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A FOURTH-ORDER PARABOLIC EQUATION WITH A LOGARITHMIC NONLINEARITY

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We consider some generalisations of the Cahn—Hilliard equation based on constitutive equations derived by M. Gurtin in (1996) with a logarithmic free energy. Compared to the classical Cahn—Hilliard equation (see [4, 5]), these models take into account the work of internal microforces and the anisotropy of the material. We obtain the existence and uniqueness of solutions results and then prove the existence of finite dimensional attractors.

1. INTRODUCTION

We consider a generalisation of the Cahn–Hilliard equation, which is a conservation law (in the sense that the average of the order parameter is conserved). This equation is based on constitutive equations proposed by M. Gurtin in [10], and describes very important qualitative features of two-phase systems, namely the transport of atoms between unit cells when we take into account the work of the internal microforces and the anisotropy of the material. These derivations are based on belief that fundamental physical laws involving energy should account for the work associated with each kinematical process (the order parameter in our case). See [10] for more details on this theory, and [12] where the full nonlinear partial differential equations are derived. Most of the mathematical literature on the Cahn—Hilliard equation (and also generalisations of the Cahn—Hilliard equation) has concentrated on a polynomial nonlinearity or on more general assumptions on the potential f excluding a logarithmic nonlinearity (see for instance [3, 12, 13] and the references therein). Some results concerning the Cahn—Hilliard equation with logarithmic potentials can be found in [1, 2, 7, 9].

Many equations arising from mechanics and physics possess a global attractor, which is a compact invariant set which attracts uniformly the trajectories as time goes to infinity, and thus appears as a suitable object for the study of the asymptotic behaviour of the system. An important issue is then to study the dimension, in the sense of the fractal dimension or the Hausdorff dimension, of the global attractor. Indeed, we would then obtain an estimate of the number of degrees of freedom of the system. To this purpose,

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one approach is to prove the existence of an exponential attractor, which is a compact positively invariant set which attracts exponentially the trajectories (and is thus more stable than the global attractor under perturbations and numerical approximations), contains the global attractor and has finite fractal dimension (see [8] for more details on exponential attractors).

We set $\Omega = \prod_{i=1}^{n}]0, L_i[$, $L_i > 0$, $i = 1, \dots, n$, $n = 2$ or 3, and consider the following system:

$$(1.1) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} - d.\nabla \dfrac{\partial \rho}{\partial t} - \operatorname{div}\Big(\widetilde{B}\nabla \dfrac{\partial \rho}{\partial t}\Big) + \alpha \operatorname{div}(B\nabla\Delta\rho) - \operatorname{div}\big(B\nabla f'(\rho)\big) = 0, \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ is } \Omega - \text{periodic}, \end{cases}$$

where B and $\widetilde{B}$ are two symmetric positive definite tensors with constant coefficients (B is called mobility tensor), d is a constant vector, ρ is the order parameter (corresponding to a density of atoms). The free energy $f : [-1, 1] \to \mathbb{R}$: is given by

$$(1.2) \qquad f(s) = \begin{cases} -\dfrac{\theta_c}{2}s^2 + \dfrac{\theta}{2}\big[(1+s)\ln(1+s) + (1-s)\ln(1-s)\big], \; s \in]-1, 1[; \\ f(-1) = f(1) = \theta \ln 2 - \dfrac{\theta_c}{2}; \end{cases}$$

where $0 < \theta < \theta_c$.

For the mathematical setting of the problem, we denote by $\|.\|$ and $(.,.)$ the usual norm and the scalar product in $L^2(\Omega)$. For each $\rho \in L^1(\Omega)$, $m(\rho)$ stands for the average of ρ, that is, $m(\rho) = 1/|\Omega| \int_\Omega \rho(x)\,dx$, and, for a space X, we denote by $\dot{X}$ the space $\{q \in X,\, m(q) = 0\}$. We set $\overline{q} = q - m(q)$, $\forall q \in L^1(\Omega)$. We define by $N = -\operatorname{div}B\nabla$ a linear, selfadjoint, strictly positive operator with compact inverse N^{-1} on $\dot{H}^2_{\text{per}}(\Omega)$. We note that N and $\dfrac{\partial}{\partial x_i}$, and thus N^{-1} and $\dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n$, commute.

The layout of this paper is as follows. In Section 2, we recall some results we already know. In Section 3 we study a regularised problem where the potential f is replaced by a regular function f_m as in [7]. In Section 4, we derive uniform a priori estimates in m for the approximate solutions which unable us to pass to the limit in the approximate problem to get the existence and uniqueness of a weak solution as stated in Theorem 4.1. Section 5 is devoted to the study of the existence of the global attractor. Finally in Section 6, we prove that the fractal dimension of the global attractor is finite by studying the existence of exponential attractors. Throughout, the same letter c (and sometimes c_i, $i = 0, 1, 2, \dots$) shall denote positive constants that may change from line to line.

2. Preliminary results

We first recall the following result which is proved in [3].

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Proposition 2.1.

(i) *The mapping $q \mapsto \|\nabla B^{1/2}\nabla q\|$ defines a norm on $\dot{H}^2_{\rm per}(\Omega)$ that is equivalent to the usual H^2-norm.*

(ii) *We have, for every $q \in \dot{L}^2(\Omega)$, $\big(q, -\mathrm{div}(\widetilde{B}\nabla N^{-1}q)\big) = \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}q\|^2$. Furthermore, the mapping $q \mapsto \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}q\|^2$ defines a norm on $\dot{L}^2(\Omega)$ that is equivalent to the usual L^2-norm.*

To study this problem, we rewrite (1.1) in the following form

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + NF\Big(\rho, \dfrac{\partial\rho}{\partial t}\Big) = 0, \\ F\Big(\rho, \dfrac{\partial\rho}{\partial t}\Big) = -N^{-1}\mathrm{div}\Big(\widetilde{B}\nabla\dfrac{\partial\rho}{\partial t}\Big) - d.\nabla N^{-1}\dfrac{\partial\rho}{\partial t} - \alpha\Delta\rho + f'(\rho), \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ is } \Omega\text{-periodic.} \end{cases} \tag{2.1}$$

We take formally, but the calculations can be easily justified, the L^2-scalar product of the first equation of (2.1) with $F\Big(\rho, \dfrac{\partial\rho}{\partial t}\Big)$. Noting that $(d.\nabla q, q) = 0, \quad \forall q \in \dot{L}^2(\Omega)$, we obtain

$$\frac{d}{dt}J(\rho) + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho}{\partial t}\Big\|^2 + \Big\|N^{1/2}F\Big(\rho, \frac{\partial\rho}{\partial t}\Big)\Big\|^2 = 0, \tag{2.2}$$

where

$$J(\rho) = \int_\Omega \Big(f(\rho) + \frac{\alpha}{2}|\nabla\rho|^2\Big)\,dx. \tag{2.3}$$

We then deduce that $J(\rho)$ is a Lyapunov function for (1.1). Apart from its interest from the physical point of view, this result is useful to obtain informations on the structure of the global attractor (see for instance [**13**]).

Due to the spectral properties of operator N, there exists a basis of eigenvectors $\{e_j\}_{\{j\in\mathbb{N}\}}$ on $H^2_{\rm per}(\Omega)$ which is orthonormal in $L^2(\Omega)$ and is associated with the eigenvalues $\{\lambda_j\}_{\{j\in\mathbb{N}\}}$ such that

$$\begin{cases} Ne_j = \lambda_j e_j, \ j = 1, 2, \dots \\ 0 = \lambda_0 < \lambda_1 \leqslant \lambda_2, \dots, \ \lambda_j \to \infty. \end{cases}$$

We then set, for $\rho = \sum_{j=0}^{+\infty} \rho_j\, e_j$, and for all $s \in \mathbb{R}$, $N^s\rho = \sum_{j=1}^{+\infty} \lambda_j^s\, \rho_j\, e_j$, and

$$V_s = \Big\{\rho = \sum_{j=0}^{+\infty} \rho_j\, e_j,\ \sum_{j=1}^{\infty} \lambda_j^s\, \rho_j^2 < +\infty\Big\}.$$

We endow V_s with the seminorm $|\rho|_s = \|N^{s/2}\rho\|$, the semiscalar product $(u, v)_s = (N^{s/2}u, N^{s/2}v)$ and the norm $\|\rho\|_s = \big(|\rho|_s^2 + m(\rho)^2\big)^{1/2}$. Note that $V_0 = L^2(\Omega)$;

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$V_1 = H^1_{\rm per}(\Omega)$, $V_2 = H^2_{\rm per}(\Omega)$ and $\|.\|_s$ is a norm equivalent to the usual norm of $H^s(\Omega)$. Also, V_{-s} is the dual space of V_s.

The following results can be found in [7].

LEMMA 2.1.

(i) *We have* $|\rho|_s^2 = \|\overline{\rho}\|_s^2 = \sum_{j=1}^{\infty} \lambda_j^s \rho_j^2$.

(ii) $\forall\, u, v \in L^2(\Omega)$, *we have*

$$\begin{cases} (u,v)_0 = (u,v) - m(u)m(v)|\Omega| \\ (u,v)_0 = (\overline{u},v) = (u,\overline{v}) = (\overline{u},\overline{v}). \end{cases}$$

(iii) *There exists a constant* $C_0 > 0$ *such that* $|u|_1 \leqslant C_0\|\nabla u\|, \forall u \in H^1_{\rm per}(\Omega)$.

(iv) *We have*

$$\begin{cases} |u|_s \leqslant c|u|_{s'},\ c > 0,\ s \leqslant s', \quad \forall u \in V_{s'}, \\ |u|_{\lambda s_1 + (1-\lambda)s_2} \leqslant |u|_{s_1}^{\lambda} |u|_{s_2}^{1-\lambda},\ s_1 \leqslant s_2,\ \lambda \in [0,1], \quad \forall u \in V_{s_2}. \end{cases}$$

The following proposition is proved in [6].

PROPOSITION 2.2. *The norms* $\|.\|_2$ *and* $\left(\|\nabla B^{1/2}\nabla\rho\|^2 + \|\rho\|^2\right)^{1/2}$ *are equivalent to the usual* H^2*-norm on* $H^2_{\rm per}(\Omega)$. *Furthermore, there exists constants* c_1 *and* c_2 *which can be chosen independently of* Ω *such that* $c_1\|\rho\|_2 \leqslant \|\rho\|_{H^2_{\rm per}(\Omega)} \leqslant c_2\|\rho\|_2$, $\forall \rho \in H^2_{\rm per}(\Omega)$.

3. A REGULARISED PROBLEM

We denote by ψ and ϕ the functions

$$\psi(s) = (1+s)\ln(1+s) + (1-s)\ln(1-s), \tag{3.1}$$

and $\phi(s) = \psi'(s)$, for $s \in]-1,1[$. We then have $f(s) = -(\theta_c/2)s^2 + (\theta/2)\psi(s)$ and $f'(s) = -\theta_c s + (\theta/2)\phi(s)$.

The major difficulty in the study of problem (1.1) is that $\phi(s)$ is singular at $s = \pm 1$ and, therefore, has no meaning if $\rho = \pm 1$ in an open set of non-zero measure. To overcome this difficulty, we consider a regularised problem as in [7]. The logarithmic free energy $f(s)$ is replaced by the polynomial function $f_m(s) = -(\theta_c/2)s^2 + (\theta/2)\psi_m(s)$, where $m \in \mathbb{N}$, and

$$\psi_m(s) = 2\sum_{k=0}^{m} \frac{s^{2k+2}}{(2k+1)(2k+2)},\ s \in]-1,1[. \tag{3.2}$$

The idea is to study the approximate problem and derive uniform estimates (in m) and then pass to the limit $m \to +\infty$. This approach is justified by the fact that f can be

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written as $f(s) = -(\theta_c/2)s^2 + \theta \sum_{k=0}^{+\infty} \big(s^{2k+2}/(2k+1)(2k+2)\big)$, for $s \in]-1,1[$. We then have $\phi_m(s) = 2\sum_{k=0}^{m} (s^{2k+1})/(2k+1)$, and consider the following problem

$$\frac{\partial \rho_m}{\partial t} + NF_m\Big(\rho_m, \frac{\partial \rho_m}{\partial t}\Big) = 0, \tag{3.3}$$

$$F_m\Big(\rho_m, \frac{\partial \rho_m}{\partial t}\Big) = -N^{-1}\mathrm{div}\Big(\widetilde{B}\nabla \frac{\partial \rho_m}{\partial t}\Big) - d.\nabla N^{-1}\frac{\partial \rho_m}{\partial t} - \alpha\Delta\rho_m - \theta_c\rho_m + \frac{\theta}{2}\phi_m(\rho_m), \tag{3.4}$$

$$\rho_m|_{t=0} = \rho_0, \tag{3.5}$$

$$\rho_m \text{ is } \Omega-\textit{periodic}. \tag{3.6}$$

The main aim of this paper is to extend the results of [7] and [6] to system (1.1). First, we note that the approximate potentials f_m satisfy conditions (3.1) to (3.5) on the potential f of [**3**, Theorem 3]; and therefore, this theorem is applicable to system (3.3)–(3.6). We then have the following result.

THEOREM 3.1. *We assume that $\rho_0 \in H^1_{\mathrm{per}}(\Omega)$. Then, for all $m \in \mathbb{N}$, (3.3)–(3.6) possesses a unique solution ρ_m satisfying $\rho_m \in \mathcal{C}\big([0,T]; H^1_{\mathrm{per}}(\Omega)\big) \cap L^2\big(0,T; H^2_{\mathrm{per}}(\Omega)\big)$ and $\dfrac{\partial \rho_m}{\partial t} \in L^2\big(0,T;L^2(\Omega)\big)$. If furthermore $\rho_0 \in H^2_{\mathrm{per}}(\Omega)$, then $\rho_m \in \mathcal{C}\big([0,T]; H^2_{\mathrm{per}}(\Omega)\big)$ and $\dfrac{\partial \rho_m}{\partial t} \in L^2\big(0,T; H^1_{\mathrm{per}}(\Omega)\big)$.*

We have the conservation property:

$$m\big(\rho_m(t)\big) = m(\rho_0), \quad \forall t \geqslant 0, \tag{3.7}$$

which follows from (3.3) by taking the L^2-scalar product with 1. We deduce from Section 2 that

$$J_m(\rho_m) = \frac{\alpha}{2}\|\nabla\rho_m\|^2 - \frac{\theta_c}{2}\int_\Omega \rho_m^2\,dx + \frac{\theta}{2}\int_\Omega \psi_m(\rho_m)\,dx. \tag{3.8}$$

We have

$$\begin{aligned}\frac{\theta}{4}\int_\Omega \psi_m(\rho_m)\,dx - \frac{\theta_c}{2}\int_\Omega \rho_m^2\,dx &= \frac{\theta}{2}\int_\Omega \sum_{k=0}^{m} \frac{\rho_m^{2k+2}}{(2k+1)(2k+2)} - \frac{\theta_c}{2}\int_\Omega \rho_m^2\,dx \\ &= \int_\Omega \Big\{\theta\Big(\frac{\rho_m^2}{4} + \frac{\rho_m^4}{24} + \frac{\rho_m^6}{60} + \cdots +\Big) - \frac{\theta_c}{2}\rho_m^2\Big\}\,dx \\ &\geqslant \int_\Omega \Big\{\frac{\theta}{24}\rho_m^4 - \frac{\theta_c}{2}\rho_m^2\Big\}\,dx,\end{aligned} \tag{3.9}$$

and, using the inequality $(\theta/24)y^2 - (\theta_c/2)y + (3\theta_c^2/2\theta) \geqslant 0$, $\forall y \in \mathbb{R}$, we obtain

$$\frac{\theta}{4}\int_\Omega \psi_m(\rho_m)\,dx - \frac{\theta_c}{2}\int_\Omega \rho_m^2\,dx \geqslant -\frac{3\theta_c^2}{2\theta}|\Omega|. \tag{3.10}$$

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Therefore,

$$J_m(\rho_m) \geqslant \frac{\alpha}{2}\|\nabla \rho_m\|^2 + \frac{\theta}{4}\int_\Omega \psi_m(\rho_m)\,dx - \frac{3\theta_c^2}{2\theta}|\Omega|. \tag{3.11}$$

From now, we suppose that $d = 0$. We take the L^2–scalar product of (3.3) with $\frac{\partial}{\partial t}F_m\Big(\rho_m, \frac{\partial \rho_m}{\partial t}\Big)$, and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 &+ \frac{1}{2}\frac{d}{dt}\Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\Big\|^2\\ &+\frac{\alpha}{2}\Big\|\nabla\frac{\partial\rho_m}{\partial t}\Big\|^2 - \theta_c\Big\|\frac{\partial\rho_m}{\partial t}\Big\|^2 + \frac{\theta}{2}\Big(\frac{\partial\rho_m}{\partial t}\phi_m'(\rho_m),\frac{\partial\rho_m}{\partial t}\Big) = 0.\end{aligned} \tag{3.12}$$

Noting that $\phi_m'(s) \geqslant 0$, $\forall s \in \mathbb{R}$; and then $\Big(\frac{\partial\rho_m}{\partial t}\phi_m'(\rho_m),\frac{\partial\rho_m}{\partial t}\Big) \geqslant 0$; we finally obtain the estimate

$$\frac{d}{dt}\Big(\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 + \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\|^2\Big) + \alpha\Big\|\nabla\frac{\partial\rho_m}{\partial t}\Big\|^2 \leqslant 2\theta_c\Big\|\frac{\partial\rho_m}{\partial t}\Big\|^2. \tag{3.13}$$

Furthermore, we have

$$\Big\|\frac{\partial\rho_m}{\partial t}\Big\|^2 \leqslant \Big|\frac{\partial\rho_m}{\partial t}\Big|_{-1}\Big|\frac{\partial\rho_m}{\partial t}\Big|_1, \text{ and } \Big|\frac{\partial\rho_m}{\partial t}\Big|_{-1}^2 = \Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2.$$

We then obtain

$$\begin{aligned}\frac{d}{dt}\Big(\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\Big\|^2\Big) &+ \frac{\alpha}{C_0}\Big|\frac{\partial\rho_m}{\partial t}\Big|_1^2\\ &\leqslant 2\theta_c\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1\Big|\frac{\partial\rho_m}{\partial t}\Big|_1,\end{aligned} \tag{3.14}$$

and, therefore

$$\begin{aligned}\frac{d}{dt}\Big(\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\Big\|^2\Big) &+ \frac{\alpha}{2C_0}\Big|\frac{\partial\rho_m}{\partial t}\Big|_1^2\\ &\leqslant \frac{2C_0\theta_c^2}{\alpha}\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2\\ &\leqslant \frac{2C_0\theta_c^2}{\alpha}\Big(\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\Big\|^2\Big).\end{aligned} \tag{3.15}$$

4. Existence and uniqueness of solutions

We want to give in this section a result on the existence and uniqueness of solution for system (1.1). To this purpose, we need to derive several a priori estimates uniform in m for solutions ρ_m, for all $m \in \mathbb{N}$. We assume that ρ_0 satisfies $\rho_0 \in H^1_{\text{per}}(\Omega)$, $\|\rho_0\|_{L^\infty(\Omega)} \leqslant 1$, and $m(\rho_0) \in]-1, 1[$. The regularised counterpart of (2.2) is

$$\frac{d}{dt}J_m(\rho_m) + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_m}{\partial t}\Big\|^2 + \Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2 = 0, \tag{4.1}$$

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where we have

$$J_m(\rho_m(t)) \leqslant J_m(\rho_0) \leqslant J(\rho_0) < +\infty, \quad \forall t \geqslant 0, \tag{4.2}$$

and, thanks to (3.11),

$$\frac{\alpha}{2}\|\nabla \rho_m\|^2 \leqslant J(\rho_0) + \frac{3\theta_c^2}{2\theta}|\Omega|. \tag{4.3}$$

We deduce that $|\rho_m|_1 \leqslant c$, and therefore

$$\|\rho_m\|_1 \leqslant c, \tag{4.4}$$

where c are independent of m. Integrating (4.1) with respect to t, $t \in [0, T]$, we obtain

$$J_m(\rho_m(t)) + \int_0^t \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial s}\right\|^2 ds + \int_0^t \left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial s}\right)\right|_1^2 ds = J_m(\rho_0), \tag{4.5}$$

therefore

$$\int_0^T \left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 dt = \int_0^T \left\|\overline{F}_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right\|_1^2 dt \leqslant c, \tag{4.6}$$

and $\int_0^T \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2 dt \leqslant c$, and therefore

$$\int_0^T \left\|\frac{\partial \rho_m}{\partial t}\right\|^2 dt \leqslant c, \tag{4.7}$$

where c is independent of m. We multiply (3.15) by t and obtain after calculation

$$\begin{aligned} \frac{d}{dt}\left(t\left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 + t\left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2\right) + \frac{\alpha t}{2C_0}\left|\frac{\partial \rho_m}{\partial t}\right|_1^2 \\ \leqslant \frac{2C_0\theta_c^2 t}{\alpha}\left(\left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 + \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2\right) \\ + \left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 + \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2. \end{aligned} \tag{4.8}$$

Using Gronwall's lemma and estimates (4.3) to (4.7), we obtain

$$\sqrt{t}\left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1 = \sqrt{t}\left\|\overline{F}_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right\|_1 \leqslant c, \tag{4.9}$$

and $\sqrt{t}\left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\| \leqslant c$, and therefore

$$\sqrt{t}\left\|\frac{\partial \rho_m}{\partial t}\right\| \leqslant c, \tag{4.10}$$

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where c is independent of m. We now take the $L^2(\Omega)$-semiscalar product of (3.4) with $\phi_m(\rho_m)$, and obtain

$$(4.11)\quad \left(F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right), \phi_m(\rho_m)\right)_0 = -\left(N^{-1}\mathrm{div}\left(\widetilde{B}\nabla \frac{\partial \rho_m}{\partial t}\right), \phi_m(\rho_m)\right)_0 - \alpha\big(\Delta\rho_m, \phi_m(\rho_m)\big)_0 - \theta_c\big(\rho_m, \phi_m(\rho_m)\big)_0 + \frac{\theta}{2}\left|\phi_m(\rho_m)\right|_0^2.$$

Noting that $\big(\Delta\rho_m, \phi_m(\rho_m)\big)_0 = -\int_\Omega |\nabla\rho_m|^2\phi_m'(\rho_m)\,dx \leqslant 0$, and the equivalence of norms $\left\|\mathrm{div}\left(\widetilde{B}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right)\right\|$ and $\left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|$, we obtain

$$(4.12)\quad \frac{\theta}{2}\left|\phi_m(\rho_m)\right|_0^2 \leqslant \frac{\theta}{4}\left|\phi_m(\rho_m)\right|_0^2 + c_1\left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_0^2 + c_2|\rho_m|_0^2 + c_3\left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2,$$

and therefore

$$(4.13)\quad \sqrt{t}\left|\phi_m(\rho_m)\right|_0 = \sqrt{t}\left\|\overline{\phi}_m(\rho_m)\right\|_0 \leqslant c,$$

where c is independent of m. We finally take the L^2-scalar product of $F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)$ with $N\rho_m$ obtaining

$$(4.14)\quad \frac{1}{2}\frac{d}{dt}\|\widetilde{B}^{1/2}\nabla\rho_m\|^2 + \alpha\|\nabla B^{1/2}\nabla\rho_m\|^2 - \theta_c|\rho_m|_1^2 + \frac{\theta}{2}(\phi_m(\rho_m), N\rho_m) = \left(F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right), N\rho_m\right).$$

Noting that $(\phi_m(\rho_m), N\rho_m) = \int_\Omega \phi_m'(\rho_m)|B^{1/2}\nabla\rho_m|^2\,dx \geqslant 0$, we deduce the estimate

$$(4.15)\quad \frac{1}{2}\frac{d}{dt}\|\widetilde{B}^{1/2}\nabla\rho_m\|^2 + \alpha\|\nabla B^{1/2}\nabla\rho_m\|^2 \leqslant (\theta_c+1)|\rho_m|_1^2 + \left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2,$$

and then $\int_0^T\left(\|\nabla B^{1/2}\nabla\rho_m\|^2 + \|\rho_m\|^2\right)dt \leqslant c$ thanks to (4.4), and therefore

$$(4.16)\quad \int_0^T \|\rho_m\|_2^2\,dt \leqslant c,$$

where c is independent of m. The L^2-scalar product of $F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)$ with $N\rho_m$ and the fact that $\left(-\mathrm{div}\left(\widetilde{B}\nabla\frac{\partial \rho_m}{\partial t}\right),\ \rho_m\right) = \left(\frac{\partial \rho_m}{\partial t}, -\mathrm{div}(\widetilde{B}\nabla\rho_m)\right)$ also give

$$(4.17)\quad \left(F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right), N\rho_m\right) = \left(\frac{\partial \rho_m}{\partial t}, -\mathrm{div}(\widetilde{B}\nabla\rho_m)\right) - \alpha(\Delta\rho_m, N\rho_m) - \theta_c(\rho_m, N\rho_m) + \frac{\theta}{2}(\phi_m(\rho_m), N\rho_m),$$

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and therefore, thanks to the equivalence of norms

$$\Big(\big\|\operatorname{div}(\widetilde{B}\nabla\rho_m)\big\|^2+\|\rho_m\|^2\Big)^{1/2},\qquad \big(\|\nabla B^{1/2}\nabla\rho_m\|^2+\|\rho_m\|^2\big)^{1/2}$$

and $\|\rho_m\|_{H^2_{\text{per}}(\Omega)}$ on $H^2_{\text{per}}(\Omega)$, and estimates (4.4), (4.9) and (4.10)

$$\begin{aligned}
\alpha t\|\nabla B^{1/2}\nabla\rho_m\|^2 &\leqslant (\theta_c+1)t|\rho_m|_1^2+t\Big|F_m\Big(\rho_m,\frac{\partial\rho_m}{\partial t}\Big)\Big|_1^2+t\Big\|\frac{\partial\rho_m}{\partial t}\Big\|\big\|\operatorname{div}(\widetilde{B}\nabla\rho_m)\big\|\\
&\leqslant c_1+c_2(\varepsilon)t\Big\|\frac{\partial\rho_m}{\partial t}\Big\|^2+\varepsilon t\Big(\big\|\operatorname{div}(\widetilde{B}\nabla\rho_m)\big\|^2+\|\rho_m\|^2\Big)\\
&\leqslant c_3(\varepsilon)+\varepsilon tc_4\big(\|\nabla B^{1/2}\nabla\rho_m\|^2+\|\rho_m\|^2\big),\quad \forall\varepsilon>0.
\end{aligned}\tag{4.18}$$

With a suitable choice of ε we deduce that $t\big(\|\nabla B^{1/2}\nabla\rho_m\|^2+\|\rho_m\|^2\big)\leqslant c$, and therefore

$$\sqrt{t}\|\rho_m\|_2\leqslant c,\tag{4.19}$$

where c is not depend on m.

We have the following result.

Theorem 4.1. *We assume that $\rho_0\in H^1_{\text{per}}(\Omega)$ and satisfy $\|\rho_0\|_{L^\infty(\Omega)}\leqslant 1$, and $m(\rho_0)\in]-1,1[$. Then, (2.1) with $d=0$, possesses a unique solution ρ such that $\rho\in\mathcal{C}\big([0,T];H^1_{\text{per}}(\Omega)\big)\cap L^2\big(0,T;H^2_{\text{per}}(\Omega)\big)$, and $\dfrac{\partial\rho}{\partial t}\in L^2\big(0,T;L^2(\Omega)\big)$, $\forall T>0$. Furthermore, we have $F\Big(\rho,\dfrac{\partial\rho}{\partial t}\Big)\in L^2\big(0,T;H^1_{\text{per}}(\Omega)\big)$, $\big\|\rho(t)\big\|_{L^\infty(\Omega)}\leqslant 1$, $\forall t\geqslant 0$, and the set $\Big\{x\in\Omega,\ \big|\rho(x,t)\big|=1\Big\}$ have zero mesure for $t>0$.*

Proof: The proof is similar to that of [7] for the classical Cahn-Hilliard equation. We just give a sketch of the proof. The uniqueness follows from the standard method which consists of studying the equation of the difference of two solutions ρ_1 and ρ_2; and noting that $(\phi(\rho_1)-\phi(\rho_2),\rho_1-\rho_2)\geqslant\theta\|\rho_1-\rho_2\|^2$. The existence of solutions follows from the limit $m\to\infty$ in the approximate problem (3.3)–(3.6). By Theorem 3.1, we know that, for all $m\in\mathbb{N}$, there exists a unique solution such that $\rho_m\in\mathcal{C}\big([0,T];H^1_{\text{per}}(\Omega)\big)\cap L^2\big(0,T;H^2_{\text{per}}(\Omega)\big)$ and $\dfrac{\partial\rho_m}{\partial t}\in L^2\big(0,T;L^2(\Omega)\big)$. The major difficulty is the passage to the limit in the nonlinear term. Due to the uniform (in m) a priori estimates we obtained, we prove that there exists a subsequence (which we still denote by $\{\rho_m\}_{m\in\mathbb{N}}$) and a function ρ such that

$$\begin{aligned}
&\rho_m\to\rho \text{ in } L^2\big(0,T;H^1_{\text{per}}(\Omega)\big) \text{ strongly and almost everywhere in } \Omega\times]0,T[;\\
&\rho_m\rightharpoonup\rho \text{ in } L^2\big(0,T;H^2_{\text{per}}(\Omega)\big) \text{ weakly,}\\
&\rho_m\rightharpoonup\rho \text{ in } L^\infty\big(0,T;H^1_{\text{per}}(\Omega)\big) \text{ weakly-star}\\
&\frac{\partial\rho_m}{\partial t}\rightharpoonup\frac{\partial\rho}{\partial t} \text{ in } L^2\big(0,T;L^2(\Omega)\big) \text{ weakly,}\\
&\phi_m(\rho_m)\rightharpoonup\phi(\rho) \text{ almost everywhere in } \Omega\times]0,T[\text{ and in } \mathcal{D}'(\Omega\times]0,T[)
\end{aligned}$$

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Passing to the limit, we obtain that ρ is a solution of (1.1). Furthermore, we have $\rho \in \mathcal{C}([0,T];L^2(\Omega))$ and the map $\rho(t) : [0,T] \to H^1_{\text{per}}(\Omega)$ is weakly continuous (see compactness results of [**11**]). To conclude that $\rho \in \mathcal{C}([0,T];H^1_{\text{per}}(\Omega))$ we prove that the map $t : [0,T] \to |\rho(t)|_1$ is continuous. The others points of theorem are justified in a similar way as in [**7**]. ∎

5. Existence of the global attractor

We first recall the definition of the global attractor for a semigroup. Let E be a Banach space.

Definition 5.1: The set $\mathcal{A}$ is called a global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$ if $\mathcal{A}$ is compact in E; $\mathcal{A}$ is strictly invariant, that is, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$; and $\mathcal{A}$ is an attracting set for $\{S(t)\}_{t\geqslant 0}$ in the following sense: for any bounded set $B \subset E$ the Hausdorff distance

$$\operatorname{dist}_E(S(t)B,\mathcal{A}) \to 0, \text{ when } t \to \infty.$$

We denote by $V_0^\sigma = \left\{\rho \in L^2(\Omega);\ |m(\rho)| \leqslant \sigma\right\}$ and by $V_1^\sigma = \left\{\rho \in H^1_{\text{per}}(\Omega);\ |m(\rho)| \leqslant \sigma\right\}$, for $\sigma \geqslant 0$. We endow these sets by the norms of $L^2(\Omega)$ and $H^1_{\text{per}}(\Omega)$ respectively. Thanks to Theorem 4.1, we can define the semigroup $\{S(t)\}_{t\geqslant 0}$: $S(t) : V_1^\sigma \to V_1^\sigma$, $\rho_0 \mapsto \rho(t)$, $\sigma < 1$. We easily prove that $S(t)$ is continuous for the norm of $H^1_{\text{per}}(\Omega)$ (and also for the norm of $L^2(\Omega)$ if we extend $S(t)$ to $L^2(\Omega)$). We actually consider the restriction of $S(t)$ to the space $\{\rho \in L^\infty(\Omega); \|\rho\|_{L^\infty(\Omega)} \leqslant 1\}$. We now prove the existence of bounded absorbing sets. We take the L^2-scalar product of (3.4) with $N^{-1}\overline{\rho}_m$, $\overline{\rho}_m = \rho_m - m(\rho_0)$, and obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{\rho}_m\|_{-1}^2 + \left(F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right), \overline{\rho}_m\right) = 0. \tag{5.1}$$

and therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\overline{\rho}_m\|_{-1}^2 + \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\overline{\rho}_m\|^2\right) + \frac{\alpha}{2}\|\nabla\rho_m\|^2 \\ - \frac{\theta_c}{2}\|\rho_m\|^2 + J_m(\rho_m) \leqslant \theta|\Omega|\sum_{k=0}^{m}\frac{m(\rho_0)^{2k+2}}{(2k+1)(2k+2)}.\end{aligned} \tag{5.2}$$

From (3.11) we have $J(\rho_m) - (\theta_c/2)\|\rho_m\|^2 \geqslant \alpha/2\|\nabla\rho_m\|^2 - (3\theta_c^2/\theta)|\Omega|$. Thanks to inequalities $\|\overline{\rho}_m\|_{-1} \leqslant c\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\overline{\rho}_m\| \leqslant c'\|\nabla\rho_m\|$, we deduce the estimate

$$\begin{aligned}\frac{d}{dt}\left(\|\overline{\rho}_m\|_{-1}^2 + \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\overline{\rho}_m\|^2\right) \\ + C_1\left(\|\overline{\rho}_m\|_{-1}^2 + \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\overline{\rho}_m\|^2\right) \leqslant D_\sigma^m,\end{aligned} \tag{5.3}$$

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where $C_1 > 0$ and

$$D_\sigma^m = 2\theta|\Omega| \sum_{k=0}^{m} \frac{\sigma^{2k+2}}{(2k+1)(2k+2)} + \frac{6\theta_c^2}{\theta}|\Omega|. \tag{5.4}$$

The following result is then obtained.

PROPOSITION 5.1. *For all $\varrho > 0$ and $\rho_0 \in L^2(\Omega)$ such that $\|\rho_0\| \leqslant \varrho$, $|m(\rho_0)| \leqslant \sigma$, there exists $T_1(\varrho) > 0$ such that the solution ρ_m of (3.3)–(3.6) satisfies*

$$\|\overline{\rho}_m(t)\|^2 \leqslant c_1 D_\sigma^m, \quad \forall t \geqslant T_1(\varrho). \tag{5.5}$$

Furthermore, if $\sigma < 1$, then the solution ρ of (1.1) satisfies

$$\|\overline{\rho}(t)\|^2 \leqslant c_1 D_\sigma, \quad \forall t \geqslant T_1(\varrho). \tag{5.6}$$

Integrating (5.2) with respect to t, we deduce that

$$\int_t^{t+1} J_m(\rho_m(s))\, ds \leqslant \frac{\theta_c}{2} \int_t^{t+1} \|\rho_m(s)\|^2 ds + \theta|\Omega| \sum_{k=0}^{m} \frac{\sigma^{2k+2}}{(2k+1)(2k+2)} + \frac{1}{2}\left(\|\overline{\rho}_m\|_{-1}^2 + \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\overline{\rho}_m\|^2\right); \tag{5.7}$$

and since J is decreasing and that $\|\rho_m\|^2 = \|\overline{\rho}_m\|^2 + m(\rho_0)^2|\Omega|$, we obtain thanks to (5.5)

$$J_m(\rho(t+1)) \leqslant C_2 D_\sigma^m + \frac{\theta_c \sigma^2}{2}|\Omega|, \quad \forall t > T_1(\varrho), \tag{5.8}$$

where $C_2 > 0$, and therefore

$$\frac{\alpha}{2}\|\nabla \rho_m(t)\|^2 \leqslant C_2 D_\sigma^m + \frac{6\theta_c^2}{\theta}|\Omega| + \frac{\theta_c\sigma^2}{2}|\Omega|, \quad \forall t > T_1(\varrho) + 1. \tag{5.9}$$

We then obtain the following result.

PROPOSITION 5.2. *We assume that ρ_0 satisfies the conditions of Proposition 5.1. Then, the solution ρ_m of (3.3)–(3.6) satisfies*

$$\|\overline{\rho}_m(t)\|_1^2 \leqslant c_2 D_\sigma^m, \quad \forall t \geqslant T_1(\varrho) + 1. \tag{5.10}$$

Furthermore, if $\sigma < 1$, then the solution ρ of (1.1) satisfies

$$\|\overline{\rho}(t)\|_1^2 \leqslant c_1 D_\sigma, \quad \forall t \geqslant T_1(\varrho) + 1. \tag{5.11}$$

Integrating (4.1) from t to $t+1$ and using (5.9), we obtain for all $t \geqslant T_1(\varrho) + 1$

$$\int_t^{t+1} \left(\left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial s}\right)\right|_1^2 + \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial s}\right\|^2 \right) ds \leqslant C_3 D_\sigma^m + \frac{\theta_c\sigma^2}{2}|\Omega|. \tag{5.12}$$

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Now, appling uniform Gronwall's lemma to (3.15), we obtain for all $t \geqslant T_1(\varrho) + 2$,

$$\left| F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 + \left\| \widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial \rho_m}{\partial t}\right\|^2 \leqslant C_4 D_\sigma^m. \tag{5.13}$$

We finally deduce from (4.17), as for (4.18), that

$$\begin{aligned} \alpha\|\nabla B^{1/2}\nabla\rho_m\|^2 &\leqslant (\theta_c+1)|\rho_m|_1^2 + \left|F_m\left(\rho_m, \frac{\partial \rho_m}{\partial t}\right)\right|_1^2 + \left\|\frac{\partial \rho_m}{\partial t}\right\| \|\operatorname{div}(\widetilde{B}\nabla\rho_m)\| \\ &\leqslant c_1 D_\sigma^m + c_2(\varepsilon)\left\|\frac{\partial \rho_m}{\partial t}\right\|^2 + \varepsilon\left(\|\operatorname{div}(\widetilde{B}\nabla\rho_m)\|^2 + \|\rho_m\|^2\right) \\ &\leqslant c_3(\varepsilon)D_\sigma^m + \varepsilon c_4\left(\|\nabla B^{1/2}\nabla\rho_m\|^2 + \|\rho_m\|^2\right), \quad \forall \varepsilon > 0. \end{aligned} \tag{5.14}$$

A proper choice of ε gives $\left(\|\nabla B^{1/2}\nabla\rho_m\|^2 + \|\rho_m\|^2\right) \leqslant cD_\sigma^m$ and therefore the following result.

PROPOSITION 5.3. *We assume that ρ_0 satisfies the conditions of Proposition 5.1. Then, the solution ρ_m of* (3.3)–(3.6) *satisfies*

$$\|\rho_m(t)\|_2^2 \leqslant c_3 D_\sigma^m, \quad \forall t \geqslant T_1(\varrho) + 2. \tag{5.15}$$

Furthermore, if $\sigma < 1$, then the solution ρ of (1.1) *satisfies*

$$\|\rho(t)\|_2^2 \leqslant c_3 D_\sigma, \quad \forall t \geqslant T_1(\varrho) + 2. \tag{5.16}$$

We have

$$D_\sigma = 2\theta|\Omega| \sum_{k=0}^{\infty} \frac{\sigma^{2k+2}}{(2k+1)(2k+2)} + \frac{6\theta_c^2}{\theta}|\Omega|. \tag{5.17}$$

and for $\sigma < 1$, we have

$$D_\sigma \leqslant \left(\frac{2\theta}{1-\sigma} + \frac{6\theta_c^2}{\theta}\right)|\Omega|; \tag{5.18}$$

and then $D_\sigma \to 0$ when $|\Omega| \to 0$.

Thanks to Proposition 5.1 to 5.3, we obtain the bounded absorbing sets in V_0^σ and in V_1^σ. Together with the continuity and the uniform compactness of the semigroup lead to the existence of the global attractor. The reader is refered to [**13**] for more details on the attractors. We have the following result.

THEOREM 5.1. *The semigroup $\{S(t)\}_{t\geqslant 0}$ possesses the global attractor $\mathcal{A}_\sigma$ in V_0^σ and in V_1^σ, $\sigma < 1$ which is bounded in $H^2_{\mathrm{per}}(\Omega)$.*

REMARK 5.1. The global attractor $\mathcal{A}_\sigma$ is the same in V_0^σ and in V_1^σ.

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6. Existence of exponential attractors

Now, we recall the definition of the exponential attractor where $\mathcal{A}$ is the global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$.

DEFINITION 6.1: A set $\mathcal{M}$ is called exponential attractor for $\{S(t)\}_{t\geqslant 0}$ if $\mathcal{M}$ is compact in E; $\mathcal{A}\subset\mathcal{M}$; $S(t)\mathcal{M}\subset\mathcal{M}$, $\forall t\geqslant 0$; for every u_0 in E, we have

$$\operatorname{dist}_E\big(S(t)u_0,\mathcal{M}\big)\leqslant c_1\exp(-c_2t),\quad \forall t\geqslant 0,$$

(here $c_1>0$, $c_2>0$ are independent of u_0); and $\mathcal{M}$ has finite fractal dimension.

To prove the existence of an exponential attractor, we use the approach of [**8**]. Rewriting (1.1) in the form

$$\frac{\partial\rho}{\partial t}+B\rho+R(\rho)=0, \tag{6.1}$$

we just have to prove that there exists a real number $\beta\in(0,1/2]$ such that, for all u,v belonging in E, we have

$$\big|R(u)-R(v)\big|_E\leqslant c\big|B^\beta(u-v)\big|_E. \tag{6.2}$$

This estimate guarantees the squeezing property for (6.1) which lead to the existence of an exponential attractor. To this purpose, we first consider a bounded absorbing set in $V_1^\sigma\cap H^2_{\text{per}}(\Omega)$ which we denote by B_σ and set

$$X_\sigma=\overline{\bigcup_{t\geqslant T_2}S(t)B_\sigma},$$

where $S(t)B_\sigma\subset B_\sigma$, $\forall t\geqslant T_2$. Since $H^2(\Omega)\subset L^\infty(\Omega)$, we deduce from Proposition 2.3 and Proposition 5.3 the following result.

PROPOSITION 6.1. *Let $\sigma\in]-1,1[$. If $|\Omega|$ is sufficiently small, there exists a constant δ, $0<\delta<1$ such that*

$$\|\rho(t)\|_{L^\infty(\Omega)}\leqslant\delta<1,\quad \forall\rho\in X_\sigma.$$

We are now in the position to give the following result.

THEOREM 6.1. *The semigroup $\{S(t)\}$ possesses an exponential attractor on X_σ. In consequence, the global attractor $\mathcal{A}_\sigma$ obtained in Theorem 5.1 has finite fractal dimension.*

PROOF: We rewrite (1.1) in the form

$$\frac{\partial\rho}{\partial t}+(L^{-1}A)\rho+L^{-1}D(\rho)=0 \tag{6.3}$$

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where $L\rho = \rho - \operatorname{div}(\widetilde{B}\nabla\rho)$, $A\rho = \rho + \alpha\operatorname{div}(B\nabla\Delta\rho)$, and $D(\rho) = -\operatorname{div}\big(B\nabla f'(\rho)\big) - \rho$. Condition (6.2) is held. Indeed, since f'' is bounded in X_σ, we have

$$\big\|f'(u) - f'(v)\big\| \leqslant c(\sigma)\|u - v\|, \quad \forall\, u, v \in X_\sigma. \tag{6.4}$$

Noting the equivalence of norms $\|L^{-1}u\|_{H^2(\Omega)}$ and $\|u\|$, and thus $\big\|(L^{-1}A)^{1/2}u\big\|$ and $\|u\|$ on X_σ, we have

$$\begin{aligned}\big\|L^{-1}D(u) - L^{-1}D(v)\big\| &\leqslant c\big\|L^{-1}(u-v)\big\|_{H^2(\Omega)} + c\Big\|L^{-1}\big(f'(u) - f'(v)\big)\Big\|_{H^2(\Omega)} \\ &\leqslant c(\sigma)\|u - v\| \\ &\leqslant c'(\sigma)\big\|(L^{-1}A)^{1/2}(u - v)\big\|,\end{aligned} \tag{6.5}$$

therefore the result. ∎

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