UNION CURVES IN A SUBSPACE V_n OF A RIEMANNIAN V_m

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1. Introduction. A union curve on a surface in a euclidean 3-space, relative to a given congruence is characterized by the property that its osculating plane at each point contains the ray of the congruence through that point. Springer **(2)** and Pan **(1)** have studied union curves in a hypersurface V_n of a Riemannian V_{n+1}. In the present paper we proceed to obtain the equations of union curves in a subspace V_n of a Riemannian V_m.

2. Subspaces of V_m. Consider a subspace V_n of co-ordinates x^i, $i = 1, 2, \ldots, n$ and positive-definite metric

$$ds^2 = g_{ij}\, dx^i dx^j \tag{2.1}$$

imbedded in a V_m of co-ordinates y^α, $\alpha = 1, 2, \ldots, m$ and positive-definite metric

$$ds^2 = a_{\alpha\beta} dy^\alpha dy^\beta. \tag{2.2}$$

For points of V_n, the y's are expressible as functions of the x's, the matrix

$$\left\| \frac{\partial y^\alpha}{\partial x^i} \right\|$$

being of order n. The coefficients of the fundamental forms of V_m and V_n are connected by the relations

$$g_{ij} = a_{\alpha\beta}\, y^\alpha_{,i} y^\beta_{,j} \tag{2.3}$$

where (,) followed by an index indicates the covariant derivative with respect to the x with that index.

In V_m there are $(m - n)$ mutually orthogonal independent unit vectors normal to V_n. If $N^\alpha_{\nu 1}$ be the contravariant components in the y's of any such system of unit normals to V_n, they must satisfy the relations

$$a_{\alpha\beta} N^\alpha_{\nu 1} N^\beta_{\nu 1} = 1, \tag{2.4}$$

$$a_{\alpha\beta} N^\alpha_{\nu 1} N^\beta_{\mu 1} = 0 \quad (\mu \neq \nu), \tag{2.5}$$

and

$$a_{\alpha\beta} N^\alpha_{\nu 1} y^\beta{}_{,i} = 0 \quad (\nu = n + 1, \ldots, m), \tag{2.6}$$

since $y^\alpha{}_{,i}$ are the components of a vector tangential to the curve of parameter x^i.

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3. Tensor derivatives. Let

$$T^{\alpha_1 \ldots i_1 \ldots}_{\beta_1 \ldots j_1 \ldots}$$

be a set of quantities over V_n having twofold tensor character indicated by the two types of indices, and let

$$\left\{ \begin{matrix} \alpha \\ \beta\nu \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$$

be the Christoffel symbols formed with the tensors $a_{\alpha\beta}$ and g_{ij} respectively. Then

$$\begin{aligned} T^{\alpha_1 \ldots i_1 \ldots}_{\beta_1 \ldots j_1 \ldots ; k} = \frac{\partial T^{\cdots}_{\cdots}}{\partial x^k} &+ \sum_{\tau} \left\{ \begin{matrix} \alpha_\tau \\ \lambda\gamma \end{matrix} \right\} \binom{\lambda}{\alpha_\tau} T^{\cdots}_{\cdots} y^{\gamma}_{,k} \\ &- \sum_{\tau} \left\{ \begin{matrix} \lambda \\ \beta_\tau \gamma \end{matrix} \right\} \binom{\beta_\tau}{\lambda} T^{\cdots}_{\cdots} y^{\gamma}_{,k} \\ &- \sum_{t} \left\{ \begin{matrix} i_t \\ lk \end{matrix} \right\} \binom{l}{i_t} T^{\cdots}_{\cdots} \\ &- \sum_{t} \left\{ \begin{matrix} l \\ j_t k \end{matrix} \right\} \binom{j_t}{l} T^{\cdots}_{\cdots} \end{aligned}$$

are the components of a tensor, called the tensor derivative of the tensor $T^{\cdots}_{\cdots}$. Here $\binom{\lambda}{\alpha_\tau} T$ denotes that the suffix α_τ of $T^{\cdots}_{\cdots}$ is to be replaced by λ.

Throughout the present paper we shall use (;) followed by an index to indicate the tensor-derivative with respect to the x with that index.

4. Congruences of curves in V_n. Let us consider a set of $(m - n)$ congruences of curves, one curve of each of which passes through each point of the subspace V_n. Let $\lambda^{\alpha}_{\tau 1}$ be the contravariant components in the y's, of a unit vector in the direction of the congruence λ_τ. Since the vector λ_τ is, in general, not normal to V_n, it may be expressed linearly in terms of $y^{\alpha}_{,i}$ and the set of normals $N^{\alpha}_{\nu 1}$. Thus

$$\lambda^{\alpha}_{\tau 1} = t^{i}_{\tau 1} y^{\alpha}_{,i} + \sum_{\nu} c_{\nu\tau} N^{\alpha}_{\nu 1} \tag{4.1}$$

where the parameters $t^{i}_{\tau 1}$ and $c_{\nu\tau}$ are such that if $\theta_{\nu\tau 1}$ is the angle between the vectors $N^{\alpha}_{\nu 1}$ and $\lambda^{\alpha}_{\tau 1}$ then

$$c_{\nu\tau} = \cos\theta_{\nu\tau 1} = a_{\alpha\beta} \lambda^{\alpha}_{\tau 1} N^{\beta}_{\nu 1} \tag{4.2}$$

and

$$1 - t^{i}_{\tau 1} t_{\tau 1 i} = \sum \cos^2 \theta_{\nu\tau 1} \tag{4.3}$$

for

$$a_{\alpha\beta} \lambda^{\alpha}_{\tau 1} \lambda^{\beta}_{\tau 1} = a_{\alpha\beta} \Big(t^{i}_{\tau 1} y^{\alpha}_{,i} + \sum_{\nu} c_{\nu\tau} N^{\beta}_{\nu 1} \Big) \Big(t^{j}_{\tau 1} y^{\beta}_{,j} + \sum_{\nu} c_{\nu\tau} N^{\beta}_{\nu 1} \Big)$$

or

$$1 = g_{ij} t^{i}_{\tau 1} t^{j}_{\tau 1} + \sum_{\nu} c^{2}_{\nu\tau 1}.$$

5. Union curves in V_n. A curve in a subspace V_n of a Riemannian V_m is a union curve relative to a set of $(m-n)$ congruences $\lambda_{\tau 1}$ if $\lambda_{\tau 1}(\tau = n+1, \ldots, m)$ are tangential to the osculating variety of C (that is, the variety determined by the tangent to C and the first curvature vector of C in V_m) at every point of C, that is, if there exists a linear relation between the vectors $\mathbf{t}$, $\lambda_{\tau 1}(\tau = n+1, \ldots, m)$ and $\mathbf{q}$, where $\mathbf{t}$ is the unit tangent vector to C and $\mathbf{q}$ is the first curvature vector of C relative to V_m.

Let $\phi_{\tau 1}$ be the angle which $\lambda_{\tau 1}$ makes with the tangent to C, then, by (4.1) we have

$$\text{(5.1)} \qquad \cos \phi_{\tau 1} = a_{\alpha\beta}\lambda^{\alpha}_{\tau 1} y^{\beta}_{,j} \frac{dx^j}{ds} = a_{\alpha\beta}\left[t^i_{\tau 1} y^{\alpha}_{,i} + \sum_{\nu} c_{\nu\tau} N^{\alpha}_{\nu 1} \right] y^{\beta}_{,j} \frac{dx^j}{ds}$$
$$= g_{ij} t^i_{\tau 1} \frac{dx^j}{ds},$$

by virtue of (2.3) and (2.6).

Let $\xi^{\alpha}_{\tau 1}$ be the contravariant components of a unit vector at a point P of the curve C in V_n which satisfies the following conditions:

(1) It is linearly dependent on $\lambda_{\tau 1}$ and the unit tangent vector $\mathbf{t}$.

(2) It is orthogonal to $\mathbf{t}$.

We may write

$$\text{(5.2)} \qquad \xi^{\alpha}_{\tau 1} = a_{\tau 1} \frac{dy^{\alpha}}{ds} + b_{\tau 1} \lambda^{\alpha}_{\tau 1}.$$

Multiplying (5.2) by

$$a_{\alpha\beta} \frac{dy^{\beta}}{ds}$$

and summing over α, we have

$$\text{(5.3)} \qquad 0 = a_{\tau 1} + b_{\tau 1} \cos \phi_{\tau 1}$$

because of (5.1) and Condition (2).

Using (5.3), we may write (5.2) as

$$\text{(5.4)} \qquad \lambda^{\alpha}_{\tau 1} = \frac{dy^{\alpha}}{ds} \cos \phi_{\tau 1} + \frac{1}{b_{\tau 1}} \xi^{\alpha}_{\tau 1}.$$

Since $\lambda^{\alpha}_{\tau 1}$ and $\xi^{\alpha}_{\tau 1}$ are components of unit vectors, from (5.4) we have

$$1 = a_{\alpha\beta}\lambda^{\alpha}_{\tau 1}\lambda^{\beta}_{\tau 1} = a_{\alpha\beta}\left(\frac{dy^{\alpha}}{ds} \cos \phi_{\tau 1} + \frac{1}{b_{\tau 1}} \xi^{\alpha}_{\tau 1}\right)\left(\frac{dy^{\beta}}{ds} \cos \phi_{\tau 1} + \frac{1}{b} \xi^{\beta}_{\tau 1}\right)$$

or

$$\text{(5.5)} \qquad \frac{1}{b_{\tau 1}} = \sin \phi_{\tau 1}.$$

(5.2) may now be written as

$$\text{(5.6)} \qquad \xi^{\alpha}_{\tau 1} = \lambda^{\alpha}_{\tau 1} \operatorname{cosec} \phi_{\tau 1} - \frac{dy^{\alpha}}{ds} \cot \phi_{\tau 1}.$$

Using (4.1), (5.6) may be written in the form

$$\xi^{\alpha}_{\tau 1} = \left(t^i_{\tau 1} \operatorname{cosec} \phi_{\tau 1} - \frac{dx^i}{ds} \cot \phi_{\tau 1}\right) y^{\alpha}_{,i} + \sum_{\nu} c_{\nu\tau 1} N^{\alpha}_{\nu 1} \operatorname{cosec} \phi_{\tau 1}. \tag{5.7}$$

The set of $(m - n)$ linear equations (5.7) in the quantities $N^{\alpha}_{\nu 1}$, when solved, yield,

$$N^{\alpha}_{\sigma 1} = \sum_{\tau} \left[\xi^{\alpha}_{\tau 1} \sin \phi_{\tau 1} - \left(t^i_{\tau 1} - \frac{dx^i}{ds} \cos \phi_{\tau 1}\right) y^{\alpha}_{,i}\right] C_{\sigma\tau} \tag{5.8}$$

where $C_{\sigma\tau}$ is the normalized cofactor of $c_{\sigma\tau}$ in the determinant $|c_{\sigma\tau}|$.

Also, we have

$$q^{\alpha} = y^{\alpha}_{,i} p^i + \sum_{\nu} \Omega_{\nu|ij} \frac{dx^i}{ds} \frac{dx^j}{ds} N^{\alpha}_{\nu 1} \tag{5.9}$$

where p^i and q^{α} are the contravariant components of the first curvature vectors **p** and **q** of C in V_n and V_m respectively **(3)**.

Using (5.8), (5.9) may be written as

$$q^{\alpha} = y^{\alpha}_{,i}\left[p^i - \sum_{\nu,\tau} \Omega_{\nu|lj} \frac{dx^l}{ds} \frac{dx^j}{ds} \left(t^i_{\tau 1} - \frac{dx^i}{ds} \cos \phi_{\tau 1}\right) C_{\nu\tau}\right] + \sum_{\nu,\tau} \xi^{\alpha}_{\tau 1} \sin \phi_{\tau 1} C_{\nu\tau} \Omega_{\nu|ij} \frac{dx^i}{ds} \frac{dx^j}{ds}. \tag{5.10}$$

Since each λ_{τ} is linearly dependent on **t** and $\xi_{\tau 1}$, by the definition of a union curve, **t**, ξ_{τ} and **q** must be linearly dependent; $\tau = n + 1, \ldots, m$. Again, since **t** is orthogonal to $\xi_{\tau 1}$ and **q**, it follows that $\xi_{\tau 1}$ and **q** must be linearly connected $(\tau = n + 1, \ldots, m)$. Therefore, from (5.10) we find that the equations of a union curve are

$$p^i - \sum_{\nu,\tau} \Omega_{\nu|lj} \frac{dx^l}{ds} \frac{dx^j}{ds} \left(t^i_{\tau 1} - \frac{dx^i}{ds} \cos \phi_{\tau 1}\right) C_{\nu\tau} = 0. \tag{5.11}$$

The quantities

$$\eta^i = p^i - \sum_{\tau,\nu} \Omega_{\nu|lj} \frac{dx^l}{ds} \frac{dx^j}{ds} \left(t^i_{\tau 1} - \frac{dx^i}{ds} \cos \phi_{\tau 1}\right) C_{\nu\tau} \tag{5.12}$$

are the components of a vector in V_n, which may be called the *union curvature vector* in analogy with the corresponding result for a curve in a hypersurface V_n of a V_{n+1} **(2)**.

The magnitude of this vector is called the union curvature of the curve whose unit tangent vector has components in $(dx^i)/(ds)$ in the x's. The union curvature vector is a null vector for a union curve, and consequently the union curvature of a union curve is zero.

6. A particular case. If $m = n + 1$, $N^{\alpha}_{\nu 1} = N^{\alpha}$, $\theta_{\nu\tau} = \theta$,

$$\phi_{\tau 1} = \phi, \qquad C_{\sigma\tau 1} = \frac{1}{\cos\theta},$$

and equations (5.1) reduce to

$$p^i - \Omega_{lj}\frac{dx^l}{ds}\frac{dx^j}{ds}\left(t^i - \frac{dx^i}{ds}\cos\phi\right)\sec\theta = 0 \tag{6.1}$$

which are the equations of a union curve in a hypersurface V_n of a Riemannian V_{n+1}. Also (5.12) reduces to

$$\eta^i \equiv p^i - \Omega_{lj}\frac{dx^l}{ds}\frac{dx^j}{ds}\left(t^i - \frac{dx^i}{ds}\cos\phi\right)\sec\theta, \tag{6.2}$$

which are the components of the union curvature vector of a curve in V_n **(2)**.

References

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