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ON LEVI-LIKE PROPERTIES AND SOME OF THEIR APPLICATIONS IN RIESZ SPACE THEORY

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ABSTRACT. Let (L, λ) be a locally solid Riesz space. (L, λ) is said to have the *Levi property* if for every increasing λ-bounded net $(x_\alpha) \subset L^+$, $\sup x_\alpha$ exists. The Levi property, appearing in literature also as *weak Fatou property* (Luxemburg and Zaanen), condition (B) or *monotone completeness* (Russian terminology), is a classical object of investigation. In this paper we are interested in some variations of the property, their mutual relationships and applications in the theory of topological Riesz spaces. In the first part of the paper we clarify the status of two problems of Aliprantis and Burkinshaw. In the second part we study ideal-injective Riesz spaces.

1. Measurable cardinals, lateral Levi properties and two problems of Aliprantis and Burkinshaw.

Our aim, in this section, is to clarify the status of the following two problems, posed in ([2], page 190).

PROBLEM 1. *Is every Hausdorff locally solid topology on a universally complete Riesz space necessarily Lebesgue*?

PROBLEM 2. *If a universally complete Riesz space admits a Hausdorff locally solid topology does it follow that it must admit a Hausdorff Lebesgue topology also*?

We start by explaining our terminology. Because we consider *lateral* properties and (even "more lateral") *disjoint* properties, in order to be consistent, we had to call *disjoint* some properties that appear in [2] as *lateral*. For instance, a Riesz space L is said to be *disjointly complete* if, for any disjoint family $(x_\alpha) \subset L^+$, $\sup x_\alpha$ exists. Thus L is universally complete if it is Dedekind complete (DC) and disjointly complete. Otherwise we tried to keep our terminology as close to [2] as possible and, in particular, all unexplained terms and notational conventions are used in the sense of [2].

Let (L, λ) be a locally solid Riesz space. We say that (L, λ), or λ alone, has the *Levi property*, if, for all increasing λ-bounded nets $(x_\alpha) \subset L^+$, $\sup x_\alpha$ exists. An

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increasing net $(x_\alpha) \subset L^+$ is said to be *laterally increasing* (notation $x_\alpha \,\lrcorner\!\!\uparrow$), if $x_\beta - x_\alpha \wedge x_\alpha = 0$ for all $\beta > \alpha$. We say that (L, λ) has the *lateral Levi property* if it has the Levi property with respect to laterally increasing nets, i.e., if sup x_α exists whenever $x_\alpha \,\lrcorner\!\!\uparrow$ and (x_α) is λ-bounded. A (formal) series $\sum x_n$, or a sequence (x_n) itself, in a topological vector space, is often said, after Orlicz, to be perfectly bounded if its set of all finite partial sums is bounded. Following this terminology, we will say that a family $\{x_\alpha : \alpha \in A\}$ in (L, λ) is *perfectly bounded* if the set $\{\sum_{\alpha \in E} x_\alpha : E \subset A, E \text{ finite}\}$ is λ-bounded. (L, λ) is said to have the *disjoint Levi property* if, for each disjoint family $(x_\alpha) \subset L^+$ that is perfectly bounded sup $x_\alpha = (o) - \sum_\alpha x_\alpha$ exists.

We will also consider the countable versions of the above properties; they will be called *σ-Levi*, *lateral σ-Levi* and *disjoint σ-Levi property*, respectively. Given a laterally increasing sequence (x_n) in L^+, we may reduce the situation to (x_n) that is actually disjoint. Thus,

1.1. *Lateral and disjoint σ-Levi properties are equivalent.*

We recall that a sequence (x_n) in L is said to be r-Cauchy if there exist $r \in L^+$ and a sequence (ϵ_n) of positive reals, $\epsilon_n \to 0$, such that

$$|x_{n+p} - x_n| \leqq \epsilon_n r \text{ for each } p \in \mathbf{N}.$$

A sequence (x_n) is r-convergent to $x \in L$ if $|x - x_n| \leqq \epsilon_n r$. An Archimedean Riesz space L is r-complete (or relatively uniformly complete) if any r-Cauchy sequence in L is r-convergent in L. Our first result may be considered as a topological analogue of the following theorem of Veksler and Geiler ([12]).

V-G. *For a Riesz space L the following are equivalent.*

(i) *L is σ-Dedekind complete.*

(ii) *L is Archimedean, r-complete and relatively σ-disjointly complete.*

REMARK. Here *relatively* σ-disjointly complete means that if $(x_n) \subset L^+$ is disjoint and *order bounded* then sup x_n exists. We shall use the word "relatively" to qualify properties that hold under additional condition of order boundedness and reserve the terms "bounded", "boundedly" for reference to the boundedness in the sense of topological vector structure.

THEOREM 1.2. *Let (L, λ) be a locally solid Riesz space. The following are equivalent.*

(i) *(L, λ) has the σ-Levi property.*

(ii) *(L, λ) is Hausdorff, r-complete and has the disjoint σ-Levi property.*

We will need two lemmas. The first one can be traced back to Abramovich (see [1] Lemma 2). L^u denotes the universal completion of L.

LEMMA 1.3. *Let (L, λ) be a σ-Dedekind complete Hausdorff locally solid Riesz space having the disjoint σ-Levi property. If $0 \leqq x_n \uparrow x \in L^u$ and the sequence $(x_n) \subset L$ is λ-bounded, then $x \in L$.*

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PROOF. Restricting ourselves to the band generated by (x_n), we may assume that L has a weak unit. By a known representation theorem (see [11] 26.2.8 and 26.2.10) we may identify L with a solid vector subspace of $C^{\infty}(\Omega)$. Here Ω is an ω-extremally disconnected compact Hausdorff space and $C^{\infty}(\Omega)$ denotes the space of continuous functions from Ω into $\bar{\mathbf{R}}$ that can take infinite values on a nowhere dense set. Define

$$E_n = \mathrm{cl}\{\omega \in \Omega: 2x_n(\omega) > x(\omega)\}$$

(with "cl" standing for the closure in Ω). We have $x1_{E_n} \nearrow x$ and, since $x1_{E_n} \leqq 2x_n$, we find that

$$x1_{E_n} \in L \text{ for all } n \in \mathbf{N}$$

and

$$\{x1_{E_n} : n \in \mathbf{N}\}$$

is a λ-bounded set. Hence by the lateral σ-Levi ($\Leftrightarrow$ disjoint σ-Levi) property, $\sup\{x1_{E_n} : n \in \mathbf{N}\} = x$ is in L.

LEMMA 1.4. *A σ-universally complete Hausdorff locally solid Riesz space (L, λ) has the σ-Levi property.*

PROOF. By [2] 24.1, λ is σ-Lebesgue and therefore λ is σ-Fatou. Also since L is σ-universally complete it cannot be further enlarged in the sense of [8] (i.e., $(L, \lambda) = (L^{\sim}, \lambda^{\sim})$). Hence the Lemma follows by Theorem A in [8].

PROOF OF THEOREM 1.2. Only the implication (ii) $\Rightarrow$ (i) is not trivial. Take $0 \leqq x_n \uparrow \subset L$ such that (x_n) is λ-bounded. As above we restrict ourselves to the band generated by (x_n) and embed it as a solid vector subspace of $C^{\infty}(\Omega)$. Suppose that (x_n) is not order bounded in $C^{\infty}(\Omega)$. Then there exists an open subset E of Ω such that $x_n 1_E \uparrow \infty 1_E$. Assume, to simplify notation, that $E = \Omega$. Take any $x \in C^{\infty}(\Omega)$. Then $x_n \wedge x \uparrow x$ and, by Lemma 1.3, $x \in L$. Hence $L = C^{\infty}(\Omega)$ and, by Lemma 1.4, (L, λ) has the σ-Levi property contradicting the fact that (x_n) is not order bounded. Hence (x_n) is order bounded in $C^{\infty}(\Omega)$. Let $x_0 = \sup x_n$ therein. By Lemma 1.3 again, $x_0 \in L$.

Interestingly enough, though the uncountable analogue of V-G holds (cf. [12], Theorem 4), the situation with its topological counterpart is much less obvious. We will now show that under the assumption of the existence of 2-measurable cardinals there exists a universally complete Hausdorff locally convex-solid Riesz space without the Levi property.

EXAMPLE 1.5. Let X be a set and let ν be a non-trivial $\{0, 1\}$-valued (countably additive) measure on $\mathscr{P}(X)$. Then

$$\mathscr{F} = \{E \subset X: \nu(E) = 1\}$$

is a non principal ($\cap\mathscr{F} = \phi$) ultrafilter on X that is closed under countable intersections. Let Y be the set of all non-empty finite sequences of elements of X. If $t \in Y$ and $u \in X$ we denote by $t*u$ the element of Y which we get by putting u behind t. A (Hausdorff) topology on Y is defined by declaring a set G in Y to be open if and only if, for all $t \in G$,

$$\{u \in X: t*u \in G\} \in \mathscr{F}.$$

This topology is extremally disconnected. Indeed, if G is open and $t \in \bar{G}$, we reason as follows. If $t \in G$ then $\{u \in X: t*u \in \bar{G}\}$ contains $\{u \in X: t*u \in G\}$ and the latter set is in $\mathscr{F}$, so the former is also in $\mathscr{F}$. If $t \in \bar{G}\backslash G$ then $\bar{G}\backslash\{t\}$ is not closed, i.e., $H := \{t\} \cup \{Y\backslash\bar{G}\}$ is not open, so there exists $s_0 \in H$ such that $\{u \in X: s_0^* u \in Y\backslash\bar{G}\} \notin \mathscr{F}$. However, $\{u \in X: s*u \in Y\backslash\bar{G}\} \in \mathscr{F}$ for s in $Y\backslash\bar{G}$ and thus $s_0 = t$, and $\{u \in X: t*u \in Y\backslash G\} \notin \mathscr{F}$. As $\mathscr{F}$ is an ultrafilter, $\{u \in X: t*u \in \bar{G}\}$ must be in $\mathscr{F}$. Thus we conclude that indeed $\bar{G}$ is an open set. For y in Y, write

$$U_y = \{t \in Y: t = y \text{ or } t \text{ extends } y\}$$

where "t extends y" means that for $n = 1, 2, \ldots, \text{length}(y)$, we have $t(n) = y(n)$ and other terms of t are arbitrary. The sets U_y are obviously open. We show that they are clopen. Suppose that $t \notin U_y$, i.e., t does not extend y. There are two possibilities:

(1) $t(n) \neq y(n)$ for some $n \leqq \text{length}(y)$

(2) length $(t) <$ length (y), but t "agrees" with y on the initial part of y.

In case (1), $\{u \in X: t*u \in Y\backslash U_y\} = X \in \mathscr{F}$.

In case (2), if length $(t) <$ length $(y) - 1$, then again $\{u \in X: t*u \in Y\backslash U_y\} = X \in \mathscr{F}$. If length $(t) =$ length $(y) - 1$ then, setting $u_0 = y$ (length (y)), $\{u \in X: t*u \in Y\backslash U_y\} = X\backslash\{u_0\}$, which is a member of $\mathscr{F}$, since $\mathscr{F}$ is non principal. We thus conclude that the complement of U_y is open. Let now $C(Y)$ be the Riesz space of all real-valued continuous functions on Y. Then, as Y is extremally disconnected, $C(Y)$ is Dedekind complete. As $\mathscr{F}$ is closed under countable intersections, G_δ-sets are open in Y. It follows that $C(Y)$ is σ-disjointly complete. As $C(Y)$ has a weak unit it is universally complete ([2] 23.24). Equip $C(Y)$ with the topology λ of pointwise convergence. We will show that $(C(Y), \lambda)$ does not have the Levi property. To this end consider the following set in $C(Y)$:

$$B = \{\text{length}(y)1_{U_y} : y \in Y\}$$

The pointwise supremum of B exists (and equals length(y_0) at y_0) but is unbounded on any open set in Y. Thus the finite suprema of elements in B form an increasing net which is λ-bounded but does not have a supremum in $C(Y)$.

We now return to Problems 1 and 2 mentioned at the beginning of this section.

It is known that if a *proof* of the non-existence of a 2-measurable cardinal in ZFC exists, then Problem 1 has the positive solution. On the other hand, the positive answer to Problem 1 easily implies the non-existence of 2-measurable cardinals. One describes such a situation by saying that the positive answer to Problem 1 is *equiconsistent* with the non-existence of measurable cardinals (see [6] for more details). We now show that

1.6. *Problems 1 and 2 are equiconsistent.*

As a positive answer to Problem 1 implies trivially a positive answer to Problem 2, we only have to show that a proof (in ZFC) of a positive answer to Problem 2 implies a positive answer to Problem 1. Suppose that such a proof can be given. Now, a Hausdorff Lebesgue locally solid topology λ on a universally complete space L has the Levi property ([2] Theorem 24.2). Further, if τ is another Hausdorff locally solid topology on L then τ is finer than λ ([2] Theorem 24.4). It follows that τ has the Levi property. Hence every Hausdorff locally solid topology τ on L has the Levi property in contradiction with Example 1.5. Thus, we have a proof of the non-existence of 2-measurable cardinals. Then, according to what is said above only Lebesgue topologies exist on L.

REMARKS 1.7. (a) The space $C(Y)$ that appears in Example 1.5 has been constructed by Fremlin in [6] (and also appeared in [3]).

(b) Every disjointly complete Riesz space L is laterally complete in the sense that if $(x_\alpha) \subset L_+$ is increasing laterally then $x_\alpha \uparrow x \in L$. Indeed, we can define

$$x(\omega) = \begin{cases} x_\alpha(\omega) \text{ for some } \alpha \text{ such that } x_\alpha(\omega) \neq 0 \\ 0 \text{ if there is no such } \alpha. \end{cases}$$

Thus every Hausdorff locally solid universally complete Riesz space has the lateral Levi property in a trivial way and Example 1.5 tells us that there is no hope to obtain, in general, that

$$DC + \textit{lateral Levi} \Rightarrow \textit{Levi}$$

Nonetheless the above implication holds under some (mild) additional assumptions. Let us recall that a locally solid Riesz space (L, τ) is pseudo-Fatou if $0 \leqq x_\alpha \uparrow$, (x_α) τ-Cauchy implies $\sup x_\alpha$ exists. It is known (cf. [9] 3.17) that if (L, τ) is pseudo-Fatou and L is universally complete then τ is Lebesgue Levi. A similar conclusion holds if τ is metrizable ([2] Theorem 24.7). Applying these two facts, a variation of the argument used above to prove Theorem 1.2 also shows the following statement.

Let (L, τ) be a Hausdorff locally solid universally complete Riesz space. Suppose that (L, τ) is either metrizable or pseudo-Fatou and has the lateral Levi property. Then (L, τ) has the Levi property.

(c) In Lemma 1.4 the fact that the topology is σ-Fatou has been used. The following example shows that some assumption of that sort is needed in order to locate the sup of an increasing sequence (x_n) in L^u.

EXAMPLE 1.8. Let S be the set of all non-empty finite sequences of natural numbers. For $s \in S$ define $\lambda(s) = \text{length}(s)$. If $s, t \in S$ define $s \leqq t$ if $\lambda(s) \leqq \lambda(t)$ and $s(i) = t(i)$ for $i = 1, \ldots, \lambda(s)$. For $s \in S$ such that $\lambda(s) = n$ and $i \in \mathbf{N}$ we define $s * i = (s(1), \ldots, s(n), i)$. Put

$$L = \left\{x \in l^\infty(S): \lim_{i\to\infty} x(s * i) = \frac{1}{2}x(s) \text{ for all } s \in S\right\}.$$

One can show that for every $t \in S$:

$$e^t(s) = \begin{cases} \left(\dfrac{1}{2}\right)^{\lambda(s)-\lambda(t)} & \text{if } t \leqq s \\ 0 & \text{in other cases} \end{cases}$$

defines an element e^t of L. Furthermore L is a Banach space under the supremum norm $\|\ \|_\infty$.

Define $f_1 = e^{(1)}$, $f_2 = e^{(1)} \vee e^{(2)} \vee e^{(1,2)} \vee e^{(2,1)}, \ldots$, generally

$$f_n = \sup\{e^t: \lambda(t) \leqq n \text{ and } t(k) \leqq n \text{ for all } k \leqq \lambda(t)\}.$$

The sequence (f_n) is increasing and $\|f_n\|_\infty \leqq 1$. We will prove that $\{f_n : n \in \mathbf{N}\}$ is not order bounded in L^u. According to Theorem 23.22 in [2] we have to show that $\{f_n : n \in \mathbf{N}\}$ is not a dominable subset of L. Suppose it is. Take any $u \in L^+\setminus\{0\}$. There exist $v \in L^+\setminus\{0\}$ and $k \in \mathbf{N}$ such that $(ku - f_n)^+ \geqq v$ for all $n \in \mathbf{N}$. In particular for all $t \in S$ such that $v(t) \neq 0$ we find $ku(t) - 1 \geqq v(t)$. Because $v > 0$, there exists $t_0 \in S$ such that $v(t_0) > 0$. Because

$$\lim_{i\to\infty} u(t_0 * i) = \frac{1}{2}u(t_0) \text{ and } \lim_{i\to\infty} v(t_0 * i) = \frac{1}{2}v(t_0),$$

we can find $i \in \mathbf{N}$ such that

$$u(t_0 * i) \leqq \frac{2}{3}u(t_0)$$

and $v(t_0 * i) \neq 0$.

Define $t_1 = t_0 * i$. Choose inductively $t_n \in S$ such that

$$u(t_{n+1}) \leqq \frac{2}{3}u(t_n),$$

t_{n+1} extends t_n and $v(t_n) \neq 0$ for all $n \in \mathbf{N}$. For all $n \in \mathbf{N}$, we find $ku(t_n) - 1 \geqq v(t_n) > 0$. However

$$ku(t_n) \leqq \left(\frac{2}{3}\right)^n u(t_0)$$

and hence for large n we have $ku(t_n) - 1 < 0$ which is a contradiction.

We remark that the space L has been constructed by Groenewegen in [7] for other purposes.

2. **Ideal-injective Riesz spaces.**

For a Riesz space L, the *order-bound topology* is the topology generated by all Riesz seminorms on L. The order-bound topology of L will be denoted by $\beta(L)$. We consider Archimedean Riesz spaces only. L is called *ideal-injective* if for every Riesz space K, every ideal $I \subset K$ and every $\beta(K)|_I - \beta(L)$ continuous Riesz homomorphism $I \to L$ there exists a Riesz homomorphic extension $K \to L$.

PROPOSITION 2.1. *If $(L, \beta(L))$ has the Levi property then L is ideal-injective.*

PROOF. Let K be a Riesz space, $I \subset K$ an ideal and $\varphi: I \to L$ a $\beta(K)|_I - \beta(L)$ continuous Riesz homomorphism. Take $g \in K^+$. Observe that $A_g = \{\varphi(f): f \in [0, g] \cap I\}$ is an increasing $\beta(L)$-bounded net in L. By the Levi property, sup A_g exists in L. Using the fact that I is an ideal in K, we obtain that the map $g \mapsto \sup A_g$ is positively homogeneous additive and preserves finite suprema on K^+. It therefore can be extended to a Riesz homomorphism $K \to L$.

We need the following notion from [4]. For an ideal $I \subset K$, a map $\varphi: I \to L$ is said to be $c(I, K, L)$-continuous if for any sequence (f_n) in I which is r-convergent to 0 in K, $(\varphi(f_n))$ r-converges to 0 in L.

PROPOSITION 2.2. *Every $c(I, K, L)$-continuous Riesz homomorphism $I \to L$ (where $I \subset K$ is an ideal) is $\beta(K)|_I - \beta(L)$ continuous.*

PROOF. We denote by K_h the ideal generated by $h \in K^+$ in K. Suppose $\varphi: I \to L$ is a $c(I, K, L)$-continuous Riesz homomorphism. Then $\varphi|K_h \cap I$ is $\beta(K_h \cap I) - \beta(L)$ continuous for every $h \in K^+$. Denoting by ρ_h the uniform norm generated by h on K_h, we would like to know whether $\varphi|_{K_h \cap I}$ is also $\rho_h|_{K_h \cap I} - \beta(L)$ continuous for every $h \in K^+$. To this end, let $h \in K^+$ and take any Riesz seminorm p on L. Suppose that $p \circ \varphi|_{K_h \cap I}$ is not $\rho_h|_{K_h \cap I} - \beta(\mathbf{R})$ continuous. Then for each $n \in \mathbf{N}$ we can find $f_n \in K_h \cap I$ such that $\rho_h(f_n) \leqq 1$ and $p \circ \varphi(f_n) \geqq n^2$. Because $((1/n)f_n)$ h-converges to 0, we know that

$$\left\{\frac{1}{n} p \circ \varphi(f_n): n \in \mathbf{N}\right\}$$

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is bounded, which gives the desired contradiction. Therefore, $\varphi: I \to L$ is continuous from the inductive limit $\varinjlim \rho_h|_{K_h \cap I}$ to $(L, \beta(L))$. We are done as soon as we can prove that the topology of $\varinjlim \rho_h|_{K_h \cap I}$ is weaker than $\beta(K)|_I$. As $\varinjlim \rho_h|_{K_h \cap I}$ is a locally convex-solid space (see for instance 4.16 page 108 of [10]), its topology is generated by a collection of seminorms. Let p be any of these seminorms. By definition of inductive limit, for all $h \in K^+$, $p|_{K_h \cap I}$ is continuous relative to $\rho_h|_{K_h \cap I}$. Define for $e \in K$,

$$\bar{p}(e) = \sup\{p(f): f \in I \text{ and } 0 \leqq f \leqq |e|\} < \infty.$$

Then $\bar{p}$ is a Riesz seminorm on K and hence is $\beta(K)$-continuous. Thus p is $\beta(K)|_I$-continuous.

We recall a definition from [10]. A Riesz space L is said to have the *Peressini property* (in [10] and [4]: the boundedness property) if it satisfies either of the following two equivalent conditions.

(i) A set B in L^+ is order bounded whenever $(\alpha_n x_n)$ order converges to 0 for every sequence (x_n) in B and $\alpha_n \downarrow 0$ in $\mathbf{R}$.

(ii) A set B in L^+ is order bounded whenever $(\alpha_n x_n)$ r-converges to 0 for every sequence (x_n) in B and $\alpha_n \downarrow 0$ in $\mathbf{R}$.

The *disjoint Peressini property* is obtained by replacing "a set B" in the above by "a disjoint set B". The following is now a consequence of Theorem 4.4 in [4].

COROLLARY 2.3. *Every ideal-injective space has the disjoint Peressini property and is relatively disjointly complete.*

We are now going to observe a similar phenomenon as in Section 1: to what extent the sufficient condition in Proposition 2.1 is necessary depends on the relation between the Levi property and the disjoint Levi property. More precisely, we have the following theorem.

THEOREM 2.4. *Let L be a Riesz space. Consider the following statements*

(i) *L is ideal-injective*

(ii) *L has the disjoint Levi property.*

If L is r-complete we have (ii) $\Rightarrow$ (i). *If L is $\beta(L)$-sequentially complete we have* (i) $\Rightarrow$ (ii).

PROOF. Suppose L is r-complete. We show (ii) $\Rightarrow$ (i). Suppose L has the disjoint Levi property and I, K are Riesz spaces such that I is an ideal in K. Suppose furthermore that $\varphi: I \to L$ is a $\beta(K)|_I - \beta(L)$ continuous Riesz homomorphism. We show the existence of a Riesz homomorphic extension $K \to L$. Let $C^\infty(\Omega)$, where Ω is an extremally disconnected compact Hausdorff space, be the universal completion of L. Choose, by Zorn's lemma, a maximal disjoint set $\{1_{U_s}: s \in S\}$ contained in the ideal generated by $\varphi(I)$ in $C^\infty(\Omega)$. Let A be the

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support in Ω of the latter ideal. It easily follows that $\cup_{s\in S} U_s$ is a dense subset of A. For every $s \in S$ choose $h_s \in I^+$ such that $1_{U_s} \leqq \varphi(h_s)$.

Let $f \in K^+$. For $x \in \Omega$ we define $l_f(x) \in [0, \infty]$ by

$$l_f(x) = \sup\{\varphi(g)(x): g \in [0, f] \cap I\}.$$

For every $s \in S$ we have

$$l_f(x) = \sup\{\varphi(f \wedge nh_s)(x): n \in \mathbf{N}\} \text{ for all } x \in U_s.$$

For every $s \in S(\varphi(f \wedge nh_s))$ is an increasing $\beta(L)$-bounded sequence. By Lemma 1.4 (here we use r-completeness of L), we have

$$C^\infty\text{-}\sup\{\varphi(f \wedge nh_s): n \in \mathbf{N}\} = L\text{-}\sup\{\varphi(f \wedge nh_s): n \in \mathbf{N}\} \in L^+.$$

Define $F_s = L\text{-}\sup\{\varphi(f \wedge nh_s): n \in \mathbf{N}\}$ and $B = \{x \in \Omega: \exists s \in S$ such that $F_s(x) < \infty$ and $x \in U_s\}$. B is an open dense subset of A and l_f is continuous at every point of B. Thus we can find $f^* \in C^\infty(\Omega)$ such that $f^*|_B = l_f|_B$ and f^* vanishes on the complement of the closure of A. In fact

$$f^* = C^\infty\text{-}\sup\{\varphi(g): g \in [0, f] \cap I\}.$$

We will now sketch the proof of the fact that $f^* \in L$. To this end, consider families of clopen subsets $\{V_t\}$ of Ω such that:

(1) $V_t \cap V_{t'} = \phi$ if $t \neq t'$.

(2) For each t there exists $h_t \in I \cap [0, f]$ with $1_{V_t} \leqq \varphi(h_t)$.

(3) For each t there exists $n \in \mathbf{N}$ such that

$$f^*|_{V_t} = \varphi(f \wedge nh_t)|_{V_t}.$$

Choose by Zorn's lemma a maximal family $\{V_t: t \in T\}$. By maximality we have

(4) $\cup_{t\in T} V_t$ is a dense subset of A.

Define $f_t = f^* 1_{V_t}$. By (3) above and Lemma 3.2 of [4] it follows that $f_t \in L$. By $\beta(K)|_I - \beta(L)$ continuity of φ, it follows that the set

$$\{\text{finite sums of distinct } f_t\}$$

is $\beta(L)$-bounded. By the disjoint Levi property, $L\text{-}\sup\{f_t: t \in T\} = f^{**}$ exists. Since $f^{**} \geqq f^*$ we can use Lemma 3.2 of [4] again to show that $f^* \in L$. It is easy to extend the map $f \mapsto f^*$ $(f \in K^+)$ to a Riesz homomorphism $K \to L$.

Conversely assume that L is $\beta(L)$-sequentially complete. We show (i) $\Rightarrow$ (ii). Because L is ideal-injective, it follows from Corollary 2.3 that L is relatively disjointly complete and has the disjoint Peressini property. By a variation of 23Nf in [5] the $\beta(L)$-sequential completeness implies that L has the disjoint Levi property.

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