AN EMBEDDING THEOREM FOR BALANCED INCOMPLETE BLOCK DESIGNS

MARSHALL HALL JR. AND W. S. CONNOR

1. Introduction. From a symmetric balanced incomplete block design we may construct a derived design by deleting a block and its varieties. But a design with the parameters of a derived design may not be embeddable in a symmetric design. Bhattacharya **(1)** has such an example with $\lambda = 3$. When $\lambda = 1$, the derived design is a finite Euclidean plane and this can always be embedded in a corresponding symmetric design which will be a finite projective plane.

In this paper it is shown that for $\lambda = 2$ as well as for $\lambda = 1$ a design with the parameters of a derived design is indeed embeddable in a symmetric design. The methods used depend on techniques developed in **(2)**. It is interesting to note that for $k = 4$ the entire embedding was carried out by Nandi **(3)**.

2. General conditions for embedding. A balanced incomplete block design with parameters v, b, r, k, and λ satisfying

$$bk = vr, \quad r(k-1) = (v-1)\lambda \tag{2.1}$$

is symmetric if $v = b$. From a symmetric design S, a derived design D' may be obtained by deleting a block B_0 and the varieties of B_0 throughout. This leaves a design D' with parameters $v' = v - k$, $b' = b - 1$, $r' = r$, $k' = k - \lambda$, and $\lambda' = \lambda$. Since $r = k$ in the symmetric design, $r' = k' + \lambda'$ in the derived design. Thus, for every derived design D' the parameters satisfy

$$r' = k' + \lambda', \quad v'\lambda' = k'(k' + \lambda' - 1),$$
$$b'\lambda' = (k' + \lambda')(k' + \lambda' - 1).$$

It is not difficult to state conditions on a design with appropriate parameters so that it is recognizable as a derived design of some symmetric design. These conditions follow:

THEOREM 2.1 *A design D with parameters satisfying*

$$r = k + \lambda, \quad v\lambda = k(k + \lambda - 1), \quad b\lambda = (k+\lambda)(k + \lambda - 1) \tag{2.2}$$

can be embedded as a derived design in a symmetric design if and only if we can find in D sets of blocks S_j, $j = 1, \ldots, k + \lambda$, such that:

Received March 19, 1953. Theorem 3.2 was proved independently by both authors. Upon learning of this by correspondence they decided to write the present joint paper, which is a synthesis of the two original manuscripts.

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(1) *Each S_j consists of $k + \lambda - 1$ blocks of D.*
(2) *The blocks of an S_j together contain each variety of D λ times.*
(3) *Any two distinct sets S_i, S_j have exactly $\lambda - 1$ blocks in common.*
(4) *Any block of D is in precisely λ sets S_j.*

Proof. Let us adjoin to D new varieties $x_1, \ldots, x_{k+\lambda}$ and a new block B_0 consisting of these varieties. We also adjoin the variety x_j to all blocks of the set S_j and to no other blocks. Then the new array S contains $b^* = b + 1$ blocks and $v^* = v + k + \lambda$ varieties. The block B_0 contains $k + \lambda$ varieties and by (4) we have adjoined just λ new varieties to each old block. Hence, in S each block contains $k^* = k + \lambda$ varieties. Each old variety appeared $r = k + \lambda$ times and each new variety x_j appears once in B_0 and in the $k + \lambda - 1$ blocks of S_j. Hence, in S each variety appears $k + \lambda = r^*$ times. Finally, in D each pair d_i, d_j occurred together λ times; by (2) a new x_j occurs with each d_i of D λ times and by (3) a new pair x_i, x_j occurs together in $\lambda - 1$ old blocks and once in the new block B_0. Thus every pair in S occurs together λ times. S is seen to be a balanced incomplete block design, and as $r^* = k^*$, S is a symmetric design.

Conversely, if we drop from a symmetric design S a block B_0 and all of the varieties of B_0, we may readily verify that the blocks of S containing a suppressed variety x_j form in the derived design D a set of blocks S_j, and the sets S_j have all of the properties mentioned in the theorem.

3. The embedding theorems for $\lambda = 1$ and $\lambda = 2$. Not every design D with parameters satisfying the relations (2.2) satisfies the embedding conditions. Since it is known that any two distinct blocks of a symmetric design intersect in λ varieties, a design D cannot possibly be embedded if it has two blocks intersecting in more than λ varieties. Such an example with $\lambda = 3$, $v = 16$, $b = 24$, $r = 9$, and $k = 6$ was found by Bhattacharya **(1)** and is listed in **(2)**. In this example there are two blocks with four varieties in common.

When $\lambda = 1$, the design D is readily seen to be a finite Euclidean plane and it is well known that every such plane can be embedded in a finite projective plane, this being the corresponding symmetric design. This known result and the absence of a counter-example for $\lambda = 2$ comparable to that for $\lambda = 3$ suggested the embedding theorem for $\lambda = 2$ which is the main part of this paper. The embedding theorem for $\lambda = 1$ will be included here not as a new result but as an indication of the general motivation of the more complicated embedding theorem for $\lambda = 2$.

THEOREM 3.1 *Every design D with parameters $v = k^2$, $b = k^2 + k$, $r = k + 1$, and $\lambda = 1$ satisfies the conditions of Theorem* 2.1 *and has a unique embedding in a symmetric design S.*

Proof. Let B_1 be an arbitrary block of D whose varieties are $a_1, a_2, \ldots, a_k$. If c is any variety of D not in B_1, then there are k blocks containing the pairs $ca_1, ca_2, \ldots, ca_k$ and these are distinct since no a_i, a_j occur together except in

B_1. This accounts for k of the $k + 1$ blocks containing c. Thus there is exactly one block B_2 containing c and no variety of B_1. Let the varieties of B_2 be c, $c_2, \ldots, c_k$. Then B_2 is, for each of $c_2, \ldots, c_k$, equally the unique block containing this variety and none from B_1. Hence, in all there will be blocks $B_2, \ldots, B_k$ containing the $k^2 - k$ varieties of D not in B_1, and no one of $B_2, \ldots, B_k$ intersects B_1 in a variety. Moreover, no two of these B's have a variety in common with each other since each is the unique block for each of its varieties not intersecting B_1. The blocks $B_1, \ldots, B_k$ form a set S_1 of k blocks which together contain each variety once. Moreover, each block determines uniquely such a set of k non-intersecting blocks and there will be in all $k + 1$ sets $S_1, \ldots, S_{k+1}$ and these sets have the properties required by Theorem 2.1. We note finally that D determines these sets uniquely and so the embedding is unique.

For $\lambda = 2$ the same result holds but the proof is much harder.

THEOREM 3.2 *Every balanced incomplete block design D with parameters*

$$v = \tfrac{1}{2}k(k + 1), \quad b = \tfrac{1}{2}(k + 1)(k + 2), \quad r = k + 2, \quad \lambda = 2$$

satisfies the conditions of Theorem 2.1 *and can be embedded uniquely in a symmetric design.*

The proof given in the following sections will not cover the case $k = 6$, shown to be impossible in **(2)**. Nor will it cover the case $k = 2$, and so we shall always exclude this case without further mention. It is easy to verify by replacement of the varieties in the blocks that the theorem is true for $k = 2$. Throughout the proof we shall suppose that D exists.

4. Some relations among the blocks of D. In this section we shall quote some lemmas from **(2)** and shall develop one new lemma, all of which will be useful later.

Let two blocks of D which have u varieties in common be called "uth associates." Then we may paraphrase Lemma 4.1 of **(2)** as follows:

LEMMA 4.1 *Any block of D has $2k$ first associates, $\frac{1}{2}k(k - 1)$ second associates, and zero sth associates* ($s \neq 1, 2$).

Next consider any two initial blocks of D, say B_1 and B_2. Any other block of D is of "type 1" if it is a second associate of both B_1 and B_2, of "type 2" if it is a second associate of one of B_1 and B_2 but a first associate of the other, and of "type 3" if it is a first associate of both B_1 and B_2. Now we may paraphrase Lemma 4.2 of **(2)**, thus:

LEMMA 4.2 *If two initial blocks of D are first associates, then there are* $\frac{1}{2}(k - 1)(k - 2)$ *blocks of type* 1, $2(k - 1)$ *blocks of type* 2, *and k blocks of type* 3. *If two initial blocks of D are second associates, then there are* $\frac{1}{2}(k - 2)(k - 3)$ *blocks of type* 1, $4(k - 2)$ *blocks of type* 2, *and* 4 *blocks of type* 3.

Next we shall consider the structural matrices which correspond to three sets of blocks. In each matrix there is an unknown element, which we shall

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determine under the condition that the corresponding set of blocks forms part of D. The first matrix is

$$(4.1)\qquad S_6 = \begin{bmatrix} k & 1 & 1 & 1 & 1 & 2 \\ & k & 1 & 2 & 2 & 1 \\ & & k & 2 & 2 & 1 \\ & & & k & s_{45} & 2 \\ & & & & k & 2 \\ & & & & & k \end{bmatrix}$$

which corresponds to a set U_6 of blocks. We desire to know whether 1 or 2 or both are admissible values for s_{45} to assume if U_6 forms a part of D.

Associated with S_6 is the characteristic matrix C_6, which has the elements $c_{jj} = 2k$ and $c_{ju} = k - 2$ or -4 $(j \neq u)$ according as $s_{ju} = 1$ or 2, where j and u refer to the jth and uth blocks of U_6. Hence, our problem is to decide whether $k - 2$ or -4 or both are admissible values for c_{45} to assume.

We obtain the determinant

$$(4.2)\qquad \begin{aligned} |C_6| &= 4(k+2)^3(2k - c_{45})[(k-2)c_{45} + 2(k-6)] \\ &= 4(k+2)^3 f_1 f_2, \end{aligned}$$

where $f_1 = (2k - c_{45})$ and $f_2 = (k-2)c_{45} + 2(k-6)$. Now by Theorem 3.1 of **(2)** it is necessary that $|C_6| \geqslant 0$. For any k, $4(k+2)^3 > 0$. Hence, either (a) $f_1 \leqslant 0$ and $f_2 \leqslant 0$, or (b) $f_1 \geqslant 0$ and $f_2 \geqslant 0$. Since (a) implies that $2k \leqslant c_{45}$, which is impossible, we must have (b), whence

$$(4.3)\qquad c_{45} \geqslant -2 + 8/(k-2),$$

which has the minimum, -2. Hence it is necessary that $c_{45} = k - 2$, and therefore that $s_{45} = 1$. This result is contained in the following lemma.

LEMMA 4.3 *If D contains the blocks of U_6, then in S_6, $s_{45} = 1$.*

We shall state two other lemmas, which can be proved by arguments analagous to those used in proving Lemma 4.3. Proofs of these lemmas are given in **(2)**. Our Lemmas 4.4 and 4.5 correspond respectively to Lemmas 4.4 and 4.5 of **(2)**.

Consider a set of blocks, $U_5^{(1)}$, which has the structural matrix

$$(4.4)\qquad S_5^{(1)} = \begin{bmatrix} k & 1 & 1 & 2 & 2 \\ & k & 1 & 1 & 1 \\ & & k & 1 & 1 \\ & & & k & s_{45} \\ & & & & k \end{bmatrix}.$$

We may state a lemma about s_{45}.

[1]In **(2)** it was stated that "the equations of 4.6 may be solved to determine the number of blocks of types 11, . . . , 32." However, since the six equations 4.6 are dependent, one cannot obtain the individual x_{ij}'s but only the needed linear combinations thereof.

LEMMA 4.4 *If D contains the blocks of $U_5^{(1)}$, then in $S_5^{(1)}$, $s_{45} = 1$.*

Finally, consider a set of blocks, $U_5^{(2)}$, which has the structural matrix

$$(4.5) \qquad S_5^{(2)} = \begin{bmatrix} k & 1 & 1 & 1 & 2 \\ & k & 1 & 2 & 1 \\ & & k & 2 & 1 \\ & & & k & s_{45} \\ & & & & k \end{bmatrix}.$$

The related lemma follows:

LEMMA 4.5 *If D contains the blocks of $U_5^{(2)}$, then in $S_5^{(2)}$, $s_{45} = 2$.*

5. The sets which are determined by a block and its first associates. We shall consider an arbitrary block B_1 of D and its $2k$ first associates. It will be shown that these $2k + 1$ blocks are uniquely separable into two sets S_1 and S_2 of $k + 1$ blocks which pairwise are first associates and which have only B_1 in common. To show this we shall determine the structural matrix of a certain set of $k + 2$ blocks.

Let any block be B_1. Then, by Lemma 4.1, B_1 has $2k$ first associates: B_2, $B_3, \ldots, B_{2k+1}$. Let us focus our attention on any one of these, say B_2. By Lemma 4.2, regarding B_1 and B_2 as initial blocks, there are $\frac{1}{2}(k-1)(k-2)$ blocks of type 1, $2(k-1)$ blocks of type 2, and k blocks of type 3 among the remaining blocks of D. So without loss of generality we may write down two rows of the structural matrix S_b of D as follows:

$$(5.1) \qquad \begin{array}{ccc|c|c|c|c} k & 1 & 1 & 1 \ldots 1 & 1 \ldots 1 & 2 \ldots 2 & 2 \ldots 2 \\ 1 & k & 1 & 1 \ldots 1 & 2 \ldots 2 & 2 \ldots 2 & 1 \ldots 1 \end{array}$$

where the columns correspond in left-to-right order to blocks $B_1, B_2, \ldots, B_b$. For convenience we have partitioned S_b in left-to-right order into submatrices A, B, C, D, E, where A contains 3 columns, D contains $\frac{1}{2}(k-1)(k-2)$ columns, and B, C, E each contain $k - 1$ columns.

Let the blocks $B_3, \ldots, B_{k+2}$ comprise a set M, and the blocks $B_{k+3}, \ldots, B_{2k+1}$ comprise a set N. We shall show that there exists one block of M which is a first associate of two or more blocks of N. To show this we shall count the number of ones in the submatrix C^* which consists of rows $3, \ldots, k+2$ of C. Since every block of N is a first associate of B_1, it follows from Lemma 4.2 and the observation that there are $k - 1$ twos in row 2 of C that there are $(k-1)(k-2)$ twos and hence $2(k-1)$ ones in C^*. But there are k rows in C^*, and so there is at least one row of C^*, say the row corresponding to B_3, which contains two or more ones.

Now consider how the third row of S_b may be filled up. By considering B_1 and B_3 as initial blocks, Lemma 4.2 applies. Also, by considering B_2 and B_3 as initial blocks, the lemma applies. These considerations do not fully determine the third row of S_b, but do exclude all possibilities except the following. If there are j twos in row 3 of C ($j = 0, \ldots, k-3$), then there are $k - j - 1$ twos in row 3 of B, $\frac{1}{2}(k-1)(k-2) - j$ twos in row 3 of D, and j twos in row 3 of E.

Now consider S_{k+2}, the structural matrix of the blocks which correspond to the columns of A, the j columns of C which have 2 in row 3, and the $k - j - 1$ columns of E which have 1 in row 3, i.e.,

$$S_{k+2} = \left[\begin{array}{ccc|c|c} k & 1 & 1 & 1 \ldots 1 & 2 \ldots 2 \\ & k & 1 & 2 \ldots 2 & 1 \ldots 1 \\ & & k & 2 \ldots 2 & 1 \ldots 1 \\ \hline & & & F & G \\ \hline & & & & H \end{array}\right] \tag{5.2}$$

where F and H have k in the main diagonal, but the other elements of F, G, and H are so far unknown but will now be determined. Comparisons of the structure of S_{k+2} with the structures of S_6, $S_5^{(1)}$, $S_5^{(2)}$ shows that Lemmas 4.3, 4.4, and 4.5 apply, and hence the non-diagonal elements of F and H are 1, and the elements of G are 2.

Corresponding to S_{k+2} is the characteristic matrix C_{k+2}, which has $2k$ in the main diagonal and $k - 2$ or -4 elsewhere, according as S_{k+2} has 1 or 2 in the corresponding position. Calculating $|C_{k+2}|$, which is readily done with the help of Lemma 3.1 of **(2)**, we obtain

$$|C_{k+2}| = j(j - k + 2)(k - 6)(k + 2)^{k+1}. \tag{5.3}$$

Now by Theorem 3.1 of **(2)**, $|C_{k+2}| = 0$. From (5.3), noting that $0 \leqslant j \leqslant k - 3$ and that $k + 2 > b - v = k + 1$, it follows that $|C_{k+2}| = 0$ when and only when $j = 0$ or $k = 6$. The case $k = 6$ was disposed of in **(2)**.

Let $j = 0$. Then by Lemma 4.4, the blocks of N pairwise are first associates, and the blocks of M other than B_3 pairwise are first associates. Thus B_1 and its $2k$ first associates uniquely determine two sets of $k + 1$ blocks which pairwise are first associates. The sets are $S_1(B_1, B_2, B_4, \ldots, B_{k+2})$ and $S_2(B_1, B_3, B_{k+3}, \ldots, B_{2k+1})$. We summarize in the following lemma.

LEMMA 5.1 *Any block B_1 and its $2k$ first associates uniquely comprise two sets S_1 and S_2 of $k + 1$ blocks which pairwise are first associates and have only one common element, B_1.*

6. Conclusion. We have shown that there exist sets S_j which satisfy (1) and (4) of Theorem 2.1. To show that there are $k + 2$ sets S_j in all, we observe that since every block occurs in precisely two sets S_j, and each set contains $k + 1$ blocks, the number n of sets satisfies $2b = n(k + 1)$, whence $n = k + 2$. Further, because any one of the $b = \frac{1}{2}(k + 1)(k + 2)$ blocks is the unique block in common to two unordered sets S_i, S_j, each of the unordered pairs of sets will have a different block in common and thus any two sets will have one and only one block in common. Thus, the sets S_j satisfy (3) of Theorem 2.1.

To prove (2) of Theorem 2.1, let m_i denote the number of treatments which are replicated i times in an S_j. Then the following relations are necessary:

$$\begin{aligned} &\sum_{i=0}^{k+1} m_i = v = \tfrac{1}{2}k(k+1), \\ (6.1) \qquad &\sum_{i=0}^{k+1} i m_i = k(k+1), \\ &\sum_{i=0}^{k+1} i(i-1) m_i = k(k+1), \end{aligned}$$

where the last relation arises because every two blocks of S_j are first associates. Now consider the function

$$Q(i) = \sum_{i=0}^{k+1} m_i (i-2)^2.$$

By (6.1), $Q(i) = 0$, which implies that $i = 2$, since $m_i \geqslant 0$ $(i = 0, 1, \ldots, k+1)$ and

$$\sum_{i=0}^{k+1} m_i > 0.$$

This completes the proof of all properties of the sets S_j required for Theorem 2.1 and incidentally their uniqueness and so in turn the uniqueness of the embedding.

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Ohio State University
National Bureau of Standards, Washington, D.C.