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ON NONINNER 2-AUTOMORPHISMS OF FINITE 2-GROUPS

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Abstract

Let G be a finite 2-group. If G is of coclass 2 or $(G, Z(G))$ is a Camina pair, then G admits a noninner automorphism of order 2 or 4 leaving the Frattini subgroup elementwise fixed.

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1. Introduction and main results

In 1964, Liebeck proved that a finite p-group G of class 2 has a noninner automorphism σ leaving the Frattini subgroup $\Phi(G)$ elementwise fixed where σ can be chosen to have order p if $p > 2$, and order 2 or 4 if $p = 2$ [12, Theorem (1)]. In 1966, Gashütz showed that every finite p-group of order greater than p admits a noninner automorphism of p-power order [8]. In 1973, Berkovich proposed the following conjecture [13, Problem 4.13].

Conjecture 1.1. Every finite nonabelian p-group admits an automorphism of order p which is not an inner one.

Conjecture 1.1 is still open. Its validity has been established for the following classes of p-groups: G is regular [6, 14], G is nilpotent of class 2 or 3 [2, 5, 12], the commutator subgroup of G is cyclic [11], $G/Z(G)$ is powerful [1], $C_G(Z(\Phi(G))) \neq \Phi(G)$ [6] and G is of coclass 2 [4]. For other results on the conjecture, see [3, 9, 10, 15].

The noninner automorphisms of order p which are found in responding to Conjecture 1.1 act trivially on the centre $Z(G)$ or the Frattini subgroup $\Phi(G)$ of G. In the case $p = 2$, there are examples of 2-groups in which every automorphism leaving $\Phi(G)$ elementwise fixed is inner. Examples of groups of nilpotency class 2 and orders 64 and 128 with the latter property are exhibited in [1] and [12], respectively. In [5, Theorem 5.4] an infinite family of finite 2-groups of class 3 with the latter property is constructed.

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In this note, motivated by the above mentioned result of Liebeck, we are interested in studying the existence of noninner 2-automorphisms of least possible order for finite nonabelian 2-groups leaving the Frattini subgroup elementwise fixed.

Our first main result is the following theorem.

Theorem 1.2. *Let G be a finite 2-group of coclass 2. Then G has a noninner automorphism of order 2 or 4 leaving the Frattini subgroup $\Phi(G)$ elementwise fixed.*

Note that every finite p-group of coclass 2 has a noninner automorphism of order p leaving the centre elementwise fixed [4].

Let G be a finite p-group and N be a nontrivial proper normal subgroup. The pair (G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G \backslash N$, where x^G denotes the conjugacy class of x in G. Our next main result is the following theorem.

Theorem 1.3. *Let G be a finite 2-group such that $(G, Z(G))$ is a Camina pair. Then G has a noninner automorphism 2 or 4 leaving $\Phi(G)$ elementwise fixed.*

Let p be odd and G be a finite p-group such that $(G, Z(G))$ is a Camina pair. Then the existence of a noninner automorphism of order p that leaves the Frattini subgroup $\Phi(G)$ elementwise fixed follows from [9].

2. Proofs of the main results

Let G be a finite p-group. By $d(G)$ and $\Omega_1(G)$ we denote the minimum number of generators of G and the subgroup of G generated by all elements of order p, respectively. Any other unexplained notation is standard and follows that of Gorenstein [8].

We need the following preliminary lemma which will be used without further reference.

Lemma 2.1. *Let G be a finite group and N be a normal subgroup of G such that G/N is abelian. Let $G/N = \langle x_1N\rangle \times \cdots \times \langle x_dN\rangle$, where $x_1, \ldots, x_d \in G$ and $d = d(G/N)$. If $u_1, \ldots, u_d \in Z(N)$ such that*

$$\begin{cases} (x_iu_i)^{n_i} = x_i^{n_i} & \text{if } 1 \leq i \leq d, \\ [x_i, u_j] = [x_j, u_i] & \text{if } 1 \leq i < j \leq d, \end{cases}$$

where $n_i = o(x_iN)$, then the mapping $x_i \mapsto x_iu_i$, $1 \leq i \leq d$, can be extended to an automorphism of G leaving N elementwise fixed.

Proof. It follows from [5, Lemma 2.2], that the mapping $x_i \mapsto x_iu_i$, $1 \leq i \leq d$, can be extended to a derivation $f : G/N \to Z(N)$. Now the mapping $\sigma_f : G \to G$ given by $g \mapsto g(gN)^f$, for $g \in G$, is the desired automorphism. □

We first prove the following proposition.

Proposition 2.2. *Let G be a finite 2-group such that G' is cyclic and $Z(G)$ is elementary abelian. Then G has a noninner automorphism of order 2 or 4 leaving $\Phi(G)$ elementwise fixed.*

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PROOF. Suppose that G has no noninner automorphism of order 2 leaving $\Phi(G)$ elementwise fixed. By [1, Lemma 2.1] we may assume that $Z(G)$ is cyclic. Thus it follows from [1, Corollary 2.3] that $d(Z_2(G)/Z(G)) = d(G)$. Let $Z_2^\star(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$. Then by [10, Theorem 1.1], we have $Z_2^\star(G)$ is abelian. Let $Z(G) = \langle z\rangle$ and $Z_2^\star(G) = \langle u_1\rangle \times \cdots \times \langle u_r\rangle$, where $o(u_1) \geq \cdots \geq o(u_r)$. Therefore $r \geq d(G)$. Now we distinguish two cases.

Case 1. $r \geq 3$. We have $o(u_2) = o(u_3) = 2$ and $\langle u_2\rangle \times \langle u_3\rangle \cap Z(G) = 1$. Set $u = u_2$, $v = u_3$, $M = C_G(u)$ and $N = C_G(v)$. It is easy to see that M and N are two distinct maximal subgroups of G. Let $x \in N\backslash M$ and $y \in M\backslash N$. Thus $[x, u] = z = [y, v]$, $[x, v] = [y, u] = 1$ and $M \cap N = C_G(u, v)$. Now the mapping $x \mapsto xv$, $y \mapsto yu$ can be extended to an automorphism α of order 2 that fixes $M \cap N$ elementwise. If α is inner, then it follows that $\langle u_2\rangle \times \langle u_3\rangle \leq G'$, which is a contradiction.

Case 2. $r = 2$. Then $d(G) = 2$. We may also assume that $o(u_1) = 4$ and $o(u_2) = 2$. Let $M = C_G(u_1)$, $N = C_G(u_2)$, $x \in N\backslash M$ and $y \in M\backslash N$. Then $[x, u_1u_2] = [y, u_1u_2] = z = u_1^2$. Thus $(xu_1u_2)^2 = x^2$ and $(yu_1u_2)^2 = y^2$. Therefore the mapping $x \mapsto xu_1u_2$ and $y \mapsto yu_1u_2$ can be extended to an automorphism α of order 4 that fixes $\Phi(G)$ elementwise. Similarly the mapping $x \mapsto xu_1$ determines an automorphism β of order 4 that fixes M elementwise. If α and β are inner, then $u_1, u_1u_2 \in G'$. Thus $\langle u_1\rangle \times \langle u_2\rangle \leq G'$, which is a contradiction. □

We remark that if G is a nonabelian finite p-group with cyclic commutator subgroup, then G has a noninner automorphism of order p fixing $\Phi(G)$ elementwise whenever $p > 2$, and fixing either $\Phi(G)$ or $Z(G)$ elementwise whenever $p = 2$ [11, Theorem 1.1].

PROOF OF THEOREM 1.2. Let G be a finite 2-group of coclass 2. Suppose that G has no noninner automorphism of order 2 leaving $\Phi(G)$ elementwise fixed. If $|G| \geq 2^6$, then we have G' is cyclic [4, Proof of Theorem 3.1]. Now the result follow from Proposition 2.2. If $|G| \leq 2^5$, then we may assume that the nilpotency class of G is 3. Hence $|G| = 2^5$, $d(G) = 2$ and $Z_2(G) = \Phi(G)$. Construct α as in Case 2 of the proof of Proposition 2.2. If $\alpha = \theta_g$ is inner, then we must have $g \in C_G(\Phi(G)) = Z(\Phi(G))$. Thus $g \in Z_2(G)$. This implies that $u_1u_2 = [x, \alpha] = [x, g] \in Z(G)$, which is a contradiction. □

PROOF OF THEOREM 1.3. Let $(G, Z(G))$ be a Camina pair and assume that G has no noninner automorphism of order 2 or 4 leaving $\Phi(G)$ elementwise fixed. The same argument as in [9, Proof of Theorem 1.1] shows that $|Z(G)| = 2$. If $d(G) \geq 3$, then construct α as in Case 1 of the proof of Proposition 2.2. If $\alpha = \theta_h$ is inner, then

$$\begin{aligned}\{[g, h] \mid g \in G\} &= \{[g, \alpha] \mid g \in G\}\\ &= \{1, [x, h], [y, h], [xy, h]\}\\ &= \{1, v, u, uz\}.\end{aligned}$$

This means that $Z(G) \nsubseteq \{[g, h] \mid g \in G\}$, and this contradicts the hypothesis that $(G, Z(G))$ is a Camina pair. If $d(G) = 2$, then consider the automorphism β as in Case 2 of the proof of Proposition 2.2 and apply the preceding argument to get a contradiction. □

We end the paper by giving an example to show that our main results are the best possible. We first make a preliminary observation. Let G be a finite nonabelian p-group such that $C_G(Z(\Phi(G))) = \Phi(G)$. If α is an automorphism of G leaving $\Phi(G)$ elementwise fixed, then

$$x = x^{\alpha} = (x^{g^{-1}g})^{\alpha} = ((x^{g^{-1}})^{\alpha})^{g^{\alpha}} = x^{g^{-1}g^{\alpha}},$$

for all $x \in \Phi(G)$ and $g \in G$. Thus $g^{-1}g^{\alpha} \in C_G(\Phi(G)) = Z(\Phi(G))$. If, in addition, α has order p, then $g^{-1}g^{\alpha} \in \Omega_1(Z(\Phi(G)))$.

Example 2.3. Let G be a group with the polycyclic presentation

$$\langle g_1,\ g_2,\ g_3,\ g_4 \mid g_3 = [g_2, g_1],\ g_4 = g_1^4 = g_2^2,\ g_3^2 = g_4^2 = 1,\ [g_3, g_1] = g_4,\ [g_3, g_2] = 1\rangle.$$

Then G is a group of order 2^5 and coclass 2. Also, $(G, Z(G))$ is a Camina pair. Moreover, every automorphism of G of order 2 that fixes the Frattini subgroup elementwise is inner.

Checking the consistency of the presentation is straightforward (see [16, page 424] and [7, Lemma 2.1]). The order of G is 2^5, since G has relative orders $(4, 2, 2, 2)$. We also have $\Phi(G) = \langle g_1^2\rangle \times \langle g_3\rangle$, $C_G(Z(\Phi(G))) = \Phi(G)$, $[G, G] = \langle g_3,\ g_4\rangle$ and $[G, G, G] = \langle g_4\rangle = Z(G)$. Hence the nilpotency class of G is 3. Thus G is of coclass 2. Let $x \in G$. Then $x = g_1^i g_2^j g_3^k g_4^l$, where $i = 0, 1, 2, 3$ and $j, k, l = 0, 1$. We have

$$g_4 = \begin{cases} [x, g_3] & i = 1, 3, \\ [x, g_2] & i = 2, \\ [x, g_1^2] & i = 0,\ j = 1, \\ [x, g_1] & i = j = 0,\ k = 1. \end{cases}$$

This shows that $(G, Z(G))$ is a Camina pair.

Let α be an automorphism of order 2 that fixes $\Phi(G)$ elementwise. Let $a_1 = g_1^{-1}g_1^{\alpha}$ and $a_2 = g_2^{-1}g_2^{\alpha}$. Thus $a_1, a_2 \in \Omega_1(Z(\Phi(G))) = \langle g_3\rangle \times \langle g_4\rangle$, as we observed above. Since $(g_1^2)^{\alpha} = g_1^2$, $(g_2^2)^{\alpha} = g_2^2$ and $[g_1, g_2]^{\alpha} = [g_1, g_2]$,

$$[g_1, a_1] = 1, \quad [g_2, a_2] = 1 \quad \text{and} \quad [g_1, a_2] = [g_2, a_1]. \tag{$*$}$$

Now let $a_1 = g_3^i g_4^j$ and $a_2 = g_3^k g_4^l$, where $i, j = 0, 1$. Then a_1, a_2 satisfy ($*$) if and only if $i = k = 0$. Thus the number of automorphisms of G of order 2 that fix $\Phi(G)$ elementwise is at most 3. On the other hand let θ_x be an inner automorphism that fixes $\Phi(G)$ elementwise. Then it follows that $x \in C_G(\Phi(G)) = \Phi(G)$. If $x \in \Phi(G)\backslash Z(G)$, then $x^2 \in Z(G)$. Thus θ_x is of order 2. Therefore the number of nontrivial such inner automorphisms is $|\Phi(G)/Z(G)| - 1 = 3$. Therefore every automorphism of G of order 2 that fixes $\Phi(G)$ elementwise is inner.

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References

[1] A. Abdollahi, 'Powerful p-groups have noninner automorphisms of order p and some cohomology', *J. Algebra* **323** (2010), 779–789.

[2] A. Abdollahi, 'Finite p-groups of class 2 have noninner automorphisms of order p', *J. Algebra* **312** (2007), 876–879.

[3] A. Abdollahi and S. M. Ghoraishi, 'Noninner automorphisms of finite p-groups leaving the center elementwise fixed', *Int. J. Group Theory* **2** (2013), 17–20.

[4] A. Abdollahi, S. M. Ghoraishi, Y. Guerboussa, M. Reguiat and B. Wilkens, 'Noninner automorphisms of order p for finite p-groups of coclass 2', *J. Group Theory* **17** (2014), 267–272.

[5] A. Abdollahi, M. Ghoraishi and B. Wilkens, 'Finite p-groups of class 3 have noninner automorphisms of order p', *Beitr. Algebra Geom.* **54** (2013), 363–381.

[6] M. Deaconescu and G. Silberberg, 'Noninner automorphisms of order p of finite p-groups', *J. Algebra* **250** (2002), 283–287.

[7] K. Dekimpe and B. Eick, 'Computational aspects of group extensions and their application in topology', *Exp. Math.* **11** (2002), 183–200.

[8] W. Gaschütz, 'Nichtabelsche p-Gruppen besitzen äussere p-Automorphismen', *J. Algebra* **4** (1966), 1–2.

[9] S. M. Ghoraishi, 'A note on automorphisms of finite p-groups', *Bull. Aust. Math. Soc.* **87** (2013), 24–26.

[10] S. M. Ghoraishi, 'On noninner automorphisms of finite nonabelian p-groups', *Bull. Aust. Math. Soc.* **89** (2014), 202–209.

[11] A. R. Jamalli and M. Viseh, 'On the existence of noninner automorphisms of order two in finite 2-groups', *Bull. Aust. Math. Soc.* **87** (2013), 278–287.

[12] H. Liebeck, 'Outer automorphisms in nilpotent p-groups of class 2', *J. Lond. Math. Soc.* **40** (1965), 268–275.

[13] V. D. Mazurov and E. I. Khukhro (eds.) 'Unsolved problems in group theory', in: *The Kourovka Notebook*, Vol. 16 (Russian Academy of Sciences, Siberian Division, Institue of Mathematics, Novosibirisk, 2006).

[14] P. Schmid, 'A cohomological property of regular p-groups', *Math. Z.* **175** (1980), 1–3.

[15] M. Shabani-Attar, 'Existence of noninner automorphisms of order p in some finite p-groups', *Bull. Aust. Math. Soc.* **87** (2013), 272–277.

[16] C. C. Sims, *Computation with Finitly Presented Groups* (Cambridge University Press, Cambridge, 1994).

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