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INSTABILITY THEOREMS FOR CERTAIN nTH ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper we study the instability of certain nth order differential equations by means of Liapunov's functions and obtain some sufficient conditions for the equations being unstable. A method of constructing Liapunov's function for instability is being perfected. In particular the problem Ezeilo posed in 1982 has been resolved to some extent.

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Introduction

Firstly we adopt the convention: if $m < l$, then

$$\sum_{p=l}^{m} g_p(x_1, x_2, \dots, x_n) \equiv 0 \qquad \text{for arbitrary } g_p \in C[R^n, R].$$

In this paper, we will consider the following type of nth-order differential equations:

$$(1.1)\quad x^{(n)} + \sum_{p=0}^{k-1} a_p x^{(n-p-1)} + f_k(x, \dot{x}, \dots, x^{(n-1)}) x^{(n-k-1)} + f_{k+1}(x^{(n-k-3)}) x^{(n-k-2)} + f_{k+2}(x, \dot{x}, \dots, x^{(n-1)}) x^{(n-k-3)} + f_{k+3}(x^{(n-k-4)}) + f_{k+4}(x^{(n-k-5)}) + \sum_{p=k+5}^{n-1} a_p x^{(n-p-1)} = 0$$

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where a_p are constants, $n-3 \geq k \geq 1$, $f_{k+1}, f_k, f_{k+2}, f_{k+3}, f_{k+4}$ are continuous, dependent only on the argument shown and $f_{k+3}(0) = f_{k+4}(0) = 0$. If $c \neq 0$, then $x = c$ is not a solution of equation (1.1).

Ezeilo [2] has discussed equation (1.1) for $n = 6$, $k = 2$ and has proved that the trivial solution $x = 0$ is unstable if $f_2(x_1, x_2, \ldots, x_6) \operatorname{sgn} a_0 > |4a_0|^{-1} f_4(x_1, x_2, \ldots, x_6)$ for arbitrary $x_1, x_2, \ldots, x_6$. When $n = 4, k = 1, x_1 f_3(x_1, \ldots, x_4 = f(x_1)$, Li Wen-jian and Yu Yuan-hong [4] considered equation (1.1) and proved that the trivial solution $x = 0$ is unstable if $f(x_1)/x_1 - f_1^2(x_1, x_2, x_3, x_4)/4 > 0$, $(x_1 \neq 0)$ for arbitrary x_1, x_2, x_3, x_4. When $n = 5$, $k = 2$, $x_1 f_4(x_1, \ldots, x_5) = f(x_1)$, they concluded that the solution is unstable if $(f(x_1)/x_1) \operatorname{sgn} a_0 - |4a_0|^{-1} f_1(x_1, \ldots, x_5) > 0$, $(x \neq 0)$ for arbitrary x_1, x_2, x_3, x_4, x_5.

In Section 1, sufficient conditions are found for which the trivial solution $x = 0$ of equation (1.1) is unstable. In Section 2, we give the proof of the main results by constructing the Liapunov's functions.

1. Statement of Main Results

For the constant-coefficient case:

$$x^{(n)} + a_0 x^{(n-1)} + a_1 x^{(n-2)} + \cdots + a_{n-2}\dot{x} + a_{n-1}x = 0, \qquad (a_{n-1} \neq 0) \tag{1.2}$$

a necessary and sufficient condition for instability of the trivial solution $x = 0$ is that the auxiliary equation

$$\psi(\lambda) \equiv \lambda^n + a_0\lambda^{n-1} + a_1\lambda^{n-2} + \cdots + a_{n-2}\lambda + a_{n-1} = 0 \tag{1.3}$$

has at least one root with positive real part. The existence of such a root naturally depends on (though not always all of) the coefficients $a_0, a_1, \ldots, a_{n-1}$. For example, we know from general theory [5] that (1.3) necessarily has a root $\lambda = \alpha + i\beta$ with $\alpha \geq 0$ if not all of the coefficients $a_0, a_1, \ldots, a_{n-1}$ are positive. Now, (1.3) has a purely imaginary root $\lambda = i\beta$ ($\beta > 0$) if, and only if, the two equations

$$\beta^n + \sum_{p=1}^{[\frac{1}{2}(n)]} (-1)^p a_{2p-1} \beta^{n-2p} = 0, \tag{1.4}$$

$$\sum_{p=0}^{[\frac{1}{2}(n-1)]} (-1)^p a_{2p} \beta^{n-1-2p} = 0 \tag{1.5}$$

are simultaneously satisfied.

By considering the equation (1.4) we conclude that (1.3) must have at least one root with positive real part if $(-1)^p a_{2p-1} \geq 0$, $p = 1, \ldots, p_0 - 1$, $a_{2p_0-1}^2 -$

$4a_{2p_0-3}a_{2p_0+1} \le 0$, $a_{2p_0-1} \ne 0$ and $(-1)^{p-1}a_{2p-1} \ge 0$, $p = p_0+2, \dots, [\frac{1}{2}n]$. Similarly, if $(-1)^{p+p_0}a_{2p}a_{2p_0} \ge 0$, $p = 0, \dots, p_0 - 1$, $a_{2p_0+2}^2 - 4a_{2p_0}a_{2p_0-2} \le 0$, $a_{2p_0} \ne 0$ and $(-1)^{p+p_0}a_{2p}a_{2p_0} \ge 0$, $p = p_0+3, \dots, [\frac{1}{2}(n-1)]$, we know from equation (1.5) that (1.3) must have at least one root with positive real part.

The above (somewhat lengthy) preamble is merely intended to give the background to the hypotheses which play a dominant role in our treatment here (as Ezeilo did). Indeed we find that, by using the well-known criteria of Krasovskii [3], it is possible not only to arrive at the same instability results for (1.2) much more quickly, but also to extend results to the more general equation (1.1). We shall in fact prove the following theorem.

Let $A \equiv a_{k-2}x_{n-k}^2 + f_k(x_1, x_2, \dots, x_n)x_{n-k}x_{n-k-2} + f_{k+2}(x_1, x_2, \dots, x_n)x_{n-k-2}^2$.

If k is even and $5 \le k+4 \le n-1$, then we have:

THEOREM A. *Let one of the following two conditions be satisfied in a neighbourhood of the origin in $(x_1, x_2, \dots, x_n)$-space when $x_{n-k-4} \ne 0$:*

(1) $f'_{k+4}(x_{n-k-4}) > 0$ (*or* $f'_{k+4}(x_{n-k-4}) < 0$), $A\,\mathrm{sgn}(f'_{k+4}(x_{n-k-4})) \le 0$; *if* $k \ge 4$, *then* $(-1)^{\frac{1}{2}(|2p-k|-2)}a_{2p}\,\mathrm{sgn}\, f'_{k+4}(x_{n-k-4})) \le 0$ $(p = 0, 1, \dots, (k-4))$; *moreover, if* $k+6 \le n-1$, *then the above inequality holds when* $p = 0, 1, \dots, (k-4)$, $(k+6), \dots, (n-1)$.

(2) $f'_{k+4}(x_{n-k-4}) \equiv 0$; $(-1)^{\frac{1}{2}(|2p_1-k|+|2p_2-k|)}a_{2p_1}\,\mathrm{sgn}\, a_{2p_2} \ge 0$, $(-1)^{\frac{1}{2}(|2p_1-k|)}A\,\mathrm{sgn}\, a_{2p_1} \ge 0$, $(p_1, p_2 = 0, 1, \dots, (k-4))$; *moreover, if* $k+6 \le n-1$, *then the two inequalities hold when* $p_1, p_2 = 0, 1, \dots, (k-4), (k+6), \dots, (n-1)$; *if* $x_{n-k-2} \ne 0$, *then* $A^2 + \sum_{p=\frac{1}{2}(k+6)}^{[\frac{1}{2}(n-1)]} a_{2p}^2 \ne 0$.

Then the trivial solution $x = 0$ of equation (1.1) *is unstable.*

If k is odd and $5 \le k+4 \le n-1$, then we have

THEOREM B. *Let one of the following two conditions be satisfied in a neighbourhood of the origin in $(x_1, x_2, \dots, x_n)$-space when $x_{n-k-4} \ne 0$:*

(1) $(-1)^{\frac{1}{2}(k-1)}f'_{k+4}(x_{n-k-4}) < 0$, $(-1)^{\frac{1}{2}(k-1)}A \ge 0$; *if* $k \ge 5$, *then* $(-1)^{p-2}a_{2p-1} \ge 0$ $(p = 1, \dots, \frac{1}{2}(k-3))$; *moreover, if* $k+6 \le n-1$, *then* $(-1)^{p-1}a_{2p+1} \ge 0$ $(p = \frac{1}{2}(k+5), \dots, [\frac{1}{2}(n-2)])$.

(2) $f'_{k+4}(x_{n-k-4}) \equiv 0$; $(-1)^{\frac{1}{2}(k-1)}A \ge 0$, $A^2 + \sum_{p=\frac{1}{2}(k+5)}^{[\frac{1}{2}(n-2)]} a_{2p+1}^2 \ne 0$ $(x_{n-k-2} \ne 0)$; $(-1)^p a_{2p-1} \ge 0$, $(p = 1, \dots, (k-3))$; *moreover, if* $k+6 \le n-1$, *the inequality also holds when* $p = \frac{1}{2}(k+7), \dots, [\frac{1}{2}(n-2)]+1$.

Then the trivial solution $x = 0$ of equation (1.1) *is unstable.*

If k is even and $k+4 > n-1$, we have

THEOREM A_2. *Let one of the following two conditions be satisfied in a neighbourhood of the origin in* $(x_1, x_2, \ldots, x_n)$*-space:*

(1) $a_0 = a_2 = \cdots = a_{k-4} = 0$; $A > 0$ *(or* $A < 0$*) when* $x_{n-k-2} \neq 0$.

(2) *there is some* a_{2p_0} *in the set* $a_0, a_2, \ldots a_{k-4}$ *such that* (a) $(-1)^{\frac{k}{2}+p_0-1} A \operatorname{sgn} a_{2p_0} \geq 0$ $(x_{n-k-2} \neq 0)$; (b) $(-1)^{p+p_0} a_{2p} \operatorname{sgn} a_{2p_0} \geq 0$ $(p = 0, \ldots, (k-4))$.

Then the trivial solution of equation (1.1) *is unstable.*

If k is odd and $k + 4 > n - 1$ then we have

THEOREM B_2. *In a neighbourhood of the origin in* $(x_1, x_2, \ldots, x_n)$*-space, suppose that* $(-1)^{\frac{k}{2}-\frac{1}{2}} A > 0$ $(x_{n-k-2} \neq 0)$, $(-1)^p a_{2p-1} > 0$, *where* $p = 1, \ldots, \frac{1}{2}(k-3)$. *Then the trivial solution* $x = 0$ *of equation* (1.1) *is unstable.*

2. Proofs of theorems

It is convenient to consider equation (1.1) in system form:

$$
(2.1)\quad \begin{cases} x_1 = x, \ x_{i+1} = \dot{x}_i, \ i = 1, 2, \ldots, n-1, \\ \dot{x}_n = -\displaystyle\sum_{p=0}^{k-1} a_p x_{n-p} - f_k(x_1, x_2, \ldots, x_n) x_{n-k} - f_{k+1}(x_{n-k-2}) x_{n-k-1} \\ \qquad -f_{k+2}(x_1, x_2, \ldots, x_n) x_{n-k-2} - f_{k+3}(x_{n-k-3}) - f_{k+4}(x_{n-k-4}) \\ \qquad -\displaystyle\sum_{p=k+5}^{n-1} a_p x_{n-p}. \end{cases}
$$

We introduce the Liapunov function:

$$
\begin{aligned} W \equiv & -x_{n-k-2}\left(x_n + \sum_{p=0}^{k-1} a_p x_{n-p-1}\right) + x_{n-k-1}\left(x_{n-1} + \sum_{p=0}^{k=2} a_p x_{n-p-2}\right) \\ & + \frac{1}{2} a_{k-1} x_{n-k-1}^2 - \int_0^{x_{n-k-2}} s f_{k+1}(s) ds - \int_o^{x_{n-k-3}} f_{k+3}(s) ds - f_{k+4}(x_{n-k-4}) x_{n-k-3} \\ & + \bar{w}_0 + \sum_{p=0}^{k-3} a_p w_p + \sum_{p=k+5}^{n-1} a_p w_p, \end{aligned}
$$

where

$$
\bar{W}_0 \equiv \begin{cases} \displaystyle\sum_{i=1}^{\frac{1}{2}(k-1)} (-1)^i x_{n-k+i-1} x_{n-1-i}, & \text{if } k \text{ is odd,} \\ \displaystyle\sum_{i=1}^{\frac{1}{2}(k-2)} (-1)^i x_{n-k+i-1} x_{n-1-i} + \frac{1}{2}(-1)^{\frac{k}{2}} x_{n-\frac{k}{2}-1}^2, & \text{if } k \text{ is even,} \end{cases}
$$

$$\dot{W}_0\Big|_{(2.1)} = \begin{cases} -x_{n-k}x_{n-1}, & \text{if } k \text{ is even,} \\ (-1)^{\frac{1}{2}(k-1)} x^2_{n-\frac{1}{2}(k-1)} - x_{n-k}x_{n-1}, & \text{if } k \text{ is odd.} \end{cases}$$

When $p = 0, 1, \dots, k-3$,

$$W_p \equiv \begin{cases} \displaystyle\sum_{i=1}^{\frac{1}{2}(k-p-3)} (-1)^i x_{n-k+i-1}x_{n-p-2-i} + \frac{1}{2}(-1)^{\frac{1}{2}(k-p-1)} x^2_{n-\frac{1}{2}(k+p+3)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } k-p-2 \text{ is odd,} \\ \displaystyle\sum_{i=1}^{\frac{1}{2}(k-p-2)} (-1)^i x_{n-k+i-1}x_{n-p-2-i}, \qquad \text{if } k-p-2 \text{ is even,} \end{cases}$$

$$\dot{W}_p\Big|_{(2.1)} = \begin{cases} -x_{n-k}x_{n-p-2}, & \text{if } k-p-2 \text{ is odd,} \\ (-1)^{\frac{1}{2}(k-p-2)} x^2_{n-\frac{1}{2}(k+p+2)} - x_{n-k}x_{n-p-2}, & \text{if } k-p-2 \text{ is even.} \end{cases}$$

When $p = k+5, \dots, n-1$,

$$W_p \equiv \begin{cases} \displaystyle\sum_{i=1}^{\frac{1}{2}(p-k-3)} (-1)^i x_{n-p+i-1}x_{n-k-2-i} + \frac{1}{2}(-1)^{\frac{1}{2}(p-k-1)} x^2_{n-\frac{1}{2}(k+p+3)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p-k-2 \text{ is odd,} \\ \displaystyle\sum_{i=1}^{\frac{1}{2}(p-k-2)} (-1)^i x_{n-p+i-1}x_{n-k-2-i}, \qquad \text{if } p-k-2 \text{ is even,} \end{cases}$$

$$\dot{W}_p\Big|_{(2.1)} = \begin{cases} (-1)^{\frac{1}{2}(p-k-2)} x^2_{n-\frac{1}{2}(k+p+2)} - x_{n-p}x_{n-k-2}, & \text{if } p-k-2 \text{ is even,} \\ -x_{n-p}x_{n-k-2}, & \text{if } p-k-2 \text{ is odd.} \end{cases}$$

Next, let $(x_1, x_2, \dots, x_n) = (x_1(t), \dots, x_n(t))$ be an arbitrary solution of (2.1). If k is even, then

$$\begin{aligned} \dot{W}|_{(2.1)} &= \frac{d}{dt} W(x_1(t), \dots, x_n(t)) \\ &= \sum_{p=0}^{\frac{1}{2}(k-4)} (-1)^{\frac{1}{2}(k-2p-2)} a_{2p} x^2_{n-\frac{1}{2}(k+2p+2)} - f'_{k+4}(x_{n-k-4})x^2_{n-k-3} \\ &\quad + f_k(x_1, x_2, \dots, x_n)x_{n-k}x_{n-k-2} + f_{k+2}(x_1, x_2, \dots, x_n)x^2_{n-k-2} \\ &\quad + a_{k-2}x^2_{n-k} + \sum_{p=\frac{1}{2}(k+1)}^{[\frac{1}{2}(n-1)]} (-1)^{\frac{1}{2}(2p-k-2)} a_{2p} x^2_{n-\frac{1}{2}(k+2p+2)}. \end{aligned}$$

If k is odd, then

$$\dot{W}|_{(2.1)} = (-1)^{\frac{1}{2}(k-1)} x^2_{n-\frac{1}{2}(k+1)} + \sum_{p=1}^{\frac{1}{2}(k-3)} (-1)^{\frac{1}{2}(k-2p-1)} a_{2p-1} x^2_{n-\frac{1}{2}(k+2p+1)}$$

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$$+\sum_{p=\frac{1}{2}(k+5)}^{[\frac{1}{2}(n-2)]}(-1)^{\frac{1}{2}(2p-k-1)}a_{2p+1}x_{n-k-2}^2+a_{k-2}x_{n-k}^2+f_{k+2}(x_1,x_2,\dots,x_n)x_{n-k-2}^2$$
$$-f'_{k+4}(x_{n-k-4})x_{n-k-3}^2+f_k(x_1,x_2,\dots,x_n)x_{n-k}x_{n-k-2}.$$

Now, we only give the proof of the Theorem A (1).

Firstly, suppose $f'_{k+4}(x_{n-k-4})<0$.

Set $x^0=(x_1^0,\dots,x_n^0)$ where $x_{n-1}^0=M\epsilon$, $x_{n-k-1}^0=\epsilon^2$, $x_i^0=0$ $(i=1,\dots,n-k-2,n-k,\dots,n-2,n)$.

If $M>0$, and $|M|$ is big enough, it is evident from the definition that $W(x^0)=M\epsilon^3+\frac{1}{2}a_{k-1}\epsilon^4>0$ for arbitrary $\epsilon>0$, so that every neighbourhood of the origin in $(x_1,x_2,\dots,x_n)$-space contains a point x^0 such that $W(x^0)>0$.

Next, since k is even, by condition (1), $\dot{W}|_{(2.1)}\geq 0$.

(a) If $k+4=n-1$, then $\dot{W}|_{(2.1)}=0$ implies that

(2.2) $$x_n=x_{n-1}=\cdots=x_2=0,\qquad x_1=\text{constant }\xi$$

The substitution of (2.2), into (2.1) leads to $f_{k+4}(\xi)=0$, which implies that $\xi=0$. Hence, $\dot{W}=0$ $(t\geq 0)$ implies $x_1=x_2=\cdots=x_n=0$ $(t\geq 0)$.

(b) If $k+4<n-1$, then $\dot{W}|_{(2.1)}=0$ implies that $x_{n-k-3}=0$, which in turn implies that

(2.3) $$x_{n-k-3}=x_{n-k-2}=\cdots=x_n=0$$

From $x_{i+1}=\dot{x}_i$ $(i=1,2,\dots,n-1)$ we know

(2.4) $$x_{n-k-4}=c_0,\qquad x_{n-k-5}=c_0t+c_1,\qquad \dots,$$
$$x_1=\frac{c_0}{(n-k-5)!}t^{n-k-5}+\frac{c_1}{(n-k-6)!}t^{n-k-6}+\cdots+c_{n-k-6}t+c_{n-k-5}$$

where $c_0,c_1,\dots,c_{n-k-5}$ are constants.

The substitution of (2.3) and (2.4) into (2.1) leads to

$$f_{k+4}(c_0)+a_{k+5}(c_0t+c_1)+a_{k+6}\left(\frac{c_0}{2!}t^2+c_1t+c_2\right)+\cdots$$
$$+a_{n-1}\left(\frac{c_0}{(n-k-5)!}t^{n-k-5}\frac{c_1}{(n-k-6)!}t^{n-k-6}+\cdots+c_{n-k-5}\right)=0\qquad(t\geq 0)$$

which implies that $c_0=c_1=\cdots=c_{n-k-5}=0$. Thus

$$x_{n-k-4}=x_{n-k-5}=\cdots=x_1=0\qquad(t\geq 0).$$

Here, we know that the function W has all the requisite properties of Krasovskii subject to the conditions in the theorem.

Hence, the trivial solution $x=0$ of equation (1.1) is unstable. In similiar ways, we can prove our other results if we take W or $-W$ as the Liapunov function and choose appropriate M. Here, we omit these proofs.

3. Discussion

Now, we consider the equation

$$(3.1)\; x^{(6)}+a_0x^{(5)}+a_1x^{(4)}+\theta(x,\dot{x},\dots,x^{(5)})x^{(3)}+f(\dot{x})x^{(2)}+g(x,\dot{x},\dots,x^{(5)})\dot{x}+h(x)=0$$

where θ, f, g, h are continuous functions dependent only on the arguments shown, and $h(0) = 0$. Ezeilo [2] has proved that the trivial solution $x = 0$ of equation (3.1) is unstable if the following conditions are satisfied,

(3.2) $h(x) \neq 0$ for $x \neq 0,\ a_0 \neq 0$

(3.3) $g(x_1, x_2, \dots, x_6)\,\mathrm{sgn}\, a_0 > |4a_0|^{-1}\theta^2(x_1, \dots, x_6)$ for arbitrary $x_1, x_2, \dots, x_6$

Using Theorem A_2, we can obtain two sufficient conditions for the instability of equation (3.1), one being that of Ezeilo [2] above, the other being $a_0 = 0$, and for $x_2 \neq 0$, $A = \theta(x_1, \dots, x_6)x_4x_2 + g(x_1, \dots, x_6)x_2^2 > 0$.

Hence, we extend the method (set forth by Ezeilo) to the case $a_0 = 0$ (as in Ezeilo [1] and extract conditions on a_1, a_3, a_5 which can be utilized in establishing some other instability theorems for equation (3.1) in which $a_0, \dots, a_5$ are not all constants by using Theorem B. So the problem left by Ezeilo [2] has been resolved to some extent.

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