COMPOSITIO MATHEMATICA

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Compositio Math. **145** (2009), 1442–1446.

doi:10.1112/S0010437X09004187

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

We prove that the Iitaka conjecture $C_{n,m}$ for algebraic fibre spaces holds up to dimension six, that is, when $n \leqslant 6$.

1. Introduction

We work over an algebraically closed field k of characteristic zero. Let X be a normal variety of dimension n The canonical divisor K_X is one of the most important objects associated with X especially in birational geometry. If $Y \subseteq X$ is the smooth locus of X, the canonical sheaf ω_Y of Y is defined as the exterior power $\bigwedge^n \Omega_Y$, where Ω_Y is the sheaf of regular differential 1-forms on Y. Since Y is smooth, $\omega_Y \simeq \mathcal{O}_Y(K_Y)$ for some divisor K_Y, which is called the canonical divisor of Y. Obviously, K_Y is not unique as a divisor but it is unique up to linear equivalence. By taking the closure of the irreducible components of K_Y, we obtain a divisor K_X on X that we call the canonical divisor of X. Since X is normal, $X \setminus Y$ is a closed subset of codimension at least two. So, K_X is uniquely determined up to linear equivalence.

If another normal variety Z is in some way related to X, it is often crucial to find a relation between K_X and K_Z. A classical example is when Z is a smooth prime divisor on a smooth X, in which case we have $(K_X + Z)|_Z = K_Z$.

An algebraic fibre space is a surjective morphism $f\colon X \to Z$ of normal projective varieties, with connected fibres. A central problem in birational geometry is the Iitaka conjecture, which attempts to relate K_X and K_Z through invariants associated with them, namely the Kodaira dimension (see § 2 for the definition of Kodaira dimension).

Conjecture 1.1 (Iitaka). Let $f\colon X \to Z$ be an algebraic fibre space, where X and Z are smooth projective varieties of dimensions n and m, respectively, and let F be a general fibre of f. Then,

$$\kappa(X) \geqslant \kappa(F) + \kappa(Z).$$

This conjecture is usually denoted by $C_{n,m}$. A strengthened version was proposed by Viehweg (cf. [Vie83]) as follows, which is denoted by $C^+_{n,m}$.

Conjecture 1.2 (Iitaka–Viehweg). Under the assumptions of Conjecture 1.1,

$$\kappa(X) \geqslant \kappa(F) + \max\{\kappa(Z), \mathrm{var}(f)\}$$

when $\kappa(Z) \geqslant 0$.

Kawamata [Kaw85a] showed that these conjectures hold if the general fibre F has a good minimal model, in particular, if the minimal model and the abundance conjectures hold in dimension $n - m$ for varieties of nonnegative Kodaira dimension. However, at the moment the

Received 27 June 2008, accepted in final form 16 December 2008, published online 27 July 2009.
2000 Mathematics Subject Classification 14E30 (primary).
Keywords: Iitaka conjecture, minimal models.

minimal model conjecture for such varieties is known only up to dimension five [Bir08] and the abundance conjecture up to dimension three [Kaw92, Miy88] and some cases in higher dimensions, which will be discussed below. Viehweg [Vie83] proved $C^+_{n,m}$ when Z is of general type. When Z is a curve, $C_{n,m}$ was settled by Kawamata [Kaw82]. Kollár [Kol87] proved $C^+_{n,m}$ when F is of general type. The latter also follows from Kawamata [Kaw85a] and the existence of good minimal models for varieties of general type by Birkar *et al.* [BCHM08]. We refer the reader to Mori [Mor87] for a detailed survey of the above conjectures and related problems. In this paper, we prove the following.

THEOREM 1.3. *The Iitaka conjecture $C_{n,m}$ holds when $n \leqslant 6$.*

THEOREM 1.4. *The Iitaka conjecture $C_{n,m}$ holds when $m = 2$ and $\kappa(F) = 0$.*

When $n \leqslant 5$ or when $n = 6$ and $m \neq 2$, $C_{n,m}$ follows immediately from theorems of Kawamata and deep results of the minimal model program.

The Iitaka conjecture is closely related to the following.

CONJECTURE 1.5 (Ueno). Let X be a smooth projective variety with $\kappa(X) = 0$. Then, the Albanese map $\alpha\colon X \to A$ satisfies the following:

(1) $\kappa(F) = 0$ for the general fibre F;

(2) there is an etale cover $A' \to A$ such that $X \times_A A'$ is birational to $F \times A'$ over A;

(3) α is an algebraic fibre space.

The Ueno conjecture is often referred to as conjecture K, and Kawamata [Kaw81] showed that part (3) holds. See Mori [Mor87, § 10] for a discussion of this conjecture.

COROLLARY 1.6. *Part* (1) *of the Ueno conjecture holds when* $\dim X \leqslant 6$.

Proof. Immediate by Theorem 1.3. □

Concerning part (1) of the Ueno conjecture, recently Chen and Hacon [CH] showed that $\kappa(F) \leqslant \dim A$.

2. Preliminaries

Nef divisors. A Cartier divisor L on a projective variety X is called nef if $L \cdot C \geqslant 0$ for any curve $C \subseteq X$. If L is a $\mathbb{Q}$-divisor, we say that it is nef if lL is Cartier and nef for some $l \in \mathbb{N}$. We need a theorem about nef $\mathbb{Q}$-divisors due to Tsuji [Tsu00] and Bauer *et al.* [BCEKPRSW02].

THEOREM 2.1. *Let L be a nef $\mathbb{Q}$-divisor on a normal projective variety X. Then, there is a dominant almost regular rational map $\pi\colon X \dashrightarrow Z$ with connected fibres to a normal projective variety, called the reduction map of L, such that*

(1) *if a fibre F of π is projective and* $\dim F = \dim X - \dim Z$, *then* $L|_F \equiv 0$;

(2) *if C is a curve on X passing through a very general point $x \in X$ with* $\dim \pi(C) > 0$, *then* $L \cdot C > 0$.

Here, by almost regular we mean that some of the fibres of π are projective and away from the indeterminacy locus of π. Using the previous theorem, one can define the nef dimension $n(L)$ of the nef $\mathbb{Q}$-divisor L to be $n(L) := \dim Z$. In particular, if $n(L) = 0$, the theorem says that $L \equiv 0$.

Kodaira dimension. For a divisor D on a normal projective variety X and $l \in \mathbb{N}$, we let

$$H^0(X, lD) = \{f \in k(X) \mid \operatorname{div}(f) + lD \geqslant 0\} \cup \{0\},$$

where $k(X)$ is the function field of X. It is well known that $H^0(X, lD)$ is a finite-dimensional vector space over the ground field k. If $r := H^0(X, lD) \neq 0$ and if $f_1, \dots, f_r$ form a basis over k, we can define a rational map

$$\phi_{lD} \colon X \dashrightarrow \mathbb{P}^{\dim_k H^0(X,lD)-1}$$

by taking x to $(f_1(x) : \cdots : f_r(x))$. Now, let

$$N(D) = \{l \in \mathbb{N} \mid H^0(X, lD) \neq 0\}.$$

If $N(D) \neq \emptyset$, we define the Kodaira dimension of D to be

$$\kappa(D) = \max\{\dim \phi_{lD}(X) \mid l \in N(D)\}$$

but if $N(D) = \emptyset$ we let $\kappa(D) = -\infty$.

If D is a $\mathbb{Q}$-divisor, one can define the Kodaira dimension $\kappa(D) := \kappa(lD)$ for any $l \in \mathbb{N}$ such that lD is an integral divisor. This does not depend on l.

Campana and Peternell [CP07] made the following interesting conjecture.

Conjecture 2.2. Let X be a smooth projective variety and suppose that $K_X \equiv A + M$, where A and M are effective and pseudo-effective $\mathbb{Q}$-divisors, respectively. Then, $\kappa(X) \geqslant \kappa(A)$.

They proved the conjecture in the case $M \equiv 0$ [CP07, Theorem 3.1]. This result is an important ingredient of the proofs below.

Minimal models. Let X be a smooth projective variety. A projective variety Y with terminal singularities is called a minimal model of X if there is a birational map $\phi \colon X \dashrightarrow Y$, such that ϕ^{-1} does not contract divisors, K_Y is nef, and finally there is a common resolution of singularities $f \colon W \to X$ and $g \colon W \to Y$ such that $f^*K_X - g^*K_Y$ is effective and its support contains the birational transform of any prime divisor on X which is exceptional over Y. If in addition lK_Y is base-point free for some $l \in \mathbb{N}$, we call Y a good minimal model.

The minimal model conjecture asserts that every smooth projective variety X has a minimal model or a Mori fibre space Y, in particular, if X has nonnegative Kodaira dimension then it should have a minimal model Y. The abundance conjecture states that every minimal model is a good one.

Note that if the dimension of X is at least three, a minimal model Y could have singular points even though we assume X to be smooth.

3. Proofs

Proof of Theorem 1.4. We are given that the base variety Z has dimension two and that $\kappa(F) = 0$. We may assume that $\kappa(Z) \geqslant 0$, otherwise the theorem is trivial. Let $p \in \mathbb{N}$ be the smallest number such that $f_*\mathcal{O}_X(pK_X) \neq 0$. By Fujino and Mori [FM00, Theorem 4.5], there is a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\tau} & Z \end{array}$$

in which g is an algebraic fibre space of smooth projective varieties, σ and τ are birational, and there are $\mathbb{Q}$-divisors B and L on Z' and a $\mathbb{Q}$-divisor $R = R^+ - R^-$ on X' decomposed into its positive and negative parts satisfying the following:

(1) $B \geqslant 0$;

(2) L is nef;

(3) $pK_{X'} = pg^*(K_{Z'} + B + L) + R$;

(4) $g_*\mathcal{O}_{X'}(iR^+) = \mathcal{O}_{Z'}$ for any $i \in \mathbb{N}$;

(5) R^- is exceptional/X and the codimension of $g(\operatorname{Supp} R^-)$ in Z' is $\geqslant 2$.

Thus, for any sufficiently divisible $i \in \mathbb{N}$, we have

(6) $g_*\mathcal{O}_{X'}(ipK_{X'} + iR^-) = \mathcal{O}_{Z'}(ip(K_{Z'} + B + L))$.

If the nef dimension $n(L) = 2$ or if $\kappa(Z) = \kappa(Z') = 2$, then $ip(K_{Z'} + L)$ is big for some i by Ambro [Amb04, Theorem 0.3]. So, $ip(K_{Z'} + B + L)$ is also big and by (6) and by the facts that σ is birational and $R^- \geqslant 0$ is exceptional/X we have

$$H^0(ipK_X) = H^0(ipK_{X'} + iR^-) = H^0(ip(K_{Z'} + B + L))$$

for sufficiently divisible $i \in \mathbb{N}$. Therefore, in this case $\kappa(X) = 2 \geqslant \kappa(Z)$.

If $n(L) = 1$, then the nef reduction map $\pi\colon Z' \to C$ is regular, where C is a smooth projective curve, and there is a $\mathbb{Q}$-divisor D' on C such that $L \equiv \pi^* D'$ and $\deg D' > 0$ by [BCEKPRSW02, Proposition 2.11]. On the other hand, if $n(L) = 0$, then $L \equiv 0$. So, when $n(L) = 1$ or $n(L) = 0$, there is a $\mathbb{Q}$-divisor $D \geqslant 0$ such that $L \equiv D$. Now, letting $M := \sigma_* g^*(D - L)$, for sufficiently divisible $i \in \mathbb{N}$, we have

$$\begin{aligned} H^0(ip(K_X + M)) &= H^0(ip(K_{X'} + g^*D - g^*L) + iR^-) \\ &= H^0(ip(K_{Z'} + B + D)) \end{aligned}$$

and by Campana and Peternell [CP07, Theorem 3.1]

$$\kappa(X) \geqslant \kappa(K_X + M) = \kappa(K_{Z'} + B + D) \geqslant \kappa(Z). \qquad \square$$

Proof of Theorem 1.3. We assume that $\kappa(Z) \geqslant 0$ and $\kappa(F) \geqslant 0$, otherwise the theorem is trivial.

If $m = 1$, then the theorem follows from Kawamata [Kaw82]. On the other hand, if $n - m \leqslant 3$, then the theorem follows from Kawamata [Kaw85a] and the existence of good minimal models in dimension $\leqslant 3$. So, from now on we assume that $n = 6$ and $m = 2$; hence, $\dim F = 4$. By the flip theorem of Shokurov [Sho03] and the termination theorem of Kawamata *et al.* [KMM87, 5-1-15], F has a minimal model (see also [Bir08]). If $\kappa(F) > 0$, by Kawamata [Kaw85b, Theorem 7.3] such a minimal model is good, so we can apply [Kaw85a] again. Another possible argument would be to apply Kollár [Kol87] when F is of general type and to use the relative Iitaka fibration otherwise.

Now, assume that $\kappa(F) = 0$. In this case, though we know that F has a minimal model, abundance is not yet known. Instead, we use Theorem 1.4. $\square$

ACKNOWLEDGEMENTS

I would like to thank Burt Totaro for many helpful conversations and comments. I am grateful to Frédéric Campana for reminding me of a beautiful theorem written by him and Thomas Peternell which considerably simplified my arguments.

References

Amb04 F. Ambro, *Nef dimension of minimal models*, Math. Ann. **330**(2) (2004), 309–322.

BCEKPRSW02 T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. Rams, T. Szemberg and L. Wotzlaw, *A reduction map for nef line bundles*, in *Complex geometry: collection of papers dedicated to Hans Grauert (Gottingen, 2000)* (Springer, Berlin, 2002), 27–36.

Bir08 C. Birkar, *On existence of log minimal models*, arXiv:0706.1792v2 (2008).

BCHM08 C. Birkar, P. Cascini, C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, arXiv:math/0610203v2 (2008).

CP07 F. Campana and T. Peternell, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold* (with an appendix by M. Toma), arXiv:math/0405093v4 (2007).

CH J. A. Chen and C. Hacon, *On Ueno's conjecture K*, Math. Ann., to appear.

FM00 O. Fujino and S. Mori, *A canonical bundle formula*, J. Differential Geom. **56** (2000), 167–188.

Kaw81 Y. Kawamata, *Characterization of abelian varieties*, Compositio Math. **43** (1981), 253–276.

Kaw82 Y. Kawamata, *Kodaira dimension of algebraic fiber spaces over curves*, Invent. Math. **66**(1) (1982), 57–71.

Kaw85a Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

Kaw85b Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), 567–588.

Kaw92 Y. Kawamata, *Abundance theorem for minimal threefolds*, Invent. Math. **108**(2) (1992), 229–246.

KMM87 Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic geometry (Sendai, 1985)*, Advanced Studies in Pure Mathematics, vol. 10 (North-Holland, Amsterdam, 1987), 283–360.

Kol87 J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, in *Algebraic geometry (Sendai, 1985)*, Advanced Studies in Pure Mathematics, vol. 10 (North-Holland, Amsterdam, 1987), 361–398.

Miy88 Y. Miyaoka, *Abundance conjecture for 3-folds: case* $\nu = 1$, Compositio Math. **68**(2) (1988), 203–220.

Mor87 S. Mori, *Classification of higher-dimensional varieties*, in *Algebraic geometry (Bowdoin, 1985)*, Proceedings of Symposia in Pure Mathematics, vol. 46, part 1 (American Mathematical Society, Providence, RI, 1987), 269–331.

Sho03 V. V. Shokurov, *Prelimiting flips*, Tr. Mat. Inst. Steklova **240** (2003), 82–219; Proc. Steklov Inst. Math. **240** (2003), 75–213 (English transl.).

Tsu00 H. Tsuji, *Numerical trivial fibrations*, arXiv:math/0001023v6 (2000).

Vie83 E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, in *Algebraic varieties and analytic varieties (Tokyo, 1981)*, Advanced Studies in Pure Mathematics, vol. 1 (North-Holland, Amsterdam, 1983), 329–353.

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