Canad. Math. Bull.Vol. 34 (4), 1991 pp. 559–562

A FIXED POINT THEOREM FOR SEMIGROUPS OF PROXIMATELY UNIFORMLY LIPSCHITZIAN MAPPINGS

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ABSTRACT. As a generalization of Kiang and Tan's proximately nonexpansive semigroups, the notion of a proximately uniformly Lipschitzian semigroup is introduced and an existence theorem of common fixed points for such a semigroup is proved in a Banach space whose characteristic of convexity is less than one.

1. **Introduction.** Let C be a nonempty closed convex subset of a Banach space X and let k be a positive number. A mapping $T: C \longrightarrow C$ is said to be *uniformly k-Lipschitzian* if for each integer $n \geq 1$

$$(1) \qquad \|T^n x - T^n y\| \leq k\|x - y\| \text{ for } x, y \text{ in } C.$$

If (1) is valid when $k = 1$, T is called *nonexpansive.* Results on these classes of mappings can be found, for example, in Goebel and Reich [4] and Lifschitz [7]. A commutative semigroup $\mathcal{F}$ of self-mappings on C is said to be a nonexpansive semigroup on C if each member of $\mathcal{F}$ is nonexpansive. An element x in C is said to be a *common fixed point* of $\mathcal{F}$ if $f(x) = x$ for every f in $\mathcal{F}$. Generalizations of nonexpansive semigroups have been studied by several authors, see, e.g., Edelstein and Kiang [2,3], Kiang [5], and Kiang and Tan [6]. Here we particularly mention that a commutative semigroup $\mathcal{F}$ of self-mappings on C is said to be *proximately nonexpansive* [6] if for every x in C and $\varepsilon > 0$ there exists f in $\mathcal{F}$ such that

$$\|fg(x) - fg(y)\| \leq (1 + \varepsilon)\|x - y\|$$

for all g in $\mathcal{F}$ and y in C. Kiang and Tan [6] proved that such a semigroup $\mathcal{F}$ has a common fixed point if C is assumed to be a closed bounded convex subset of a uniformly convex Banach space. We now introduce a more general notion for semigroups of mappings than the one of Kiang and Tan's in [6] as follows.

DEFINITION. Let k be a positive number. A commutative semigroup $\mathcal{F}$ of self-mappings on a closed convex subset C of a Banach space X is said to be *proximately uniformly k-Lipschitzian* if for every x in C and $\varepsilon > 0$ there exists f in $\mathcal{F}$ such that

$$\|fg(x) - fg(y)\| \leq k(1 + \varepsilon)\|x - y\|$$

Received by the editors December 9, 1989.
AMS subject classification: Primary: 47H10.
Key words and phrases: Common fixed point, proximately uniformly Lipschitzian semigroup.

for all g in $\mathcal{F}$ and y in C.

It is immediately clear that a proximately nonexpansive semigroup is a proximately uniformly 1-Lipschitzian semigroup.

In this short note, we shall show that if the characteristic of convexity of X is less than one, then there is a constant $\kappa(X) > 1$ such that every proximately uniformly k-Lipschitzian semigroup on a closed bounded convex subset C of X has a fixed point provided $k < \kappa(X)$. This extends to some extent Theorem 1 of Kiang and Tan [6].

2. **The Result.** Let X be a Banach space. We recall that the characteristic of convexity of X is defined by

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\},$$

where $\delta_X(\varepsilon) := \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1 = \|y\| \text{ and } \|x-y\| = \varepsilon\}$ is the modulus of convexity of X. It is easy to see that X is uniformly convex if and only if $\varepsilon_0(X) = 0$. We also recall that a positive number c is said to have the Property (P) (cf. [4, p. 35]) if for every $0 < k < c$ there are positive numbers μ and $\alpha < 1$ such that

$$B\big(x, (1+\mu)r\big) \cap B\big(y, k(1+\mu)r\big) \subseteq B(z, \alpha r) \tag{2}$$

for some $z \in [x, y]$, the segment linking x and y, whenever x, y in X and $r > 0$ satisfy $\|x - y\| \geq (1-\mu)r$, where $B(v, r)$ denotes the closed ball with center v and radius r. Then we define the number

$$\kappa(X) := \sup\{c > 0 : c \text{ has Property (P)}\}.$$

It is now known (cf. [4 and 1]) that for a Hilbert space H, $\kappa(H) = 2^{1/2}$ and that $\varepsilon_0(X) < 1$ if and only if $\kappa(X) > 1$. It is also known [8] that $\varepsilon_0(X) < 1$ implies that X is super-reflexive.

Let us now state and prove the main result of this paper.

THEOREM. *Let X be a Banach space such that $\varepsilon_0(X) < 1$, C a closed convex subset of X, and $\mathcal{F}$ a proximately uniformly k-Lipschitzian semigroup on C with $k < \kappa(X)$. Suppose there is some x_0 in C such that the orbit $\{f(x_0) : f \in \mathcal{F}\}$ of $\mathcal{F}$ at x_0 is bounded. Then there exists an element z in C such that $f(z) = z$ for every f in $\mathcal{F}$, i.e., z is a common fixed point of $\mathcal{F}$.*

PROOF. Since $k < \kappa(X)$, there exist positive numbers μ and $\alpha < 1$ satisfying Property (P), i.e., (2) holds. For each x in C, we set

$$r(x) := \inf\{r > 0 : \text{ there exist } y_0 \text{ in } C \text{ and } g_0 \text{ in } \mathcal{F} \text{ such that } \|x - f(y_0)\| \leq r \text{ for all } f \text{ in } \mathcal{F}g_0\},$$

where $\mathcal{F}g_0 = \{fg_0 : f \in \mathcal{F}\}$. It is easy to see that $r(x)$ is well-defined for all x in C since $\{f(x_0) : f \in \mathcal{F}\}$ is bounded (this fact implies the boundedness of $\{f(x) : f \in \mathcal{F}\}$ for every $x \in C$). Now for this x_0 and the positive number $(1+\mu)^{1/2}-1$, using the definition of a proximately uniformly k-Lipschitzian semigroup, we find a g_1 in $\mathcal{F}$ such that

$$\|fg_1(x_0) - fg_1(y)\| \le k(1+\mu)^{1/2}\|x_0 - y\| \tag{3}$$

for all f in $\mathcal{F}$ and y in C. If $r(x_0) = 0$, then x_0 is a fixed point of $\mathcal{F}$ and we are done. In fact, in this case, for any $\varepsilon > 0$, by definition of $r(x_0)$, there are g_ε in $\mathcal{F}$ and y_ε in C such that $\|x_0 - f(y_\varepsilon)\| < \varepsilon$ for all $f \in \mathcal{F}g_\varepsilon$. It thus follows that for each $f \in \mathcal{F}$,

$$\begin{aligned}\|fg_1(x_0) - x_0\| &\le \|fg_1(x_0) - fg_1(fg_\varepsilon)y_\varepsilon\| + \|fg_1fg_\varepsilon(y_\varepsilon) - x_0\| \\ &\le k(1+\mu)^{1/2}\|x_0 - fg_\varepsilon(y_\varepsilon)\| + \varepsilon \\ &\le \left(1 + k(1+\mu)^{1/2}\right)\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $fg_1(x_0) = x_0$ for each f in $\mathcal{F}$. Therefore, $f(x_0) = ffg_1(x_0) = x_0$ and x_0 is a common fixed point of $\mathcal{F}$. Assume now $r(x_0) > 0$. In this case, we claim that there exists g_2 in $\mathcal{F}$ such that

$$\|x_0 - g_1g_2(x_0)\| \ge (1-\mu)r(x_0). \tag{4}$$

Indeed, if there were no such g_2 in $\mathcal{F}$, one would have $\|x_0 - g_1g(x_0)\| < (1-\mu)r(x_0)$ for all g in $\mathcal{F}$ and hence $r(x_0) \le (1-\mu)r(x_0)$, yielding a contradiction to the fact $r(x_0) > 0$. Consequently, there must be a g_2 in $\mathcal{F}$ satisfying (4). On the other hand, by definition of $r(x_0)$, one can find a $y_0 \in C$ and a $g_3 \in \mathcal{F}$ satisfying the following

$$\|x_0 - f(y_0)\| \le (1+\mu)^{1/2}r(x_0) \tag{5}$$

for all $f \in \mathcal{F}g_3$. From (3) and (5), it follows that for each $f \in \mathcal{F}$,

$$\|g_1g_2(x_0) - g_1g_2g_3f(y_0)\| \le k(1+\mu)r(x_0). \tag{6}$$

Combining (2), (4) and (6), we get by Property (P) that

$$\begin{aligned}D := g_1g_2g_3(y_0) &\subset B\big(x_0, (1+\mu)r(x_0)\big) \cap B\big(g_1g_2(x_0), k(1+\mu)r(x_0)\big) \\ &\subseteq B\big(x_1, \alpha\, r(x_0)\big)\end{aligned}$$

for some x_1 in $[x_0, g_1g_2(x_0)] \subset C$, where $\mathcal{F}f(x) = \{gf(x) : g \in \mathcal{F}\}$ for $f \in \mathcal{F}$ and $x \in C$. This shows that

$$r(x_1) \le \alpha\, r(x_0) \text{ and } \|x_1 - x_0\| \le \mathrm{Ar}(x_0),$$

where $A = 1 + \alpha + \mu$ is a constant independent of x in C. Continuing the above process in an obvious manner, we construct a sequence $\{x_n\}_{n\ge1}$ in C such that

$$r(x_{n+1}) \le \alpha\, r(x_n) \text{ and } \|x_{n+1} - x_n\| \le \mathrm{Ar}(x_n) \tag{7}$$

for $n \ge 0$. Since $\alpha < 1$, (7) indicates that $\lim_{n\to\infty} r(x_n) = 0$ and $\{x_n\}$ is norm-Cauchy and hence convergent. Let $z = \lim_{n\to\infty} x_n$. Then, since r is continuous, it is readily seen that $r(z) = 0$ and thus z is a common fixed point of $\mathcal{F}$. The proof is complete.

REMARK. We do not require any continuity assumption on the semigroup $\mathcal{F}$ in the above theorem.

COROLLARY 1 (KIANG AND TAN [6]). *Let C be a closed convex subset of a uniformly convex Banach space X and let $\mathcal{F}$ be a proximately nonexpansive semigroup on C such that $\{f(x_0) : f \in \mathcal{F}\}$ is bounded for some x_0 in C. Then $\mathcal{F}$ has a common fixed point.*

Since $\kappa(H) = 2^{\frac{1}{2}}$ for a Hilbert space H, we have the following.

COROLLARY 2. *Let C be a closed convex subset of a Hilbert space H and let $\mathcal{F}$ be a proximately uniformly k-Lipschitzian semigroup on C with $k < 2^{1/2}$. Suppose there exists an x_0 in C such that the orbit $\{f(x_0) : f \in \mathcal{F}\}$ is bounded. Then $\mathcal{F}$ has a common fixed point.*

When the semigroup $\mathcal{F}$ is singly generated, we have

COROLLARY 3. *Let T, C be as in the theorem and let $T: C \longrightarrow C$ be a mapping satisfying the property: for each x in C and $\varepsilon > 0$, there is $N = N(x, \varepsilon)$ such that*

$$\|T^n x - T^n y\| \leq k(1 + \varepsilon)\|x - y\|$$

for all y in C and $n \geq N$, where $k < \kappa(X)$ is a constant. Suppose also that there is an x_0 in C for which $\{T^n x_0\}$ is bounded. Then T has a fixed point.

ACKNOWLEDGEMENT. The author is indebted to the referee for his careful reading and good suggestions for this paper.

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