ (\partial^\mu \delta A^\nu) F_{\mu\nu} - J^\mu \delta A_\mu \right\} \\ &= \int (\mathrm{d}x) \left\{ \delta A^\nu (-\partial^\mu F_{\mu\nu}) - J^\mu \delta A_\mu \right\} \\ &+ \int \mathrm{d}\sigma^\mu \left\{ \delta A^\nu F_{\mu\nu} \right\}. \end{aligned} \tag{21.8}$$

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Thus, the field equations $\delta S = 0$ are given by

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \tag{21.9}$$

and the continuity condition tells us that the conjugate momentum ($d\sigma^\mu = d\sigma^0$) is

$$\Pi_i = F_{0i}. \tag{21.10}$$

If the surface σ is taken to separate two regions of space rather than time, one has the standard continuity conditions for the electromagnetic field in a vacuum:

$$\begin{aligned} \Delta F_{i0} &= 0 \\ \Delta F_{ij} &= 0, \end{aligned} \tag{21.11}$$

and we have assumed that δA_μ is a continuous function. The momentum conjugate to the field A_μ is

$$\Pi_\mu^\sigma = \frac{\delta \mathcal{L}}{\delta(\partial^\sigma A^\mu)} = F_{\sigma\mu}, \tag{21.12}$$

where σ points outward from a spacelike hyper-surface. The canonical choice for this momentum is $\sigma = 0$, where one has

$$\Pi_\mu = \frac{\delta \mathcal{L}}{\delta(\partial^0 A^\mu)} = F_{0i}, \tag{21.13}$$

which means that μ can only take values $i = 1, \ldots, n$ in n spatial dimensions, owing to the anti-symmetry of $F_{\mu\nu}$.

The velocity analogous to $\dot{q}$ is given by the derivative of the field $\partial_\sigma A^\mu = \partial_0 A^\mu$. Thus, the canonical definition of the Hamiltonian is

$$\mathcal{H} = F_{0\mu}(\partial_0 A^\mu) - g_{\mu\nu} \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{21.14}$$

However, this expression is not gauge-invariant, whereas the Hamiltonian must be. The problem lies in the naive interpretation of the Legendre transform. The problem may be cured by defining the Hamiltonian in terms of the variation of the action:

$$H = -\frac{\delta S}{\delta t}. \tag{21.15}$$

This is a special case (the zero–zero component θ_{00}) of the energy–momentum tensor, which is discussed below in more general terms. The result for the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\left(\epsilon_0 E_i E_i + \mu_0^{-1} B_i B_i\right) \tag{21.16}$$

where $i = 1, \ldots, n$.

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21.1.3 The Jacobi–Bianchi identity

The Bianchi identity in $n+1$ dimensions provides two of Maxwell's equations. The equations implied by this identity are different in each new number of dimensions. In $3+1$ dimensions, we have

$$\epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0. \tag{21.17}$$

Separating out the space and time components of μ, we obtain, for $\mu = 0$,

$$\epsilon^{ijk}\partial_i F_{jk} = \partial^i B_i = \text{div}\mathbf{B} = 0. \tag{21.18}$$

For $\mu = i$, i.e. the spatial components, we have

$$\epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ik0j}\partial_k F_{0j} + \epsilon^{ijk0}\partial_j F_{k0} = 0, \tag{21.19}$$

which may be re-written as

$$2\frac{1}{c}\partial_t B_i - \epsilon^{ijk}\partial_k F_{0j} + \epsilon^{ijk}\partial_j F_{k0} = 0. \tag{21.20}$$

Thus, using the definition of the electric field in eqn. (21.12), together with the anti-symmetry of $F_{\mu\nu}$, we obtain

$$(\text{curl}\,\mathbf{E})_i = -\frac{\partial B_i}{\partial t}, \tag{21.21}$$

which completes the proof.

In $2+1$ dimensions, eqn. (21.18) is absent, since the Bianchi identity now has the form

$$\epsilon^{\mu\nu\lambda}\partial_\mu F_{\nu\lambda} = 0. \tag{21.22}$$

The full expansion of this equation is

$$\epsilon^{0jk}\partial_0 F_{jk} + \epsilon^{k0j}\partial_k F_{0j} + \epsilon^{jk0}\partial_j F_{k0} = 0, \tag{21.23}$$

which can be written as

$$\epsilon^{jk}\partial_j E_k = -\frac{\partial B}{\partial t}. \tag{21.24}$$

Note that, in $2+1$ dimensions, the B field is a pseudo-scalar.

21.1.4 Formal solution by Green functions

The formal solution to the equations of motion is most conveniently expressed in terms of the vector potential. Re-writing the field equation (21.9) in terms of the vector potential, we have

$$-\Box\; A_\nu + \partial^\mu\partial_\nu A_\mu = \mu_0 J_\nu \tag{21.25}$$

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or

$$(-\Box\, \delta^{\mu}_{\ \nu} + \partial^{\mu}\partial_{\nu})A_{\mu} = \mu_0 J_{\nu}. \tag{21.26}$$

The formal solution therefore requires the inverse of the operator on the left hand side of this equation. This presents problem, though: the determinant of this matrix-valued operator vanishes! This is easily seen by separating space and time components as a 2×2 matrix,

$$\begin{vmatrix} -\Box\ + \partial_0\partial^0 & \partial_0\partial^i \\ \partial_i\partial^0 & -\Box\ + \partial_i\partial^i \end{vmatrix} = 0. \tag{21.27}$$

The problem here is related to the gauge symmetry, or non-uniqueness, of A_{μ} and can be fixed by choosing a gauge for the potential. The choice of gauge is arbitrary, but *two* conditions are required in general to fix the gauge freedom fully (see chapter 9), and ensure a one-to-one correspondence between the potentials and the physical fields.

21.1.5 Lorentz gauge

To solve the inverse problem in the 'Lorentz gauge', it is sufficient to take

$$\partial^{\mu} A_{\mu} = 0. \tag{21.28}$$

This is only a single condition, so it does not fix the gauge completely, but it is sufficient for our purposes. Using this condition directly in eqn. (21.26) we get the modified field equation,

$$-\Box\ A_{\mu} = \mu_0 J_{\mu}. \tag{21.29}$$

This equation now presents the appearance of a massless Klein–Gordon field. The formal inverse of the differential operator is therefore the scalar Green function $G(x, x')$, for $m = 0$, giving the solution

$$A_{\mu} = \mu_0 \int (\mathrm{d}x') G(x, x') J_{\mu}(x'). \tag{21.30}$$

Another way of imposing this condition, which is frequently used in the literature, is to add a Lagrange multiplier term to the action:

$$S' = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu} + \frac{1}{2\alpha} \mu_0^{-1} (\partial^{\mu} A_{\mu})^2 \right\}, \tag{21.31}$$

where α^{-1} is the Lagrange multiplier. The field equations and continuity conditions resulting from the variation of the action are now

$$\left[-\Box\, \delta^{\mu}_{\ \nu} + \left(1 - \frac{1}{\alpha}\right) \partial^{\mu}\partial_{\nu} \right] A_{\mu} = \mu_0 J_{\nu} \tag{21.32}$$

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and

$$\Delta(F_{\sigma\nu} + \partial_\sigma A_\nu) = 0. \tag{21.33}$$

The inverse of the differential operator in eqn. (21.32) is found by solving the equation

$$\left[-\Box\, \delta^{\mu}_{\ \nu} + \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial_\nu \right] D_\mu^{\ \lambda}(x, x') = \delta_\nu^{\ \lambda} \delta(x, x'), \tag{21.34}$$

which gives a formal solution for the potential

$$A_\mu(x) = \mu_0 \int (\mathrm{d}x') D_{\mu\nu}(x, x') J^\nu(x'). \tag{21.35}$$

21.1.6 Coulomb/radiation gauge

The Coulomb gauge is based on the condition

$$\partial^i A_i = 0. \tag{21.36}$$

Again, this is only a single condition, and it is usually supplemented by the condition $A_0 = 0$, or by the use of the zeroth-component field equation to eliminate A_0 entirely. Using eqn. (21.36) in eqn. (21.26), we separate the space and time parts of the field equations (a step backwards from covariance):

$$(-\Box\ \ A_0 + \partial_0(\partial^0 A_0)) = \mu_0\ J_0 \tag{21.37}$$

$$(-(\partial_0\partial^0 + \nabla^2)A_i + \partial_i\partial^0 A_0 + \partial_i(\partial^0 A_0)) = \mu_0 J_i, \tag{21.38}$$

or, simplifying,

$$-\nabla^2 A_0 = \mu_0 J_0 \tag{21.39}$$

$$-\Box\ A_i + \partial_i(\partial^0 A_0) = \mu_0 J_i. \tag{21.40}$$

At this point, it is usual to use the first of these equations to eliminate A_0 from the second, thereby fixing the gauge completely. Formally, we may write

$$-\Box\ \ A_i + \partial_i \partial^0 \left(\frac{J_0}{-\nabla^2} \right) = \mu_0 J_i, \tag{21.41}$$

where $(-\nabla^2)^{-1}$ really implies the inverse (or Green function) for the differential operator $-\nabla^2$, which we denote $g(x, x')$ and which satisfies the equation

$$-\nabla^2 g(x, x') = \delta(x, x'). \tag{21.42}$$

Thus, eqn. (21.41) is given (still formally, but more explicitly) by

$$-\Box\ \ A_i + \mu_0 \partial_i \partial^0 \int \mathrm{d}\sigma_{x'} g(x, x') J_0(x') = \mu_0 J_i. \tag{21.43}$$

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21.1.7 Retarded Green function in $n = 3$

In the Lorentz gauge, with $\alpha = 1$, we have

$$D_{\mu\nu}(x, x') = g_{\mu\nu} G(x, x'). \tag{21.44}$$

From Cauchy's residue theorem in eqn. (5.76), we have

$$G_{\rm r}(x, x') = -2\pi {\rm i}(\hbar^2 c)^{-1} \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^4} \left[\frac{{\rm e}^{{\rm i}(\mathbf{k}\cdot\Delta\mathbf{x} - \omega_k \Delta t)}}{2\omega_k} - \frac{{\rm e}^{{\rm i}(\mathbf{k}\cdot\Delta\mathbf{x} + \omega_k \Delta t)}}{2\omega_k} \right]. \tag{21.45}$$

Using the derivation in section 5.4.1, this evaluates to

$$\begin{aligned} G_{\rm r}(x, x') &= \frac{1}{4\pi\hbar^2 c \Delta X} \delta(ct - \Delta X) \\ &= \frac{1}{4\pi\hbar^2 c|\mathbf{x} - \mathbf{x}'|} \, \delta\left(c(t' - t_{\rm ret})\right), \end{aligned} \tag{21.46}$$

where the retarded time is defined by

$$t_{\rm ret} = t - |\mathbf{x} - \mathbf{x}'|. \tag{21.47}$$

Note that the retarded time is a function of the position.

21.1.8 The energy–momentum tensor

The gauge-invariant definition of the energy–momentum tensor is (see section 11.5),

$$\begin{aligned} \theta'_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\alpha)} F_\nu{}^\alpha - \mathcal{L} g_{\mu\nu} \\ &= 2 \frac{\partial \mathcal{L}}{\partial F^{\mu\alpha}} F_\nu{}^\alpha - \mathcal{L} g_{\mu\nu} \\ &= \mu_0^{-1} F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4\mu_0} F^{\lambda\rho} F_{\lambda\rho} g_{\mu\nu}. \end{aligned} \tag{21.48}$$

This result is manifestly gauge-invariant and can be checked against the traditional expressions obtained from Maxwell's equations for the energy density and the momentum flux.

The zero–zero component, in 3 + 1 dimensions, evaluates to:

$$\begin{aligned} \theta_{00} &= \mu_0^{-1} \left(F_{0i} F_0{}^i - \mathcal{L} g_{00} \right) \\ &= \frac{E_i E_i}{c^2 \mu_0} + \frac{1}{2\mu_0} \left(B_i B_i - \frac{E_i E_i}{c^2} \right) \\ &= \frac{1}{2\mu_0} (\mathbf{E}^2/c^2 + \mathbf{B}^2) \\ &= \frac{1}{2} (\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \mu_0^{-1} \mathbf{B} \cdot \mathbf{B}), \end{aligned} \tag{21.49}$$

which has the interpretation as an energy or Hamiltonian density. The spacetime off-diagonal components are given by

$$\begin{aligned}\theta_{0j} = \theta_{j0} &= \mu_0^{-1} F_{0i} F_j^{\ i} \\ &= \mu_0^{-1} \epsilon_{ijk} E_i B_k / c \\ &= -\frac{(\mathbf{E} \times \mathbf{H})_k}{c},\end{aligned} \tag{21.50}$$

which have the interpretation of a momentum density for the field. This vector is also known as Poynting's vector. The off-diagonal space parts are

$$\begin{aligned}\mu_0 \theta_{ij} = F_{i\mu} F_j^{\ \mu} &= F_{i0} F_j^{\ i} + F_{ik} F_j^{\ k} \\ &= -\frac{E_i}{c}\frac{E_j}{c} - 2 B_i B_j,\end{aligned} \tag{21.51}$$

with $i \neq j$. The diagonal terms with i not summed are

$$\begin{aligned}\mu_0 \theta_{ii} &= F_{i0} F_i^{\ 0} + F_{ij} F_i^{\ j} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= -\frac{E_i^2}{c^2} + 2 B_i^2 - \frac{1}{2}(B_j B_j - E_j E_j / c^2).\end{aligned} \tag{21.52}$$

The invariant trace of this tensor in $n + 1$ dimensions is

$$\theta^{\mu}_{\ \mu} = \mu_0^{-1} F_{\mu\alpha} F^{\mu\alpha} - \frac{(n+1)}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \tag{21.53}$$

which vanishes when $n = 3$, indicating that Maxwell's theory is conformally invariant in $3 + 1$ dimensions.

21.2 Effective theory of dielectric and magnetic media

This section contains a brief summarial discussion of the effective fields for the radiation field in the presence of a passive medium. The dielectric approach to electromagnetism in near-neutral media is often used since it offers an enormous simplification of very many systems. Its main weaknesses are that it makes two assumptions: namely, that the response of background matter is dipole-like and linear in the applied fields, and that the background matter is smoothly homogeneous throughout a given region. The first of these assumptions breaks down for strong fields, and the latter breaks down on very small length scales; thus, the theory provided in this section must be treated as a long-wavelength approximation to electromagnetism for weak fields.

Maxwell's equations in a dielectric/magnetic medium are most conveniently written in terms of the dielectric displacement vector **D**, defined by any one of

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the equivalent relations

$$\mathbf{D} = \epsilon_0\epsilon_\mathrm{r}\mathbf{E} + \mathbf{P} \tag{21.54}$$

$$= \epsilon_0(1 + \chi_\mathrm{e})\mathbf{E} \tag{21.55}$$

$$= \epsilon_0\epsilon_\mathrm{r}\mathbf{E}, \tag{21.56}$$

where $\mathbf{P}$ is the dielectric polarization and χ_e is the electric susceptibility, described below. The magnetic field intensity $\mathbf{H}$, defined by the equivalent forms

$$\mathbf{H} = \frac{1}{\mu_0\mu_\mathrm{r}}\mathbf{B} - \mathbf{M} \tag{21.57}$$

$$= \frac{\mathbf{B}}{\mu_0(1 + \chi_\mathrm{m})} \tag{21.58}$$

$$= \frac{\mathbf{B}}{\mu_0\mu_\mathrm{r}}. \tag{21.59}$$

$\mathbf{M}$ is called the magnetization and χ_m is the magnetic susceptibility, also defined below. In terms of these quantities, Maxwell's equations take on the form

$$\begin{aligned}
\vec{\nabla} \cdot \mathbf{D} &= \rho_e \\
\vec{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\vec{\nabla} \cdot \mathbf{B} &= 0 \\
\vec{\nabla} \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}.
\end{aligned} \tag{21.60}$$

This form of Maxwell's equations is valid inside any linear medium. As a further point, the energy density of the electromagnetic field is given by

$$\mathcal{E} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \tag{21.61}$$

This must agree with the Hamiltonian density.

21.2.1 The Maxwell action and Hamiltonian in a medium

To express Maxwell's equations in *covariant* form, we had to introduce the fields $\mathbf{D}$ and $\mathbf{H}$. We should therefore expect that, in the covariant description, we need to introduce a new tensor. We shall call this tensor $G_{\mu\nu}$, and define it by

$$G_{\mu\nu} = \begin{pmatrix} 0 & -cD_1 & -cD_2 & -cD_3 \\ cD_1 & 0 & H_3 & -H_2 \\ cD_2 & -H_3 & 0 & H_1 \\ cD_3 & H_2 & -H_1 & 0 \end{pmatrix}. \tag{21.62}$$

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We see that this tensor has the same structure as $F_{\mu\nu}$, but with $c\mathbf{D}$ replacing $\mathbf{E}/c$ and $\mathbf{H}$ replacing $\mathbf{B}$.

In terms of this tensor, we can write the action in the form

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4} F^{\mu\nu} G_{\mu\nu} - J^{\mu} A_{\mu} \right\}. \tag{21.63}$$

The canonical momentum can be written

$$\Pi_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial^0 A^{\mu})} = G_{0i}. \tag{21.64}$$

Again, one has the same problem of gauge invariance with the generalized 'velocity' as in the vacuum case. The Hamiltonian is best computed from the energy–momentum tensor. It is given by

$$\mathcal{H} = \frac{1}{2}(B_i H_i + E_i D_i) + J^{\mu} A_{\mu}. \tag{21.65}$$

21.2.2 Field equations and continuity

Using the property of the trace that

$$\delta F^{\mu\nu} G_{\mu\nu} = \delta G^{\mu\nu} F_{\mu\nu}, \tag{21.66}$$

for linear equations of motion (i.e. $G_{\mu\nu}$ does not depend on $F_{\mu\nu}$), we find the variation of the action is given by

$$\delta S = \int (\mathrm{d}x) \left\{ -\delta A^{\nu}\, \partial^{\mu} G_{\mu\nu} - J^{\mu} \delta A_{\mu} \right\} + \int \mathrm{d}\sigma^{\mu} \left\{ \delta A_{\nu} G_{\mu\nu} \right\} = 0. \tag{21.67}$$

We see immediately that the field equations are given by

$$\partial^{\mu} G_{\mu\nu} = -J_{\nu}, \tag{21.68}$$

which may be compared with eqn. (21.9), and that the continuity condition implies that the canonical momentum is ($\mu = 0$)

$$\Pi_{\mu} = D_{0\mu}, \tag{21.69}$$

and that the condition for continuity across a surface dividing two regions of space is $\mu = \mathrm{i}$ divides into two cases,

$$\begin{aligned} \Delta D_{i0} &= \Delta \mathbf{D} = 0 \\ \Delta D_{ij} &= \Delta \mathbf{H} = 0. \end{aligned} \tag{21.70}$$

These are the well known continuity conditions for the field at a dielectric boundary.

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In terms of the vector potential, we can write the field equations

$$c^2\epsilon\left\{\partial_0(\partial_i A^i) + \nabla^2 A^0\right\} + J^0 = 0$$
$$c^2\epsilon\left\{\partial_0\partial^0 A^i - \partial^i(\partial^0 A_0)\right\} + \frac{1}{\mu}(\partial^j\partial_j A^i - \partial^i(\partial_j A^j)) + J^i = 0. \tag{21.71}$$

This ugly mess can be compared with eqn. (21.26) for the vacuum case. At first sight, it appears that covariance is irretrievably lost in these expressions, but this is only an illusion caused by the spurious factors of ϵ and μ as explained below.

21.2.3 Reinstating covariance with $c \to c/n$

To reinstate covariance, we note that the introduction of a modified gauge condition helps to unravel the equations:

$$c^2\epsilon\mu(\partial^0 A_0) + \partial_j A^j = 0. \tag{21.72}$$

This can also be written as

$$n^2(\partial^0 A_0) + \partial_j A^j = 0, \tag{21.73}$$

which suggests that we re-define the derivative as

$$\hat{\partial}_\mu = \left(\frac{n}{c}\partial_t, \vec{\nabla}\right). \tag{21.74}$$

In terms of this equation, the field equations now combine to give

$$-\hat{\Box}\, A^\mu = \mu J^\mu. \tag{21.75}$$

To re-write the gauge condition in terms of this new derivative, we must also define $\hat{A}_\mu$ and $\hat{J}_\mu$, replacing c by c/n in each case:

$$\hat{A}^\mu = \begin{pmatrix} \frac{n}{c}\phi \\ \mathbf{A} \end{pmatrix} \qquad \hat{J}^\mu = \begin{pmatrix} \rho_e \frac{c}{n} \\ \mathbf{J} \end{pmatrix}. \tag{21.76}$$

Then we have the complete, covariant $(n+1)$-vector form of the field equations in a medium:

$$-\hat{\Box}\,\hat{A}^\mu = \mu\hat{J}^\mu$$
$$\hat{\partial}^\mu\hat{A}_\mu = 0. \tag{21.77}$$

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21.2.4 Green function

The Green function is easily obtained by direct analogy to the vacuum case. The most elegant form is obtained using the careted notation. The photon Green function in a dielectric satisfies the equation

$$\left[\hat{\Box}\, g_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)\hat{\partial}_\mu \hat{\partial}_\nu\right] \hat{D}^{\nu\rho}(x, x')\delta_\mu^{\ \rho}\delta(x, x'). \tag{21.78}$$

Thus, defining the careted momentum by

$$\mathrm{i}\hat{k}_\mu \equiv \hat{\partial}_\mu \mathrm{e}^{\mathrm{i}kx}, \tag{21.79}$$

i.e. such that $\hat{k}_\mu = (-n\omega/c, \mathbf{k})$, one has straightforwardly that

$$\hat{D}_{\mu\nu}(x, x') = \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}k(x-x')} \left[\frac{g_{\mu\nu}}{\hat{k}^2} + (\alpha - 1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^4}\right]. \tag{21.80}$$

Note carefully which of the quantities are careted and which are not. This Green function relates the careted field to the careted source,

$$\hat{A}_\mu(x) = \int (\mathrm{d}x')\hat{D}_{\mu\nu}(x, x')\hat{J}^\nu(x'). \tag{21.81}$$

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