A NEW INVERSION AND REPRESENTATION THEORY FOR THE LAPLACE TRANSFORM

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1. Introduction. If

$$(1.1) \qquad f(s) = \int_0^\infty e^{-st} F(t)dt \qquad s > 0$$

and

$$(1.2) \qquad L^a_{k,t}[f(.)] = \frac{(at)^{ak-1}}{\Gamma(ak-1)} \int_0^\infty e^{-atx} x^{ak-1} L_k^{(ak-1)}(atx)f(x)dx,$$

$a > 0$, $k = 1, 2, 3, \ldots$; where $L_k^{(v)}$ is the Laguerre polynomial of order v, defined by

$$(1.3) \qquad L_k^{(v)}(z) = \frac{e^z z^{-v}}{k!} \frac{d^k}{dz^k}(e^{-z} z^{k+v}) \qquad v > -1,$$

then we shall show that under certain conditions

$$\lim_{k\to\infty} L^a_{k,t}[f(.)] = F(t).$$

Following the inversion theory, two representation theorems are given. The proofs of these theorems follow easily along the lines of Widder [4, Ch. VII] and are therefore omitted.

The operator (1.2) can be written in different forms. Substitution of (1.3) in (1.2) and integration by parts yields

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$$L^a_{k,t}[f(.)] = \frac{(at)^{ak-1}}{k!\Gamma(ak-1)} \int_0^\infty \frac{d^k}{dx^k}[e^{-atx} x^{ak+k-1}] f(x)\,dx$$

(1.4)

$$= \frac{(-1)^k (at)^{ak-1}}{k!\,\Gamma(ak-1)} \int_0^\infty e^{-atx} x^{ak+k-1} f^{(k)}(x)\,dx ,$$

provided the intermediate terms obtained by integration by parts vanish. The inversion operator in this form was given by A. Erdélyi [1] without developing the resulting inversion and representation theories. For $a = 1$, it is the Widder-Boas inversion operator [4, § 25].

Another form of (1.2) is obtained by observing that if a function g of xt has derivatives of all orders in both x and t then

$$(1.5) \qquad x^{-k} \frac{\partial^k}{\partial t^k} g(x,t) = t^{-k} \frac{\partial^k}{\partial x^k} g(xt) .$$

Let $g(xt) = e^{-atx}(tx)^{ak+k-1}$, $a > 0$, then by (1.4) and (1.5)

$$L^a_{k,t}[f(.)] = \frac{a^{ak-1} t^{-k}}{k!\Gamma(ak-1)} \int_0^\infty \frac{\partial^k}{\partial x^k}[e^{-atx}(tx)^{ak+k-1}] f(x)\,dx$$

$$= \frac{a^{ak-1}}{k!\Gamma(ak-1)} \int_0^\infty x^{-k} \frac{\partial^k}{\partial t^k}[e^{-atx}(tx)^{ak+k-1}] f(x)\,dx$$

$$= \frac{a^{ak-1}}{k!\Gamma(ak-1)} \frac{d^k}{dt^k}\{t^{ak+k-1} \int_0^\infty x^{ak-1} e^{-atx} f(x)\,dx\} .$$

The following results are frequently employed in the sequel:

$$(1.6) \qquad \int_0^\infty e^{-zt} t^v L_k^{(v)}(t)\,dt = \frac{\Gamma(v+k+1)(z+1)^k}{k!\; z^{v+k+1}}$$

$$(1.7) \qquad |L_k^{(v)}(z)| \le \frac{e^{z/2}\,\Gamma(v+k+1)}{k!\; z^{v+k+1}} \qquad (0 < z < \infty)$$

See for example [2, Ch. 10, § 12(32), and § 18(14)].

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2. Existence and Properties of the Operator.

THEOREM 2.1. If $F(t) \in L_1(0, R)$ for each $R > 0$ and

$$\int_R^\infty t^{-\lambda_o} |F(t)| dt < \infty,$$

for some $\lambda_o > 0$, then (1.1) exists for $s > 0$; if $k > k_o = \frac{\lambda_o}{a}$, $L^a_{k,t}[f(.)]$ exists for each $t > 0$ and

$$(2.1) \qquad L^a_{k,t}[f] = \frac{\Gamma(ak+k)}{k!\,\Gamma(ak-1)} \int_0^\infty \frac{u^k (at)^{ak-1} F(u)du}{(at+u)^{ak+k}} .$$

Proof. Since $e^{-st} t^{\lambda_o}$ attains its maximum at $t = \lambda_o s^{-1}$, it follows that for $R > 0$

$$\int_0^\infty e^{-st} |F(t)| dt \leq \int_0^R |F(t)| dt + e^{-\lambda_o} \lambda_o s^{-\lambda_o} \int_R^\infty t^{-\lambda_o} |F(t)| dt < \infty.$$

Hence $f(s) = \mathcal{L}(F;s)$ exists for $s > 0$. By hypotheses and (1.7) with $v = ak-1$ and $z = atx$ it is easily seen that for $k > k_o = \frac{\lambda_o}{a}$

$$\frac{1}{\Gamma(ak-1)} \int_0^\infty e^{-atx} (atx)^{ak-1} |L_k^{(ak-1)}(atx)| dx \int_0^\infty e^{-xu} |F(u)| du$$

exists for each $t > 0$. Therefore, if $k > k_o = \frac{\lambda_o}{a}$, $L^a_{k,t}[f(.)]$ exists for each $t > 0$. Finally, the use of Fubini's theorem and an application of (1.6) with $z = (\frac{u}{at} - 1)$ and $v = ak-1$ yields

$$L^a_{k,t}[f] = \frac{1}{\Gamma(ak-1)} \int_0^\infty e^{-atx} (atx)^{ak-1} L_k^{(ak-1)}(atx) f(x)dx$$

$$= \frac{1}{\Gamma(ak-1)} \int_0^\infty e^{-atx} (atx)^{ak-1} L_k^{(ak-1)}(atx)dx \int_0^\infty e^{-xu} F(u)du$$

$$= \frac{1}{\Gamma(ak-1)} \int_0^\infty F(u)du \int_0^\infty e^{-atx(1+u/at)} (atx)^{ak-1} L_k^{(ak-1)}(atx)dx$$

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$$= \frac{\Gamma(ak+k)}{k!\,\Gamma(ak-1)} \int_0^\infty \frac{u^k (at)^{ak-1} F(u)du}{(at+u)^{ak+k}} .$$

The next theorem yields a relation involving the Laplace transform of the operator.

THEOREM 2.2. If $x^{-1}f(x) \in L_1(R, \infty)$ and $x^{\lambda_o - 1} f(x) \in L_1(0, R)$ for each $R > 0$ and some $\lambda_o > 0$, then

$$(2.2) \qquad g(x) = \int_0^\infty e^{-xu} u^{-1} f(u^{-1}) du$$

exists for $x > 0$.

(2.3) If in addition $k > k_o = \frac{\lambda}{a}$, $\sigma^{-1} L^a_{k, \sigma^{-1}}[g(.)]$ exists for $\sigma > 0$, $L^a_{k, t}[f(.)]$ exists for almost all $t > 0$, and

$$(2.4) \qquad \int_0^\infty e^{-\sigma t} L^a_{k, t}[f(.)] dt = \sigma^{-1} L^a_{k, \sigma^{-1}}[g(.)].$$

Proof. Since

$$\int_0^R x^{-1} |f(x)| dx = \int_0^{1/R} v^{-1} |f(v^{-1})| dv \qquad (v = x^{-1})$$

and

$$\int_0^R x^{\lambda_o - 1} |f(x)| dx = \int_{1/R}^\infty v^{-\lambda_o} v^{-1} |f(v^{-1})| dv$$

the hypotheses of Theorem 2.1 with R replaced by R^{-1} are satisfied by the function $x^{-1} f(x^{-1})$. Hence (2.2) exists for $x > 0$ and if $k > k_o = \frac{\lambda_o}{a}$, $\sigma^{-1} L^a_{k, \sigma^{-1}}[g(.)]$ exists for $\sigma > 0$. Now, since $|f(x^{-1})|$ satisfies the same hypotheses as $f(x^{-1})$,

$$\frac{a^{ak-1}}{\sigma^{ak}\Gamma(ak-1)} \int_0^\infty e^{-\frac{ax}{\sigma}} x^{ak-1} \left|L_k^{(ak-1)}\left(\frac{ax}{\sigma}\right)\right| dx \int_0^\infty e^{-xu} u^{-1} |f(u^{-1})| du < \infty.$$

Also, by (1.7) with $v = (ak-1)$, $z = ats$ and hypotheses

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$$\int_0^\infty |f(s)|\,ds \int_0^\infty e^{-ats-\sigma t}(ats)^{ak-1}\,|L_k^{(ak-1)}(ats)|\,dt$$

$$\leq \frac{\Gamma(ak+k)}{k!\,\Gamma(ak)} \int_0^\infty |f(s)|\,ds \int_0^\infty e^{-ats(\frac{1}{2}+\frac{\sigma}{as})}(ats)^{ak-1}\,dt$$

$$= \frac{\Gamma(ak+k)}{k!} \left\{ \int_0^R + \int_R^\infty \right\} \frac{|f(s)|\,ds}{(\frac{1}{2}+\frac{\sigma}{as})^{ak-1}(\frac{as}{2}+\sigma)}$$

$$\leq \frac{\Gamma(ak+k)}{k!} \left\{ \frac{2^{ak}}{a} \int_0^R s^{-1}|f(s)|\,ds + \frac{a^{ak-1}}{\sigma^{ak}} \int_R^\infty s^{ak-1}|f(s)|\,ds \right\} < \infty.$$

Hence, applying Fubini's theorem twice, we obtain

$$\sigma^{-1} L^a_{k,\sigma^{-1}}[g(.)] = \frac{a^{ak-1}}{\sigma^{ak}\Gamma(ak-1)} \int_0^\infty e^{-\frac{ax}{\sigma}} x^{ak-1} L_k^{(ak-1)}(\frac{ax}{\sigma})\, g(x)dx$$

$$= \frac{a^{ak-1}}{\sigma^{ak}\Gamma(ak-1)} \int_0^\infty e^{-\frac{ax}{\sigma}} x^{ak-1} L_k^{(ak-1)}(\frac{ax}{\sigma})dx \int_0^\infty e^{-xu} u^{-1} f(u^{-1})du$$

$$= \frac{a^{ak-1}}{\sigma^{ak}\Gamma(ak-1)} \int_0^\infty e^{-\frac{ax}{\sigma}} x^{ak-1} L_k^{(ak-1)}(\frac{ax}{\sigma})dx \int_0^\infty e^{-\frac{x}{s}} s^{-1} f(s)ds \quad (s=u^{-1})$$

$$= \frac{a^{ak-1}}{\sigma^{ak}\Gamma(ak-1)} \int_0^\infty s^{-1} f(s)ds \int_0^\infty e^{-\frac{ax}{\sigma}-\frac{x}{s}} x^{ak-1} L_k^{(ak-1)}(\frac{ax}{\sigma})dx$$

$$= \frac{a^{ak-1}}{\Gamma(ak-1)} \int_0^\infty f(s)ds \int_0^\infty e^{-ast-\sigma t}(st)^{ak-1} L_k^{(ak-1)}(ats)dt \quad (\frac{x}{\sigma s}=t)$$

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$$=\frac{1}{\Gamma(ak-1)}\int_0^\infty e^{-\sigma t}dt\int_0^\infty e^{-ats}(ats)^{ak-1}L_k^{(ak-1)}(ats)f(s)ds.$$

Since $\sigma^{-1}L^a_{k,\sigma^{-1}}[g(.)]$, $\sigma > 0$, exists for $k > k_o = \frac{\lambda_o}{a}$, it follows that the inner integral above exists for almost all $t > 0$ and $k > k_o = \frac{\lambda_o}{a}$. That is, for $k > k_o = \frac{\lambda_o}{a}$, $L^a_{k,t}[f(.)]$ exists for almost all $t > 0$ and

$$\int_0^\infty e^{-\sigma t}L^a_{k,t}[f(.)]\,dt = \sigma^{-1}L^a_{k,\sigma^{-1}}[g(.)] .$$

3. <u>Inversion Theory and "Fundamental" Representation Theorem.</u>

THEOREM 3.1. If the hypotheses of Theorem 2.1 are satisfied, then at each point $t > 0$ of the Lebesgue set of F

$$\lim_{k\to\infty} L^a_{k,t}[f] = F(t).$$

<u>Proof.</u> By Theorem 2.1, $f(s) = \mathcal{L}(F:s)$ exists; for $k > k_o = \frac{\lambda_o}{a}$, $\lambda_o > 0, L^a_{k,t}[f(.)]$ exists for each $t > 0$ and

$$L^a_{k,t}[f] = \frac{\Gamma(ak+k)(at)^{ak-1}}{k!\,\Gamma(ak-1)}\int_0^\infty \frac{u^k F(u)du}{(at+u)^{ak+k}}$$

$$= \frac{\Gamma(ak+k)(at)^{ak-1}}{k!\,\Gamma(ak-1)}\left\{\int_0^{t-\delta}+\int_{t-\delta}^{t}+\int_t^{t+\delta}+\int_{t+\delta}^\infty\right\}\frac{u^k F(u)du}{(at+u)^{ak+k}}$$

$$= I_1 + I_2 + I_3 + I_4 ,$$

where $t > \delta > 0$. Since $u^k(at+u)^{-ak-k}$ is an increasing function of u in $(o, t-\delta)$ and for sufficiently large k, $u^{k+\lambda_o}(at+u)^{-ak-k}$

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a decreasing function of u for $u \geq t+\delta$, a straight forward calculation shows that both I_1 and I_4 tend to zero as $k \to \infty$. Now

$$I_3 = \frac{\Gamma(ak+k)(at)^{ak-1}}{k!\,\Gamma(ak-1)} \int_t^{t+\delta} \frac{u^k F(u)\,du}{(at+u)^{ak+k}}$$

$$= \frac{\Gamma(ak+k)(at)^{ak-1}}{k!\,\Gamma(ak-1)} \int_t^{t+\delta} e^{k\cdot h(u)} F(u)\,du,$$

where $h(u) = \log u - (a+1)\log(at+u)$. Since

$$h'(u) = \frac{1}{u} - \frac{a+1}{at+u}, \quad h'(t) = 0$$

and

$$h''(u) = -\frac{1}{u^2} + \frac{a+1}{(at+u)^2}, \quad h''(t) < 0,$$

Widder [4, Theorem 2b, Chapter VII, § 2, pp. 278] applies, so that

$$\lim_{k\to\infty} I_3 = \frac{F(t)}{2} \quad \text{a.e.}$$

The same argument is applicable to the remaining integral I_2, only now Widder [4, Corollary 2b, 2, Chapter VII, § 2; pp. 279] must be used to obtain

$$\lim_{k\to\infty} I_2 = \frac{F(t)}{2} \quad \text{a.e.}$$

Hence

$$\lim_{k\to\infty} L^a_{k,t}[f] = F(t) \quad \text{a.e.}$$

which proves the theorem.

The following theorem is fundamental in the representation theory.

THEOREM 3.2. If $x^{-1} f(x) \in L_1(R,\infty)$ and

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$x^{\lambda_o - 1} f(x) \in L_1(0, R)$ for each $R > 0$ and some $\lambda_o > 0$, then

$$\lim_{k \to \infty} \int_0^\infty e^{-\sigma t} L^a_{k,t}[f(.)]\,dt = f(\sigma) \quad \text{a.e.}$$

<u>Proof.</u> By Theorem 2.2, for $k > k_o = \frac{\lambda_o}{a}$

$$(3.1) \qquad \int_0^\infty e^{-\sigma t} L^a_{k,t}[f(.)]\,dt = \sigma^{-1} L^a_{k,\sigma^{-1}}[g(.)]$$

where $g(x)$ is defined by (2.2). Also since

$$\int_0^{1/R} u^{-1} |f(u^{-1})|\,du = \int_R^\infty v^{-1} |f(v)|\,dv < \infty \quad (u^{-1} = v)$$

and

$$\int_{1/R}^\infty u^{-\lambda_o} u^{-1} |f(u^{-1})|\,du = \int_0^R v^{\lambda_o - 1} |f(v)|\,dv < \infty,$$

Theorem 3.1 with R replaced by R^{-1} is applicable to the function $u^{-1} f(u^{-1})$. Therefore, replacing t by σ^{-1}, we obtain

$$\lim_{k \to \infty} L^a_{k,\sigma^{-1}}[g(.)] = \sigma f(\sigma) \quad \text{a.e.}$$

Hence by (3.1)

$$\lim_{k \to \infty} \int_0^\infty e^{-\sigma t} L^a_{k,t}[f(.)]dt = \lim_{k \to \infty} \sigma^{-1} L^a_{k,\sigma^{-1}}[g(.)] = f(\sigma) \quad \text{a.e.}$$

4. <u>Representation Theorems.</u> The following theorems are now easily obtained from the previous section and some well known weak compactness arguments (See e.g. [4, p. 33]).

THEOREM 4.1. A set of necessary and sufficient conditions for f to have a representation (1.1) with $F(t) \in L_p(0, \infty)$, $p > 1$, is that

(4.1) $f(x)$ is continuous in $0 < x < \infty$,

(4.2) $$f(x) = O(x^{(1-p)/p}), \quad x \to 0, \quad x \to \infty$$

and

(4.3) $$\| L^{a}_{k,\cdot} [f] \| L_p \leq M,$$

where M is independent of k.

THEOREM 4.2. A set of necessary and sufficient conditions for f to have a representation (1.1) with

$$\operatorname*{ess\,sup}_{0<t<\infty} |F(t)| \leq M$$

is (4.1),

(4.4) $$f(x) = O(x^{-1}) \text{ as } x \to \infty, \quad x \to 0+$$

and

(4.5) $$| L^{a}_{k,t} [f(\cdot)] | \leq M, \quad 0 < t < \infty .$$

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