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Ideals with approximate unit in semicrossed products

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Abstract. We characterize the ideals of the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$, associated with suitable sequences of closed subsets of X, with left (resp. right) approximate unit. As a consequence, we obtain a complete characterization of ideals with left (resp. right) approximate unit under the assumptions that X is metrizable and the dynamical system (X, ϕ) contains no periodic points.

1 Introduction and notation

The semicrossed product is a nonself-adjoint operator algebra which is constructed from a dynamical system. We recall the construction of the semicrossed product we will consider in this work. Let X be a locally compact Hausdorff space, and let $\phi : X \to X$ be a continuous and proper surjection (recall that a map ϕ is *proper* if the inverse image $\phi^{-1}(K)$ is compact for every compact $K \subseteq X$). The pair (X, ϕ) is called a *dynamical system*. An action of $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ on $C_0(X)$ by isometric $*$-endomorphisms α_n, $n \in \mathbb{Z}_+$ is obtained by defining $\alpha_n(f) = f \circ \phi^n$. We write the elements of the Banach space $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $A = \sum_{n\in\mathbb{Z}_+} U^n f_n$ with the norm given by $\|A\|_1 = \sum_{n\in\mathbb{Z}_+} \|f_n\|_{C_0(X)}$. Multiplication on $\ell^1(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$(U^n f)(U^m g) = U^{n+m}(\alpha^m(f)g),$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\mathbb{Z}_+, C_0(X))$ is a Banach algebra.

The Banach algebra $\ell^1(\mathbb{Z}_+, C_0(X))$ can be faithfully represented as a (concrete) operator algebra on a Hilbert space. This is achieved by assuming a faithful action of $C_0(X)$ on a Hilbert space $\mathcal{H}_0$. Then we can define a faithful contractive representation π of $\ell_1(\mathbb{Z}_+, C_0(X))$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(\mathbb{Z}_+)$ by defining $\pi(U^n f)$ as

$$\pi(U^n f)(\xi \otimes e_k) = \alpha^k(f)\xi \otimes e_{k+n}.$$

The *semicrossed product* $C_0(X) \times_\phi \mathbb{Z}_+$ is the closure of the image of $\ell^1(\mathbb{Z}_+, C_0(X))$ in $\mathcal{B}(\mathcal{H})$ in the representation just defined. We will denote an element $\pi(U^n f)$ of $C_0(X) \times_\phi \mathbb{Z}_+$ by $U^n f$ to simplify the notation.

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For $A = \sum_{n\in\mathbb{Z}_+} U^n f_n \in \ell^1(\mathbb{Z}_+, C_0(X))$, we call $f_n \equiv E_n(A)$ the *nth Fourier coefficient* of A. The maps $E_n : \ell^1(\mathbb{Z}_+, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $C_0(X) \times_\phi \mathbb{Z}_+$, and therefore they extend to contractions $E_n : C_0(X) \times_\phi \mathbb{Z}_+ \to C_0(X)$. An element A of the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ is 0 if and only if $E_n(A) = 0$, for all $n \in \mathbb{Z}_+$, and thus A is completely determined by its Fourier coefficients. We will denote A by the formal series $A = \sum_{n\in\mathbb{Z}_+} U^n f_n$, where $f_n = E_n(A)$. Note, however, that the series $\sum_{n\in\mathbb{Z}_+} U^n f_n$ does not in general converge to A [6, II.9]. The *kth arithmetic mean* of A is defined to be $\bar{A}_k = \frac{1}{k+1}\sum_{l=0}^{k} S_l(A)$, where $S_l(A) = \sum_{n=0}^{l} U^n f_n$. Then, the sequence $\{\bar{A}_k\}_{k\in\mathbb{Z}_+}$ is norm convergent to A [6, Remark, p. 524]. We refer to [3, 4, 6] for more information about the semicrossed product.

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of closed subsets of X satisfying

$$(*) \qquad X_{n+1} \cup \phi(X_{n+1}) \subseteq X_n,$$

for all $n \in \mathbb{N}$. Peters proved in [7] that the subspace $\mathcal{I} = \{A \in C_0(X) \times_\phi \mathbb{Z}_+ : E_n(A)(X_n) = \{0\}\}$ is a closed two-sided ideal of $C_0(X) \times_\phi \mathbb{Z}_+$. We will write this as $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$. We note that if $A \in \mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$, then $U^n E_n(A) \in \mathcal{I}$ for all $n \in \mathbb{Z}_+$. Peters proved in [7] that there is a one-to-one correspondence between closed two-sided ideals $\mathcal{I} \subseteq C_0(X) \times_\phi \mathbb{Z}_+$ and sequences $\{X_n\}_{n=0}^{\infty}$ of closed subsets of X satisfying $(*)$, under the assumptions that X is metrizable and the dynamical system (X, ϕ) contains no periodic points. Moreover, he characterizes the maximal and prime ideals of the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ in this case.

Donsig, Katavolos, and Manousos obtained in [4] a characterization of the Jacobson radical for the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$, where X is a locally compact metrizable space and $\phi : X \to X$ is a continuous and proper surjection. Andreolas, Anoussis, and the author characterized in [2] the ideal generated by the compact elements and in [1] the hypocompact and the scattered radical of the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$, where X is a locally compact Hausdorff space and $\phi : X \to X$ is a homeomorphism. All these ideals are of the form $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ for suitable families of closed subsets $\{X_n\}_{n=0}^{\infty}$.

In the present paper, we characterize the closed two-sided ideals $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ of $C_0(X) \times_\phi \mathbb{Z}_+$ with left (resp. right) approximate unit. As a consequence, we obtain a complete characterization of ideals with left (resp. right) approximate unit under the additional assumptions that X is metrizable and the dynamical system (X, ϕ) contains no periodic points.

Recall that a *left* (resp. *right*) *approximate unit* of a Banach algebra $\mathcal{A}$ is a net $\{u_\lambda\}_{\lambda\in\Lambda}$ of elements of $\mathcal{A}$ such that:

(1) for some positive number r, $\|u_\lambda\| \le r$ for all $\lambda \in \Lambda$,
(2) $\lim u_\lambda a = a$ (resp. $\lim a u_\lambda = a$), for all $a \in \mathcal{A}$, in the norm topology of $\mathcal{A}$.

A net which is both a left and a right approximate unit of $\mathcal{A}$ is called an *approximate unit* of $\mathcal{A}$. A left (resp. right) approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ that satisfies $\|u_\lambda\| \le 1$ for all $\lambda \in \Lambda$ is called a *contractive left* (resp. *right*) *approximate unit*.

We will say that an ideal $\mathcal{I}$ of a Banach algebra $\mathcal{A}$ has a left (resp. right) approximate unit if it has a left (resp. right) approximate unit as an algebra.

2 Ideals with approximate unit

In the following theorem, the ideals $\mathcal{J} \sim \{X_n\}_{n=0}^{\infty}$ with right approximate unit are characterized.

Theorem 2.1 *Let $\mathcal{J} \sim \{X_n\}_{n=0}^{\infty}$ be a nonzero ideal of $C_0(X) \times_\phi \mathbb{Z}_+$. The following are equivalent:*

(1) *$\mathcal{J}$ has a right approximate unit.*
(2) *$X_n = X_{n+1}$, for all $n \in \mathbb{Z}_+$.*

Proof We start by proving that (1) $\Rightarrow$ (2). Let $\mathcal{J} \sim \{X_n\}_{n=0}^{\infty}$ be an ideal with right approximate unit $\{V_\lambda\}_{\lambda\in\Lambda}$. We suppose that there exists $n \in \mathbb{Z}_+$ such that $X_{n+1} \subsetneq X_n$. Let

$$n_0 = \min\{n \in \mathbb{Z}_+ : X_{n+1} \subsetneq X_n\},$$

$x_0 \in X_{n_0} \backslash X_{n_0+1}$, and $f \in C_0(X)$ such that $f(x_0) = 1$, $f(X_{n_0+1}) = \{0\}$, and $\|f\| = 1$. Then, for $A = U^{n_0+1} f$, we have $A \in \mathcal{J}$ and

$$\|AV_\lambda - A\| \geq \|E_{n_0+1}(AV_\lambda - A)\| = \|fE_0(V_\lambda) - f\| \geq |(fE_0(V_\lambda) - f)(x_0)| = 1,$$

for all $\lambda \in \Lambda$, since $x_0 \in X_{n_0}$ and $E_0(V_\lambda)(X_{n_0}) = 0$, which is a contradiction. Therefore, $X_n = X_{n+1}$ for all $n \in \mathbb{Z}_+$.

For (2) $\Rightarrow$ (1), assume that $X_n = X_{n+1}$ for all $n \in \mathbb{Z}_+$. By ($*$), we get that $\phi(X_0) \subseteq X_0$. We will show that if $\{u_\lambda\}_{\lambda\in\Lambda}$ is a contractive approximate unit of the ideal $C_0(X\backslash X_0)$ of $C_0(X)$, then $\{U^0 u_\lambda\}_{\lambda\in\Lambda}$ is a right approximate unit of $\mathcal{J}$. Since $\|u_\lambda\| \leq 1$, we have $\|U^0 u_\lambda\| \leq 1$.

Let $A \in \mathcal{J}$ and $\varepsilon > 0$. Then there exists $k \in \mathbb{Z}_+$ such that

$$\|A - \bar{A}_k\| < \frac{\varepsilon}{4},$$

where $\bar{A}_k$ is the kth arithmetic mean of A. Since $X_n = X_0$, $E_n(\bar{A}_k) \in C_0(X\backslash X_0)$ and $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit of $C_0(X\backslash X_0)$, there exists $\lambda_0 \in \Lambda$ such that

$$\|E_l(\bar{A}_k)u_\lambda - E_l(\bar{A}_k)\| < \frac{\varepsilon}{2(k+1)},$$

for all $l \leq k$ and $\lambda > \lambda_0$. So, for $\lambda > \lambda_0$, we get that

$$\begin{aligned}
\|AU^0 u_\lambda - A\| &= \|AU^0 u_\lambda - \bar{A}_k U^0 u_\lambda + \bar{A}_k U^0 u_\lambda - \bar{A}_k + \bar{A}_k - A\| \\
&\leq \|AU^0 u_\lambda - \bar{A}_k U^0 u_\lambda\| + \|\bar{A}_k U^0 u_\lambda - \bar{A}_k\| + \|A - \bar{A}_k\| \\
&< \|\bar{A}_k U^0 u_\lambda - \bar{A}_k\| + \frac{\varepsilon}{2} \\
&\leq \sum_{l=0}^{k} \|E_l(\bar{A}_k)u_\lambda - E_l(\bar{A}_k)\| + \frac{\varepsilon}{2} \\
&< \varepsilon,
\end{aligned}$$

which concludes the proof. ∎

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In the following theorem, the ideals $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ with left approximate unit are characterized.

Theorem 2.2 *Let $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ be a nonzero ideal of $C_0(X) \times_\phi \mathbb{Z}_+$. The following are equivalent:*

(1) *$\mathcal{I}$ has a left approximate unit.*
(2) *$X_0 \subsetneq X$ and $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$.*
(3) *$\phi(X \backslash X_1) = X \backslash X_0$ and $\phi(X_{n+1} \backslash X_{n+2}) = X_n \backslash X_{n+1}$, for all $n \in \mathbb{Z}_+$.*

Proof We start by proving that (1) $\Rightarrow$ (2). Let $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ be an ideal with left approximate unit $\{V_\lambda\}_{\lambda \in \Lambda}$.

First, we prove that $X_0 \subsetneq X$. We suppose that $X_0 = X$. Then $E_0(V_\lambda) = 0$, for all $\lambda \in \Lambda$, and hence for every $U^n f \in \mathcal{I}$, we have

$$\|V_\lambda U^n f - U^n f\| \geq \|E_n(V_\lambda U^n f - U^n f)\| = \|E_0(V_\lambda) \circ \phi^n f - f\| = \|f\|,$$

for all $\lambda \in \Lambda$, which is a contradiction. Therefore, $X_0 \subsetneq X$.

Now, we prove that $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$. We suppose that there exists $n \in \mathbb{Z}_+$ such that $\phi^n(X \backslash X_n) \nsubseteq X \backslash X_0$ and let

$$n_0 = \min\{n \in \mathbb{Z}_+ : \phi^n(X \backslash X_n) \nsubseteq X \backslash X_0\}.$$

The set $X \backslash X_{n_0}$ is nonempty, since $X_{n_0} \subseteq X_0 \subsetneq X$. Then, there exist $x_0 \in X \backslash X_{n_0}$ such that $\phi^{n_0}(x_0) \in X_0$ and a function $f \in C_0(X)$ such that $f(x_0) = 1$, $f(X_{n_0}) = \{0\}$, and $\|f\| = 1$. If $A = U^{n_0} f$, by the choice of f, we have that $A \in \mathcal{I}$, $\|A\| = 1$ and

$$\begin{aligned} \|V_\lambda A - A\| &\geq \|E_{n_0}(V_\lambda A - A)\| \\ &= \|E_0(V_\lambda) \circ \phi^{n_0} f - f\| \\ &\geq |(E_0(V_\lambda) \circ \phi^{n_0} f - f)(x_0)| \\ &= 1, \end{aligned}$$

for all $\lambda \in \Lambda$, since $\phi^{n_0}(x_0) \in X_0$ and $E_0(V_\lambda)(X_0) = \{0\}$, which is a contradiction. Therefore, $\phi^n(X \backslash X_n) \subseteq X \backslash X_0$. Furthermore, by ($*$), we get that $\phi^n(X_n) \subseteq X_0$, for all $n \in \mathbb{Z}_+$, and hence

$$X = \phi^n(X) = \phi^n(X_n \cup (X \backslash X_n)) = \phi^n(X_n) \cup \phi^n(X \backslash X_n) \subseteq X_0 \cup \phi^n(X \backslash X_n).$$

Since $\phi^n(X \backslash X_n) \subseteq X \backslash X_0$ and ϕ is surjective, $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$.

For (2) $\Rightarrow$ (1), assume that $X_0 \subsetneq X$ and $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$. We will show that if $\{u_\lambda\}_{\lambda \in \Lambda}$ is a contractive approximate unit of the ideal $C_0(X \backslash X_0)$ of $C_0(X)$, then $\{U^0 u_\lambda\}_{\lambda \in \Lambda}$ is a left approximate unit of $\mathcal{I}$. Since $\|u_\lambda\| \leq 1$, we have $\|U^0 u_\lambda\| \leq 1$.

Let A be a norm-one element of $\mathcal{I}$ and $\varepsilon > 0$. Then there exists $k \in \mathbb{Z}_+$ such that

$$\|A - \bar{A}_k\| < \frac{\varepsilon}{4},$$

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where $\bar{A}_k$ is the kth arithmetic mean of A. For $l \leq k$, let

$$D_\varepsilon(E_l(\bar{A}_k)) = \left\{ x \in X : |E_l(\bar{A}_k)(x)| \geq \frac{\varepsilon}{4(k+1)} \right\}.$$

Since $A \in \mathcal{J}$, we have $E_l(\bar{A}_k)(X_l) = \{0\}$ and hence $D_\varepsilon(E_l(\bar{A}_k)) \subseteq X \backslash X_l$. Furthermore, since $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$, we have that $\phi^l(D_\varepsilon(E_l(\bar{A}_k))) \subseteq X \backslash X_0$. Moreover, the set $D_\varepsilon(E_l(\bar{A}_k))$ is compact, since $E_l(\bar{A}_k) \in C_0(X)$, and hence the set $\phi^l(D_\varepsilon(E_l(\bar{A}_k)))$ is also compact. By Urysohn's lemma for locally compact Hausdorff spaces [8, p. 39], there is a norm-one function $v_l \in C_0(X)$ such that

$$v_l(x) = \begin{cases} 1, & x \in \phi^l(D_\varepsilon(E_l(\bar{A}_k))), \\ 0, & x \in X_0. \end{cases}$$

Then, there exists $\lambda_0 \in \Lambda$ such that

$$\|u_\lambda v_l - v_l\| < \frac{\varepsilon}{2(k+1)},$$

for all $l \leq k$ and $\lambda > \lambda_0$, and hence

$$|u_\lambda(x) - 1| < \frac{\varepsilon}{2(k+1)},$$

for all $x \in \cup_{l=0}^k \phi^l(D_\varepsilon(E_l(\bar{A}_k)))$ and $\lambda > \lambda_0$. Therefore, if $x \in \cup_{l=0}^k (D_\varepsilon(E_l(\bar{A}_k)))$, then $\phi^l(x) \in \cup_{l=0}^k \phi^l(D_\varepsilon(E_l(\bar{A}_k)))$ and hence

$$\|((u_\lambda \circ \phi^l)E_l(\bar{A}_k) - E_l(\bar{A}_k))(x)\| < \frac{\varepsilon}{2(k+1)},$$

for all $l \leq k$ and $\lambda > \lambda_0$. On the other hand, if $x \notin \cup_{l=0}^k (D_\varepsilon(E_l(\bar{A}_k)))$, then

$$|E_l(\bar{A}_k)(x)| < \frac{\varepsilon}{4(k+1)},$$

for all $l \leq k$, and hence

$$\|((u_\lambda \circ \phi^l)E_l(\bar{A}_k) - E_l(\bar{A}_k))(x)\| < \frac{\varepsilon}{2(k+1)}.$$

From what we said so far, we get that

$$\begin{aligned} \|U^0 u_\lambda A - A\| &< \|U^0 u_\lambda \bar{A}_k - \bar{A}_k\| + \frac{\varepsilon}{2} \\ &\leq \sum_{l=0}^{k} \|(u_\lambda \circ \phi^l)E_l(\bar{A}_k) - E_l(\bar{A}_k)\| + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

for all $\lambda > \lambda_0$.

Now, we show that (2) $\Rightarrow$ (3). We assume that $\phi^n(X \backslash X_n) = X \backslash X_0$, for all $n \in \mathbb{Z}_+$. Then, $\phi(X \backslash X_{n+2}) \subseteq X \backslash X_{n+1}$. Indeed, if $x \in X \backslash X_{n+2}$ and $\phi(x) \in X_{n+1}$, then $\phi^{n+2}(x) \in X_0$, by ($*$), which is a contradiction. Furthermore, by ($*$), we know that $\phi(X_{n+1}) \subseteq X_n$ and hence $\phi(X_{n+1} \backslash X_{n+2}) \subseteq X_n \backslash X_{n+1}$ for all $n \in \mathbb{Z}_+$.

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To prove that $\phi(X_{n+1}\backslash X_{n+2}) = X_n\backslash X_{n+1}$ for all $n \in \mathbb{Z}_+$, we suppose that there exists $n \in \mathbb{Z}_+$ such that $\phi(X_{n+1}\backslash X_{n+2}) \subsetneq X_n\backslash X_{n+1}$. If

$$n_0 = \min\{n \in \mathbb{Z}_+ : \phi(X_{n+1}\backslash X_{n+2}) \subsetneq X_n\backslash X_{n+1}\},$$

then

$$\begin{aligned}\phi(X_{n_0+1}) &= \phi(X_{n_0+2} \cup (X_{n_0+1}\backslash X_{n_0+2}))\\ &= \phi(X_{n_0+2}) \cup \phi(X_{n_0+1}\backslash X_{n_0+2})\\ &\subseteq X_{n_0+1} \cup \phi(X_{n_0+1}\backslash X_{n_0+2})\\ &\subsetneq X_{n_0+1} \cup (X_{n_0}\backslash X_{n_0+1})\\ &= X_{n_0},\end{aligned}$$

and hence

$$\begin{aligned}\phi(X) &= \phi(X_{n_0+1} \cup (X\backslash X_{n_0+1}))\\ &= \phi(X_{n_0+1}) \cup \phi(X\backslash X_{n_0+1})\\ &\subseteq \phi(X_{n_0+1}) \cup (X\backslash X_{n_0})\\ &\subsetneq X,\end{aligned}$$

which is a contradiction, since ϕ is surjective. Therefore, $\phi(X_{n+1}\backslash X_{n+2}) = X_n\backslash X_{n+1}$ for all $n \in \mathbb{Z}_+$.

Finally, we show that (3) $\Rightarrow$ (2). We assume that $\phi(X\backslash X_1) = X\backslash X_0$ and $\phi(X_{n+1}\backslash X_{n+2}) = X_n\backslash X_{n+1}$, for all $n \in \mathbb{Z}_+$. Then, $X_0 \subsetneq X$. Indeed, if $X_0 = X$, then $\mathcal{I} \equiv \{0\}$, which is a contradiction. If $n > 1$, we have that

$$\begin{aligned}\phi(X\backslash X_n) &= \phi\left[(X\backslash X_1) \cup (X_1\backslash X_2) \cup \cdots \cup (X_{n-1}\backslash X_n)\right]\\ &= \phi(X\backslash X_1) \cup \phi(X_1\backslash X_2) \cup \cdots \cup \phi(X_{n-1}\backslash X_n)\\ &= (X\backslash X_0) \cup (X_0\backslash X_1) \cup \cdots \cup (X_{n-2}\backslash X_{n-1})\\ &= X\backslash X_{n-1},\end{aligned}$$

and hence $\phi^n(X\backslash X_n) = X\backslash X_0$, for all $n \in \mathbb{Z}_+$. ■

Remark 2.3 It follows from the proofs of Theorems 2.1 and 2.2 that if $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ is an ideal of $C_0(X) \times_\phi \mathbb{Z}_+$ with a left (resp. right) approximate unit, then it has a contractive left (resp. right) approximate unit of the form $\{U^0 u_\lambda\}_{\lambda\in\Lambda}$ where $\{u_\lambda\}_{\lambda\in\Lambda}$ a contractive approximate unit of the ideal $C_0(X\backslash X_0)$ of $C_0(X)$.

By Theorem 2.2, if $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ is an ideal of $C_0(X) \times_\phi \mathbb{Z}_+$ with a left approximate unit, then $X_{n+1} = X_n$ or $X_{n+1} \subsetneq X_n$ for all $n \in \mathbb{Z}_+$. If $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ and $X_{n+1} = X_n$, for all $n \in \mathbb{Z}_+$, we will write $\mathcal{I} \sim \{X_0\}$. We obtain the following characterization.

Corollary 2.4 *Let $\mathcal{I} \sim \{X_0\}$ be a nonzero ideal of $C_0(X) \times_\phi \mathbb{Z}_+$. The following are equivalent:*

(1) *$\mathcal{I}$ has a left approximate unit.*
(2) *$\phi(X_0) = X_0$ and $\phi(X\backslash X_0) = X\backslash X_0$.*

Proof By Theorem 2.2, we have $\phi(X\backslash X_0) = X\backslash X_0$. By $(*)$, we have $\phi(X_0) \subseteq X_0$, and since ϕ is surjective, we get $\phi(X_0) = X_0$. ■

In the following proposition, the ideals $\mathcal{I} \sim \{X_n\}_{n=1}^{\infty}$ of $C_0(X) \times_\phi \mathbb{Z}_+$ with a left approximate unit are characterized, when ϕ is a homeomorphism.

Proposition 2.5 *Let $\mathcal{I} \sim \{X_n\}_{n=1}^{\infty}$ be a nonzero ideal of $C_0(X) \times_\phi \mathbb{Z}_+$, where ϕ is a homeomorphism. The following are equivalent:*

(1) *$\mathcal{I}$ has a left approximate unit.*
(2) *There exist $S, W \subsetneq X$ such that S is closed and $\phi(S) = S$, the sets $\phi^{-1}(W)$, $\phi^{-2}(W), \dots$ are pairwise disjoint and $\phi^k(W) \cap S = \emptyset$, for all $k \in \mathbb{Z}$, and*
$$X_n = S \cup (\cup_{k=n}^{\infty} \phi^{-k}(W)),$$
for all $n \in \mathbb{Z}_+$.

Proof The second condition implies the second condition of Theorem 2.2 and hence the implication (2) ⇒ (1) is immediate. We will prove the implication (1) ⇒ (2).

We set $S = \cap_{n=0}^{\infty} X_n$. Clearly, the set S is closed and, by $(*)$, we have $\phi(S) \subseteq S$. We will prove that $\phi(S) = S$. We suppose $\phi(S) \subsetneq S$. Since ϕ is surjective, there exists $x \in X\backslash S$ such that $\phi(x) \in S$. Moreover, $\phi^n(x) \in S$ for all $n \geq 1$. However, since $x \notin S$, there exists n_0 such that $x \notin X_{n_0}$ and hence $\phi^{n_0}(x) \in X\backslash X_0$, by Theorem 2.2, which is a contradiction since $S \cap (X\backslash X_0) = \emptyset$.

By Theorem 2.2, $\phi(X_{n+1}\backslash X_{n+2}) = X_n\backslash X_{n+1}$ for all $n \in \mathbb{Z}_+$ and hence $\phi^n(X_n\backslash X_{n+1}) = X_0\backslash X_1$ or equivalently $X_n\backslash X_{n+1} = \phi^{-n}(X_0\backslash X_1)$ since ϕ is a homeomorphism. Furthermore, the sets $\phi^{-1}(X_0\backslash X_1), \phi^{-2}(X_0\backslash X_1), \dots$ are pairwise disjoint.

We set $W = X_0\backslash X_1$. Clearly, $\phi^k(W) \cap S = \emptyset$ for all $k \in \mathbb{Z}$, since $\phi(S) = S$ and $\phi(W) \subseteq X\backslash X_0$. Also, $X_0 = S \cup (X_0\backslash X_1) \cup (X_1\backslash X_2) \cup \cdots$ and hence
$$X_0 = S \cup (\cup_{k=0}^{\infty} \phi^{-k}(W)).$$
Finally, for all $n \in \mathbb{Z}_+$ we have that
$$X_0 = X_n \cup (\cup_{k=1}^{n}(X_{k-1}\backslash X_k)) = X_n \cup (\cup_{k=1}^{n} \phi^{-k+1}(W)) = X_n \cup (\cup_{k=0}^{n-1} \phi^{-k}(W)),$$
and so
$$X_n = X_0\backslash(\cup_{k=0}^{n-1} \phi^{-k}(W)) = S \cup (\cup_{k=n}^{\infty} \phi^{-k}(W)).$$ ■

In the following corollary, the ideals with an approximate unit are characterized.

Corollary 2.6 *Let $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ be a nonzero ideal of $C_0(X) \times_\phi \mathbb{Z}_+$. The following are equivalent:*

(1) *$\mathcal{I}$ has an approximate unit.*
(2) *$X_n = X_{n+1}$, for all $n \in \mathbb{Z}_+$, and $\phi(X\backslash X_0) = X\backslash X_0$.*

Proof (1) ⇒ (2) is immediate from Theorem 2.1 and Corollary 2.4.

We show (2) ⇒ (1). If $X_n = X_{n+1}$, by $(*)$, we have $\phi(X_0) \subseteq X_0$. Since $\phi(X\backslash X_0) = X\backslash X_0$ and ϕ surjective, we have $\phi(X_0) = X_0$. Theorem 2.1 and Corollary 2.4 conclude the proof. ■

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Let B be a Banach space, and let C be a subspace of B. The set of linear functionals that vanish on a subspace C of B is called the *annihilator* of C. A subspace C of a Banach space B is an M-*ideal* in B if its annihilator is the kernel of a projection P on B^* such that $\|y\| = \|P(y)\| + \|y - P(y)\|$, for all y, where B^* is the dual space of B.

Effros and Ruan proved that the M-ideals in a unital operator algebra are the closed two-sided ideals with an approximate unit [5, Theorem 2.2]. Therefore, we obtain the following corollary about the M-ideals of a semicrossed product.

Corollary 2.7 *Let $\mathcal{I} \sim \{X_n\}_{n=0}^{\infty}$ be a nonzero ideal of $C(X) \times_\phi \mathbb{Z}_+$, where X is compact. The following are equivalent:*

(1) *$\mathcal{I}$ is an M-ideal.*
(2) *$\mathcal{I}$ has an approximate unit.*
(3) *$X_n = X_{n+1}$, for all $n \in \mathbb{Z}_+$, and $\phi(X \backslash X_0) = X \backslash X_0$.*

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