Canad. Math. Bull. Vol. **20** (3), 1977

INFINITE GEOMETRIC PRODUCTS

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SUMMARY. This paper is concerned with the infinite geometric products

$$A_j=\prod_{\substack{k=0\\k\neq j}}^{\infty}(1-p^{k-j})^{-1}\qquad (j=0,1,\ldots,\infty;\ 0<p<1)$$

and their generalizations to higher dimensions. Some new expressions and identities are derived for these products by using stochastic theory. The function A_0^{-1} is tabulated for $p=0(0.01)1$.

1. **Introduction.** The infinite geometric product

$$A_0=\prod_{k=1}^{\infty}(1-p^k)^{-1},\qquad 0<p<1 \tag{1}$$

attracted the attention of Euler some two centuries ago, while earlier this century Hardy and Ramanujan published further important properties of this product. It occurs in the theory of elliptic functions and may be used as the generating function of unrestricted partitions [2]. Thus,

$$A_0=\sum_{k=0}^{\infty}r(k)p^k=f(p) \tag{2}$$

where $r(k)$ is the number of decompositions of k into integer summands without regard to order. As tables of $r(k)$ are available [1] and because of the existence of a rapidly converging asymptotic formula for $r(k)$ of Hardy and Ramanujan ([3], [4]) and a convergent series for $r(k)$ by Rademacher [8], A_0 may be computed from (2) for various p. However the approach is tedious for even reasonably large p. More recently, different expansions for A_0 have been developed by Andrews et al. [2].

Whilst investigating the stationary distributions of patients in hospital wards, where arriving patients were geometrically distributed, the authors [9] have encountered the sequence of infinite products A_{ij} $(i=1,2,\ldots\ell;\ j=0,1,\ldots\infty)$ where

$$A_{ij}=\prod_{\substack{k=0\\k\neq j}}^{\infty}(1-\alpha_{ik}/\alpha_{ij})^{-1}. \tag{3}$$

In these products, α_{ij} is the ith element of

$$\boldsymbol{\alpha}_j'=\mathbf{p}_0'\mathbf{P}^j \tag{4}$$

Received by the editors May 4, 1976.

where $\mathbf{p}_0' = (p_{01}, p_{02}, \ldots, p_{0l})$; $0 \le p_{0i} \le 1$, $\sum_{i=1}^{l} p_{0i} = 1$ and $\mathbf{P}$ is an $l \times l$ sub-stochastic matrix.

The infinite product (1) is a particular case of (3) when $l = 1$ and $j = 0$. It has therefore been necessary to consider certain generalized forms of the product (1) which involve

(i) extension of the index of p giving the sequence of products,

$$A_j = \prod_{\substack{k=0 \\ k \neq j}}^{\infty} (1 - p^{k-j})^{-1}, \qquad j = 0, 1, 2, \ldots \infty, \tag{5}$$

(ii) generalization of the power p^{k-j} in (5) to $(\alpha_{ik}/\alpha_{ij})$ as in (3),

(iii) displacement of the least possible value of k in (1) to an arbitrary integer $(n+1)$ as in

$$T_n(p) = \prod_{k=n+1}^{\infty} (1 - p^k)^{-1}. \tag{6}$$

In this paper, three new expressions for A_0 are given together with suitably connected expressions for corresponding truncated products. Some new expressions and identities for A_{ij}, A_j are also derived, methods of calculating the products are discussed and A_0^{-1} is evaluated for $p = 0(0.01)1$.

2. **New expressions and identities for** A_{ij}. The probability generating function of the number of patients in the ith ward $(i = 1, 2, \ldots, l)$ of a hospital on any day for a certain stochastic model has been obtained by Staff and Vagholkar [9] as

$$G_\infty(t) = \prod_{j=0}^{\infty} [(1-\theta)/\{1 - \theta(\alpha_{ij}t + 1 - \alpha_{ij})\}], \qquad 0 < \theta < 1 \tag{7}$$

$$= (1-\theta) \sum_{j=0}^{\infty} A_{ij}/\{1 - \theta + \theta\alpha_{ij}(1-t)\}, \tag{8}$$

where A_{ij} are defined in (3) and t is an arbitrary variable with $|t| \le 1$. The α_{ij}'s are assumed to be distinct.

To deal with all the A_{ij} simultaneously, introduce the function

$${}_i\phi(n) = \sum_{j=0}^{\infty} A_{ij}\alpha_{ij}^n, \qquad n = 0, 1, 2, \ldots. \tag{9}$$

From (8), the nth factorial moment of the variate whose probability generating function is given by (7), is found to be

$${}_i\mu_{(n)} = [\theta/(1-\theta)]^n n!\, {}_i\phi(n), \tag{10}$$

whilst from (7), the nth factorial cumulant of this variate is derived as,

$${}_i\kappa_{(n)} = [\theta/(1-\theta)]^n (n-1)! \sum_{j=0}^{\infty} \alpha_{ij}^n. \tag{11}$$

https://doi.org/10.4153/CMB-1977-054-0 Published online by Cambridge University Press

The standard relationship between factorial moments and factorial cumulants [7] may be used to obtain connecting formulae between the ${}_i\phi(n)$, viz.,

$$(12)\qquad (n+1){}_i\phi(n+1)=\sum_{s=0}^{\infty}{}_i\phi(s)\left(\sum_{j=0}^{\infty}\alpha_{ij}^{n-s+1}\right)\qquad n=0,1,2,\ldots\infty.$$

Together with

$$(13)\qquad {}_i\phi(0)=\sum_{j=0}^{\infty}A_{ij}=1$$

from (8), (12) completely specifies the function ${}_i\phi(n)$, which is straightforward to programme for computation once $\sum_{j=0}^{\infty}\alpha_{ij}^{n}$ $(n=1,2,\ldots\infty)$ are known. It is important to realize that $\lim_{n\to\infty}{}_i\phi(n)=A_{i0}$. The function ${}_i\phi(n)$ has already proved useful in the calculation of stationary probability functions [9] and is used again in the following section but with $l=1$.

The α_{ij}'s as defined in (4) are probabilities and would therefore lie in the closed interval (0, 1). The sequence α_{ij} $(j=0,1,2,\ldots,\infty)$ for any i would in general increase, reach a modal value and then strictly decrease. It is extremely unlikely that two α_{ij} would be equal for varying j. Let the modal value of α_{ij} $(j=0,1,\ldots\infty)$ be written as α_{ij^i} for a particular i. It is possible that $j'=0$ in which case α_{ij} $(j=0,1,\ldots,\infty)$ is a strictly decreasing sequence. Now let j^* be the non-negative integer corresponding to j defined by the following inequalities

$$(14)\qquad \begin{aligned}&\alpha_{ij^*}<\alpha_{ij}<\alpha_{ij^*-1}\quad\text{for}\quad j<j',\\ &\alpha_{ij^*}<\alpha_{ij}<\alpha_{ij^*+1}\quad\text{for}\quad j>j'\quad\text{and}\quad \alpha_{i0}<\alpha_{ij}.\end{aligned}$$

No such j^* exists if $j>j'$ and $\alpha_{i0}>\alpha_{ij}$.

Consider a single term $(1-\alpha_{ik}/\alpha_{ij})^{-1}$ of the infinite product A_{ij} defined in (3). One can write this term as

$$(15)\qquad \begin{aligned}(1-\alpha_{ik}/\alpha_{ij})^{-1}&=\exp\{-\ln(1-\alpha_{ik}/\alpha_{ij})\}\\ &=\exp\left\{\sum_{x=1}^{\infty}(\alpha_{ik}/\alpha_{ij})^x/x\right\}\quad\text{if}\quad \alpha_{ik}<\alpha_{ij}\end{aligned}$$

and as

$$(16)\qquad \begin{aligned}(1-\alpha_{ik}/\alpha_{ij})^{-1}&=-(\alpha_{ik}/\alpha_{ij})^{-1}(1-\alpha_{ij}/\alpha_{ik})^{-1}\\ &=-(\alpha_{ij}/\alpha_{ik})\exp\left\{\sum_{x=1}^{\infty}(\alpha_{ij}/\alpha_{ik})^x/x\right\}\end{aligned}$$

if $\alpha_{ik}>\alpha_{ij}$.

Using (15) and (16) one gets the following expressions for A_{ij},

$$A_{ij}=\begin{cases} (-1)^{j^*-j-1}\prod_{k=j+1}^{j^*-1}(\alpha_{ij}/\alpha_{ik})\exp\left[\sum_{x=1}^{\infty}\frac{1}{x}\left(\left\{\sum_{k=0}^{j-1}+\sum_{k=j^*}^{\infty}(\alpha_{ik}/\alpha_{ij})^x\right\}\right.\right. \\ \qquad \left.\left.+\sum_{k=j+1}^{j^*-1}(\alpha_{ij}/\alpha_{ik})^x\right)\right] \quad \text{when} \quad j<j'<j^*, \\ \exp\left[\sum_{x=1}^{\infty}\sum_{\substack{k=0\\k\neq j'}}^{\infty}\frac{1}{x}(\alpha_{ik}/\alpha_{ij'})^x\right] \quad \text{when} \quad j=j', \\ (-1)^{j-j^*-1}\prod_{k=j^*+1}^{j-1}(\alpha_{ij}/\alpha_{ik})\exp\left[\sum_{x=1}^{\infty}\frac{1}{x}\left(\left\{\sum_{k=0}^{j^*}+\sum_{k=j+1}^{\infty}(\alpha_{ik}/\alpha_{ij})^x\right\}\right.\right. \\ \qquad \left.\left.+\sum_{k=j^*+1}^{j-1}(\alpha_{ij}/\alpha_{ik})^x\right)\right] \quad \text{when} \quad j^*<j'<j \quad \text{and} \quad \alpha_{i0}<\alpha_{ij}, \\ (-1)^j\prod_{k=0}^{j-1}(\alpha_{ij}/\alpha_{ik})\exp\left[\sum_{x=1}^{\infty}\frac{1}{x}\left\{\sum_{k=j+1}^{\infty}(\alpha_{ik}/\alpha_{ij})^x\right.\right. \\ \qquad \left.\left.+\sum_{k=0}^{j-1}(\alpha_{ij}/\alpha_{ik})^x\right\}\right] \quad \text{when} \quad j'<j \quad \text{and} \quad \alpha_{i0}>\alpha_{ij}. \end{cases} \tag{17}$$

In (17) it is assumed that $\alpha_{i0}\neq 0$. Should $\alpha_{i0}=0$ for certain i, then $A_{i0}=0$ for these values of i and in these cases the continued product (3) defining A_{ij} should start from $k=1$ and α_{i1} plays the role of α_{i0} in (17).

3. Special case: $l=1$. When $l=1$, let $p_{11}=p$. Then $\alpha_{1j}=p^j$ and ${}_1\phi(n)$ reduces to

$$H(n)=\sum_{j=0}^{\infty}A_jp^{nj}, \qquad 0<p<1, \tag{18}$$

where A_j is defined in (5). The recurrence formulae connecting the $H(n)$ are now

$$(n+1)H(n+1)=\frac{1}{(1-p^{n+1})}+\sum_{s=1}^{n}\left\{\frac{H(s)}{(1-p^{n-s+1})}\right\} \qquad n=1,2,\ldots \tag{19}$$

and $H(0)=1$, $H(1)=(1-p)^{-1}$.

If a comparison is now made with the algebraic identity

$$\frac{(n+1)}{\prod_{k=1}^{n+1}(1-p^k)}=\frac{1}{(1-p^{n+1})}+\sum_{s=1}^{n}\frac{1}{[\prod_{k=1}^{s}(1-p^k)](1-p^{n-s+1})} \qquad n=1,2,\ldots \tag{20}$$

it follows immediately that

$$H(n)=\prod_{k=1}^{n}(1-p^k)^{-1}, \qquad n=1,2,\ldots \tag{21}$$

and

$$H(0)=\sum_{j=0}^{\infty} A_j = 1. \tag{22}$$

The simple recurrence relationship between two successive terms of the sequence $A_0, A_1, \ldots$ is

$$A_j = A_{j-1}/(1-p^{-j}), \qquad j=1,2,\ldots. \tag{23}$$

Upon substitution for A_j ($j\geq 1$), in terms of A_0 in (18) and (22), A_0 is found as

$$A_0=\prod_{k=1}^{n}(1-p^k)^{-1}\Big/\left[1+\sum_{j=1}^{\infty}\frac{p^{nj}}{\prod_{k=1}^{j}(1-p^{-k})}\right] \qquad n=1,2,\ldots \tag{24a}$$

and

$$A_0=1\Big/\left[1+\sum_{j=1}^{\infty}\frac{1}{\prod_{k=1}^{j}(1-p^{-k})}\right]. \tag{24b}$$

Hence,

$$A_j=\frac{(-1)^j p^{j(j+1)/2}\prod_{k=1}^{j}(1-p^k)^{-1}\prod_{k=1}^{n}(1-p^k)^{-1}}{\left[1+\sum_{j=1}^{\infty}\frac{p^{nj}}{\prod_{k=1}^{j}(1-p^{-k})}\right]} \tag{25a}$$

$$j=1,2,\ldots \quad n=1,2,\ldots$$

and

$$A_j=\frac{(-1)^j p^{j(j+1)/2}\prod_{k=1}^{j}(1-p^k)^{-1}}{\left[1+\sum_{j=1}^{\infty}\frac{1}{\prod_{k=1}^{j}(1-p^{-k})}\right]}. \tag{25b}$$

Various expansions for A_0, developed by Euler, are discussed in some detail by Hardy and Wright [6, Ch. 19] and are listed below.

$$A_0=\left\{1+\sum_{j=1}^{\infty}1\Big/\prod_{k=1}^{j}(1-p^{-k})\right\}^{-1}, \tag{26}$$

$$=\exp\left[\sum_{i=1}^{\infty}p^i/i(1-p^i)\right], \tag{27}$$

$$=\sum_{i=-\infty}^{\infty}(-1)^i p^{i(3i+1)/2}, \tag{28}$$

$$=1+\sum_{j=1}^{\infty}\left[p^{j^2}\Big/\prod_{k=1}^{j}(1-p^k)^2\right]. \tag{29}$$

The expansion (24a) generalizes (24b) which is in fact the same as (26), one of Euler's expansions. Expansion (24a) may be more satisfactorily written as a truncated infinite product

$$T_n(p)=\prod_{k=n+1}^{\infty}(1-p^k)^{-1}=\left\{1+\sum_{j=1}^{\infty}\frac{p^{nj}}{\prod_{k=1}^{j}(1-p^{-k})}\right\}^{-1}. \tag{30}$$

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This extension also easily follows from the expansion of

$$\prod_{k=0}^{\infty}(1+p^k t)=1+\sum_{j=1}^{\infty}\frac{t^j p^{j(j-1)/2}}{\prod_{k=1}^{j}(1-p^k)} \tag{31}$$

provided that $|p|<1$, $|pt|<1$. Put $t=-p^{n+1}$ in (31). Hence

$$\prod_{k=0}^{\infty}(1-p^{k+n+1})=\prod_{k=n+1}^{\infty}(1-p^k)$$

$$=1+\sum_{j=1}^{\infty}\frac{(-1)^j p^{j(2n+j+1)/2}}{\prod_{k=1}^{j}(1-p^k)} \tag{32}$$

which is equivalent to (30). A_0 and A_j can be efficiently evaluated using (24) and (25), employing a value of n appropriate to the value of p. While the expansions (24b) and (25b) will mainly suffice (for small or moderate values of p), as $p\to 1$, it will be necessary to use (24a) and (25a) and increase n substantially in order to make the series in (24a) and (25a) converge rapidly.

If a large number of significant figures is required in the computation of A_0 for values of $p>0.9$, it is simpler to employ the well-known and elegant functional relationship for $f(p)$ [4],

$$f(p)=\frac{p^{1/24}\sqrt{(\ln(1/p))}}{\sqrt{(2\pi)}}\exp[\pi^2/6\ln(1/p)]$$

$$\times f[\exp(-4\pi^2/\ln(1/p))]. \tag{33}$$

A_0^{-1} for $p=0(0.01)1$ is listed in Table 1.

Finite products. It is expected that most finite products of the form $\prod_{k=r}^{s}(1-p^k)$ would be dealt with by direct multiplication. Alternatively, it may be more satisfactory to express such products as $T_{r-1}(p)/T_s(p)$ and use (30).

4. **New expansions for A_0.** Let X_j be a random variable with the following probability function,

$$\Pr(X_j=0)=1-p^j, \qquad 0<p<1,$$

$$\Pr(X_j=1)=p^j, \qquad j=0,1,2,\ldots\infty. \tag{34}$$

Now write $Y=\sum_{j=0}^{\infty}X_j$, where the X_j are mutually independent. The probability generating function of Y, the convolution of an infinite number of Bernoulli variates with parameters $(1, p^j)$ is

$$G(t)=\prod_{j=0}^{\infty}(1-p^j+p^j t)$$

$$=\mathbf{t}'\mathbf{Bp}, \tag{35}$$

https://doi.org/10.4153/CMB-1977-054-0 Published online by Cambridge University Press

Table 1.

$$A_0^{-1} = \prod_{k=1}^{\infty} (1-p^k) = c(10)^{-d}$$

p	c	d	p	p	d
0.01	0.989900	0	0.51	0.273026	0
0.02	0.979641	0	0.52	0.257456	0
0.03	0.969100	0	0.53	0.242103	0
0.04	0.958405	0	0.54	0.226995	0
0.05	0.947500	0	0.55	0.212161	0
0.06	0.936401	0	0.56	0.197629	0
0.07	0.925102	0	0.57	0.183429	0
0.08	0.913603	0	0.58	0.169590	0
0.09	0.901906	0	0.59	0.156144	0
0.10	0.890010	0	0.60	0.143121	0
0.11	0.877916	0	0.61	0.130552	0
0.12	0.865625	0	0.62	0.118465	0
0.13	0.853138	0	0.63	0.106891	0
0.14	0.840455	0	0.64	0.958576	1
0.15	0.827578	0	0.65	0.853911	1
0.16	0.814508	0	0.66	0.755159	1
0.17	0.801246	0	0.67	0.662540	1
0.18	0.787795	0	0.68	0.576241	1
0.19	0.774157	0	0.69	0.496412	1
0.20	0.760333	0	0.70	0.423158	1
0.21	0.746326	0	0.71	0.356538	1
0.22	0.732140	0	0.72	0.296551	1
0.23	0.717778	0	0.73	0.243138	1
0.24	0.703242	0	0.74	0.196170	1
0.25	0.688537	0	0.75	0.155450	1
0.26	0.673668	0	0.76	0.120708	1
0.27	0.658639	0	0.77	0.916018	2
0.28	0.643456	0	0.78	0.677198	2
0.29	0.628123	0	0.79	0.485880	2
0.30	0.612648	0	0.80	0.336799	2
0.31	0.597037	0	0.81	0.224311	2
0.32	0.581298	0	0.82	0.142574	2
0.33	0.565438	0	0.83	0.857684	3
0.34	0.549466	0	0.84	0.483262	3
0.35	0.533392	0	0.85	0.251690	3
0.36	0.517225	0	0.86	0.119122	3
0.37	0.500977	0	0.87	0.501064	4
0.38	0.484658	0	0.88	0.181828	4
0.39	0.468282	0	0.89	0.546609	5
0.40	0.451860	0	0.90	0.128604	5
0.41	0.435409	0	0.91	0.218117	6
0.42	0.418942	0	0.92	0.235712	7
0.43	0.402476	0	0.93	0.133660	8
0.44	0.386027	0	0.94	0.287652	10
0.45	0.369614	0	0.95	0.131061	12
0.46	0.353256	0	0.96	0.392908	16
0.47	0.336972	0	0.97	0.505596	22
0.48	0.320785	0	0.98	0.768372	34
0.49	0.304716	0	0.99	0.207128	69
0.50	0.288788	0			

where,

t is the infinite column vector whose kth element is t^{k-1},
B is the upper-triangular infinite matrix (b_{kl}) with

$$b_{kl}=(-1)^{l-k}\binom{l-1}{k-1}, \qquad k,l=1,2,\ldots; \qquad k\ge l,$$

$$b_{kl}=0, \qquad k<l$$

and $\mathbf{p}'$ is the infinite row vector

$$\left(1,\frac{1}{(1-p)},\frac{p}{(1-p)(1-p^2)},\ldots,\frac{p^{s(s-1)/2}}{(1-p)\cdots(1-p^s)},\ldots\right).$$

It is easy to see that $G(t)$ can be written as

$$G(t)=\sum_{i=1}^{\infty}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\binom{j+i}{j}p^{(j+i)(j+i-1)/2}}{\prod_{k=1}^{j+i}(1-p^k)}\right]t^i.$$

Substitution of $(t+1)$ for t yields

$$\prod_{k=1}^{\infty}(1+p^kt)=\sum_{i=1}^{\infty}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\binom{j+i}{j}p^{(j+i)(j+i-1)/2}}{\prod_{k=1}^{j+i}(1-p^k)}\right](t+1)^{i-1}, \tag{37}$$

provided that $|pt|<1$. Put $t=-p^n$ in (37), where n is a non-negative integer. Thus

$$\prod_{k=n+1}^{\infty}(1-p^k)=\sum_{i=1}^{\infty}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\binom{j+i}{i}p^{(j+i)(j+i-1)/2}(1-p^n)^{i-1}}{\prod_{k=1}^{j+i}(1-p^k)}\right]. \tag{38}$$

In particular with $n=0$ we get

$$\Pr(Y=1)=A_0^{-1}=\prod_{k=1}^{\infty}(1-p^k)=\frac{1}{1-p}+\sum_{j=2}^{\infty}\left[\frac{(-1)^{j-1}jp^{\binom{j}{2}}}{\prod_{k=1}^{j}(1-p^k)}\right]. \tag{39}$$

Because $G(t)$ may be rewritten as

$$G(t)=A_0^{-1}t\prod_{j=1}^{\infty}\left(1+\frac{p^j}{1-p^j}t\right), \tag{40}$$

it follows that

$$\Pr(Y=n+1)=A_0^{-1}\ \text{Coeff. of } t^n \text{ in } \prod_{j=1}^{\infty}\left(1+\frac{p^j}{1-p^j}t\right).$$

From (36),

$$A_0^{-1}=\frac{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j\binom{j+n+1}{j}p^{(j+n+1)(j+n)/2}}{\prod_{k=1}^{j+n+1}(1-p^k)}}{\displaystyle\sum_{\substack{j_1,j_2,\ldots,j_n=1\\ j_1<j_2\cdots<j_n}}^{\infty}\frac{p^{j_1+j_2+\cdots j_n}}{\prod_{i=1}^{n}(1-p^{j_i})}}. \tag{41}$$

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Hence for $n=1$,

$$A_0^{-1}=\frac{\sum_{j=0}^{\infty}\left[(-1)^j\binom{j+2}{j}p^{\binom{j+2}{2}}/\prod_{k=1}^{j+2}(1-p^k)\right]}{\sum_{j=1}^{\infty}p^j/(1-p^j)}. \tag{42}$$

It is clear that further expansions may be obtained from (41) for increasing n but with a corresponding increase in complexity and decrease in utility.

The approach adopted in this paper to develop infinite geometric products has been based on probabilistic considerations. However, it is possible as with expansion (26) that the new expansions and extensions can also be obtained by other methods.

The authors wish to thank the referee for his comments.

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