Bull. Aust. Math. Soc. **88** (2013), 243–249
doi:10.1017/S0004972712001049

THE DUAL STRUCTURE OF CROSSED PRODUCT C^*-ALGEBRAS WITH FINITE GROUPS

FIRUZ KAMALOV

(Received 10 August 2012; accepted 19 September 2012; first published online 18 January 2013)

Abstract

We study the space of irreducible representations of a crossed product C^*-algebra $A \rtimes_\sigma G$, where G is a finite group. We construct a space $\widetilde{\Gamma}$ which consists of pairs of irreducible representations of A and irreducible projective representations of subgroups of G. We show that there is a natural action of G on $\widetilde{\Gamma}$ and that the orbit space $G \setminus \widetilde{\Gamma}$ corresponds bijectively to the dual of $A \rtimes_\sigma G$.

2010 *Mathematics subject classification*: primary 46L55; secondary 46L05.

Keywords and phrases: irreducible representations, crossed product C^*-algebra, finite groups, dual structure.

1. Introduction

Let A be a C^*-algebra and let G be a locally compact group acting as automorphisms of A via a homomorphism σ into Aut(A). It has been a long-standing problem to describe the ideal structure of the crossed product $A \rtimes_\sigma G$. One approach to describing Prim($A \rtimes_\sigma G$) is to construct a set X whose structure can be understood and then realise Prim($A \rtimes_\sigma G$) as the quotient space of X. Perhaps the best example of such an approach is given by Williams in [7], where A and G are assumed to be abelian. In this case, Prim($A \rtimes_\sigma G$) can be realised as the quotient space of $X = \widehat{A} \times \widehat{G}$. In general, the problem of constructing the appropriate space X seems to be very difficult. Even in special cases where A is Type I or G is amenable the problem remains open [2].

The purpose of this paper is to describe the dual space $\widehat{A \rtimes_\sigma G}$ of $A \rtimes_\sigma G$, that is, the set of all unitary equivalence classes of irreducible representations of $A \rtimes_\sigma G$, when G is finite. The study of crossed products involving finite groups goes back to Rieffel [5]. More recently, it was shown by Arias and Latremoliere in [1] that every irreducible representation of $A \rtimes_\sigma G$ is induced from an irreducible representation of a certain subsystem. In Section 2 we construct a space $\widetilde{\Gamma}$ which consists of pairs of unitary equivalence classes of irreducible representations of A and irreducible projective representations of certain subgroups of G. There is a natural action of G on $\widetilde{\Gamma}$. We define a map Φ from $\widetilde{\Gamma}$ into the set of equivalence classes of irreducible covariant representations of the dynamical system (A, G, σ). In Section 3 we show that the map Φ is surjective. This result is also proved in [1, Theorem 3.4], but we provide an alternative approach. Our main result is Theorem 3.3, where we identify $\widehat{A \rtimes_\sigma G}$ with the set of orbits in $\widetilde{\Gamma}$.

Recall that a covariant representation of (A, G, σ) on a Hilbert space $\mathcal{H}$ is a pair (π, U), where π is a nondegenerate representation of A on $\mathcal{H}$ and U is a homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ such that

$$U(s)\pi(a)U(s)^* = \pi(\sigma_s a)$$

for all $a \in A$ and $s \in G$. There exists a one-to-one correspondence between the covariant representations of the system (A, G, σ) and the nondegenerate representations of $A \rtimes_\sigma G$. Therefore, the study of representations of $A \rtimes_\sigma G$ is equivalent to that of covariant representations of (A, G, σ).

2. The action of G on Γ

Let (A, G, σ) be a dynamical system, where G is a finite group. The action of G on A induces a natural action of G on $\widehat{A}$ given by $[\pi] \mapsto [\pi \circ \sigma_s]$ for all $[\pi] \in \widehat{A}$ and $s \in G$. Define $G_\pi = \{s \in G : [\pi] = [\pi \circ \sigma_s]\}$ to be the stability group for each $[\pi] \in \widehat{A}$. Then for each $s \in G_\pi$ there is a unitary V_s such that $V_s \pi V_s^* = \pi \circ \sigma_s$. A routine calculation shows that the map $s \mapsto V_s$ defines a projective representation of G_π. Let ω be the multiplier of the projective representation V. The multiplier ω and the projective representation V do not depend on the choice of π but only on the equivalence class $[\pi]$. Let W_ω be an ω-representation of G_π. Then according to [4], $\overline{W_\omega}$, the adjoint of W_ω, is an ω^{-1}-representation. We can construct a covariant representation of (A, G_π, σ) by

$$\pi_\omega = \pi \otimes 1 \quad \text{and} \quad U_\omega = V \otimes \overline{W_\omega}. \tag{2.1}$$

The map $W_\omega \mapsto (\pi_\omega, U_\omega)$ sets up a one-to-one correspondence between the set of ω-representations of G_π and the set of all covariant representations of (A, G_π, σ) of the form $(\pi \otimes 1, V \otimes \overline{W_\omega})$. Moreover, the commutant of (π_ω, U_ω) is isomorphic to the commutant of W_ω under the canonical correspondence [6, Lemma 5.2]. In particular, if W_ω is irreducible, then so is (π_ω, U_ω).

Let Γ be the set of all pairs (π, W_ω), where π is an irreducible representation of A and W_ω is an irreducible ω-representation of G_π. There exists a natural action of G on the set Γ which we now describe. For each $s \in G$, we have $G_{\pi\circ\sigma_s} = s^{-1}G_\pi s$. So given a projective representation W_ω of G_π we can construct a projective representation of $G_{\pi\circ\sigma_s}$ by $(s \cdot W_\omega)(s^{-1}ts) = W_\omega(t)$ for all $t \in G_\pi$. Thus we can define the action of G on Γ by

$$(\pi, W_\omega) \mapsto (\pi \circ \sigma_s, s \cdot W_\omega).$$

In order to establish a connection between Γ and $\widehat{A \rtimes_\sigma G}$ we need to extend a representation of (A, G_π, σ) to a representation of (A, G, σ). We will use the Mackey–Takesaki construction of induced representations for this purpose. Since we are working with a finite group G, induced representations are easy to describe. Let H

be a subgroup of G and let (π, U) be a covariant representation of (A, H, σ) on a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}$ be the space of all $\mathcal{H}_0$-valued functions ξ on G satisfying $\xi(ts) = U(t)\xi(s)$ for all $t \in H$ and all $s \in G$. Define $\overline{U}$ to be the homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ given by

$$(\overline{U}(t)\xi)(s) = \xi(st)$$

for all $\xi \in \mathcal{H}$ and $s, t \in G$. For each $a \in A$, define an operator $\overline{\pi}(a)$ on $\mathcal{H}$ by

$$(\overline{\pi}(a)\xi)(s) = \pi(\sigma_s a)\xi(s)$$

for all $\xi \in \mathcal{H}$ and $s \in G$. Then $(\overline{\pi}, \overline{U})$ is the induced covariant representation of (A, G, σ).

Let H be a subgroup of G and let (π, U) be a representation of (A, H, σ). Let $s \in G$. Define a representation $(\pi \circ \sigma_s, U_s)$ of $(A, s^{-1}Hs, \sigma)$ by $U_s(s^{-1}ts) = U(t)$ for all $t \in H$. We want to establish that (π, U) and $(\pi \circ \sigma_s, U_s)$ lead to equivalent representations.

Lemma 2.1. *Let (A, G, σ) be a dynamical system, where G is a finite group. Let H be a subgroup of G and $s \in G$. Suppose that (π, U) and $(\pi \circ \sigma_s, U_s)$ are as above and that $(\overline{\pi}, \overline{U})$ and $(\overline{\pi \circ \sigma_s}, \overline{U_s})$ are the corresponding induced representations of (A, G, σ). Then $(\overline{\pi}, \overline{U})$ is unitarily equivalent to $(\overline{\pi \circ \sigma_s}, \overline{U_s})$.*

Proof. Let $\mathcal{H}$ denote the representation space for $(\overline{\pi}, \overline{U})$ and $\mathcal{H}_s$ denote the representation space for $(\overline{\pi \circ \sigma_s}, \overline{U_s})$. Define a unitary V from $\mathcal{H}$ to $\mathcal{H}_s$ by $(V\xi)(r) = \xi(sr)$ for all $\xi \in \mathcal{H}$ and $r \in G$. For each $\eta \in \mathcal{H}_s$,

$$\begin{aligned}(V\overline{\pi}(a)V^*\eta)(r) &= (\overline{\pi}(a)V^*\eta)(sr)\\ &= \pi(\sigma_{sr}a)(V^*\eta)(sr)\\ &= \pi(\sigma_{sr}a)\eta(r) = (\overline{\pi \circ \sigma_s}(a)\eta)(r)\end{aligned}$$

for all $r \in G$ and $a \in A$. Similarly,

$$(V\overline{U}(t)V^*\eta)(r) = \eta(rt) = (\overline{U_s}(t)\eta)(r)$$

for all $t, r \in G$. It follows that $(\overline{\pi}, \overline{U})$ is equivalent to $(\overline{\pi \circ \sigma_s}, \overline{U_s})$ via the unitary V. □

Let (π_ω, U_ω) be a representation of (A, G_π, σ) as in (2.1). For each representation of the form (π_ω, U_ω), we can induce a representation $(\overline{\pi_\omega}, \overline{U_\omega})$ of (A, G, σ). The commutant of (π_ω, U_ω) is isomorphic to the commutant of $(\overline{\pi_\omega}, \overline{U_\omega})$. In particular, if (π_ω, U_ω) is irreducible, then so is $(\overline{\pi_\omega}, \overline{U_\omega})$. Let (π_1, W_{ω_1}) and $(\pi_2, W_{\omega_2}) \in \Gamma$. We will say that (π_1, W_{ω_1}) is equivalent to (π_2, W_{ω_2}) if π_1 is unitarily equivalent to π_2 and W_{ω_1} is unitarily equivalent to W_{ω_2}. Let $\widetilde{\Gamma}$ be the set of all equivalence classes in Γ. Note that the action of G on Γ induces the action of G on $\widetilde{\Gamma}$.

Lemma 2.2. *Let (A, G, σ) be a dynamical system, where G is a finite group. Let $(\pi_1, W_{\omega_1}), (\pi_2, W_{\omega_2}) \in \Gamma$ and let $(\overline{\pi_{\omega_1}}, \overline{U_{\omega_1}}), (\overline{\pi_{\omega_2}}, \overline{U_{\omega_2}})$ be the corresponding representations of (A, G, σ). If $(\overline{\pi_{\omega_1}}, \overline{U_{\omega_1}})$ is unitarily equivalent to $(\overline{\pi_{\omega_2}}, \overline{U_{\omega_2}})$, then (π_1, W_{ω_1}) is equivalent to $(\pi_2 \circ \sigma_s, s \cdot W_{\omega_2})$ for some $s \in G$.*

https://doi.org/10.1017/S0004972712001049 Published online by Cambridge University Press

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be representation spaces for $(\overline{\pi_{\omega_1}}, \overline{U_{\omega_1}})$ and $(\overline{\pi_{\omega_2}}, \overline{U_{\omega_2}})$ respectively. Let $\{r_i\}$ be the set of right coset representatives of G_{π_1} in G. Define $\mathcal{H}_i = \{\xi \in \mathcal{H} : \xi(t) = 0 \text{ for all } t \notin G_{\pi_1} r_i\}$, that is, $\mathcal{H}_i$ is the set of functions in $\mathcal{H}$ supported on the coset $G_{\pi_1} r_i$. Then $\overline{\pi_{\omega_1}}_{|\mathcal{H}_i}$ is equivalent to $\pi_{\omega_1} \circ \sigma_{r_i}$ for each r_i and $\overline{\pi_{\omega_1}}$ decomposes as a direct sum of disjoint representations

$$\overline{\pi_{\omega_1}} = \bigoplus_i \pi_{\omega_1} \circ \sigma_{r_i}.$$

Similarly, $\overline{\pi_{\omega_2}} = \bigoplus_j \pi_{\omega_2} \circ \sigma_{s_j}$, where $\{s_j\}$ is the set of right coset representatives of G_{π_2} in G. Since $(\overline{\pi_{\omega_1}}, \overline{U_{\omega_1}})$ is unitarily equivalent to $(\overline{\pi_{\omega_2}}, \overline{U_{\omega_2}})$ there is a unitary V such that $V\overline{\pi_{\omega_1}} = \overline{\pi_{\omega_2}}V$ and $V\overline{U_{\omega_1}} = \overline{U_{\omega_2}}V$. We can view V as a matrix operator with respect to decomposition $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ and $\mathcal{K} = \bigoplus_j \mathcal{K}_j$. Since $\{\pi_1 \circ \sigma_{r_i}\}_i$ are mutually inequivalent representations and $\{\pi_2 \circ \sigma_{s_j}\}_j$ are also mutually inequivalent, V is a permutation matrix whose nonzero entries are unitaries. Therefore, there exists a unitary V_{j1} such that $V_{j1}\pi_{\omega_1} = (\pi_{\omega_2} \circ \sigma_{s_j})V_{j1}$ for some s_j. It follows that π_1 is equivalent to $\pi_2 \circ \sigma_{s_j}$ and $G_{\pi_1} = s_j^{-1} G_{\pi_2} s_j$. Observe that the restriction of $\overline{U_{\omega_1}}_{|\mathcal{H}_1}$ to G_{π_1} is equivalent to the representation U_{ω_1} and the restriction of $\overline{U_{\omega_2}}_{|\mathcal{K}_1}$ to G_{π_2} is equivalent to the representation U_{ω_2}. Since $V\overline{U_{\omega_1}} = \overline{U_{\omega_2}}V$, $V_{j1}\overline{U_{\omega_1}}_{|\mathcal{H}_1}(r) = \overline{U_{\omega_2}}_{|\mathcal{K}_j}(r)V_{j1}$ for all $r \in G_{\pi_1}$. Also $\overline{U_{\omega_2}}_{|\mathcal{K}_j}(s_j^{-1} t s_j)$ is equivalent to $\overline{U_{\omega_2}}_{|\mathcal{K}_1}(t)$ for all $t \in G_{\pi_2}$. Therefore, $U_{\omega_1}(s_j^{-1} t s_j)$ is equivalent to $U_{\omega_2}(t)$ for all $t \in G_{\pi_2}$. It follows that (π_1, W_{ω_1}) is equivalent to $(\pi_2 \circ \sigma_{s_j}, s_j \cdot W_{\omega_2})$. □

Define a map Φ from $\widetilde{\Gamma}$ into the set of equivalence classes of irreducible covariant representations of (A, G, σ) by

$$\Phi(\pi, W_\omega) = (\overline{\pi_\omega}, \overline{U_\omega}). \tag{2.2}$$

If (π_1, W_{ω_1}) is equivalent to (π_2, W_{ω_2}), then $\Phi(\pi_1, W_{\omega_1})$ is equivalent to $\Phi(\pi_2, W_{\omega_2})$. So Φ is well defined. The next result follows directly from Lemmas 2.1 and 2.2.

Corollary 2.3. *Let (A, G, σ) be a dynamical system, where G is a finite group. Suppose that (π_1, W_{ω_1}) and $(\pi_2, W_{\omega_2}) \in \widetilde{\Gamma}$. Then $\Phi(\pi_1, W_{\omega_1}) = \Phi(\pi_2, W_{\omega_2})$ if and only if $(\pi_2, W_{\omega_2}) = (\pi_1 \circ \sigma_s, s \cdot W_{\omega_1})$ for some $s \in G$.*

3. The main result

The remaining step in obtaining our main result is to show that the map Φ, as defined in (2.2), is surjective. We first need the following elementary lemma about projections.

Lemma 3.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathbb{A}$ be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$. Let p_1 and p_2 be a pair of projections in $\mathbb{A}$. Suppose that $q = p_1 - (p_1 \wedge p_2)$. Then $q \wedge p_2 = 0$. Moreover, if p_2 is a minimal projection, then $(p_1 \vee p_2) - p_1$ is a minimal projection in $\mathbb{A}$.*

https://doi.org/10.1017/S0004972712001049 Published online by Cambridge University Press

PROOF. Suppose that $qh_1 = p_2h_2$ for some $h_1, h_2 \in \mathcal{H}$. Since $q \le p_1$, we have $p_1p_2h_2 = p_2h_2$. Hence, $(p_1 \wedge p_2)h_2 = p_2h_2$. It follows that $(p_1 \wedge p_2)h_2 = qh_1$. But $q \wedge (p_1 \wedge p_2) = 0$, so $qh_1 = 0$.

To prove the second part of the statement let $e = (p_1 \vee p_2) - p_1$. Suppose that there exists a nonzero projection $e' \in \mathbb{A}$ such that $e' \lneq e$. Then $p_2e' \neq 0$ and $p_2e'\mathcal{H} \subsetneq p_2\mathcal{H}$. Let p_2' be the projection onto the closure of the range of p_2e'. Then $p_2' \in \mathbb{A}$ and $p_2' \lneq p_2$, which is a contradiction. It follows that e is a minimal projection. □

Let (π, U) be a covariant representation of (A, G, σ) on a Hilbert space $\mathcal{H}$. There is a natural action of G on the von Neumann algebra $\pi(A)'$ given by $T \mapsto U(s)TU(s)^*$ for all $T \in \pi(A)'$. We say that the action of G on a von Neumann algebra $\mathbb{A}$ is ergodic if the only elements of $\mathbb{A}$ that are fixed by the group action are the scalar multiples of the identity operator. It was shown in [1, Theorem 3.1], using a powerful result of [3], that von Neumann algebras which admit ergodic action by a finite group are necessarily finite-dimensional. We present this result below with an alternative proof.

PROPOSITION 3.2. *Let U be a unitary representation of a finite group G on a Hilbert space $\mathcal{H}$. Suppose that G acts ergodically on a von Neumann algebra $\mathbb{A}$ in $\mathcal{B}(\mathcal{H})$. Then there exists a finite family of minimal projections $p_i \in \mathbb{A}$ such that $\bigoplus p_i = 1_{\mathcal{H}}$.*

PROOF. We will first show that there exists a minimal projection $p \in \mathbb{A}$ together with a subset $S \subseteq G$ such that $\bigvee_{s_j \in S} U(s_j)pU(s_j)^* = 1_{\mathcal{H}}$ and $(\bigvee_{j \le i-1} U(s_j)pU(s_j)^*) \wedge U(s_i)pU(s_i)^* = 0$ for all $s_i \in S$. To this end, let $p \in \mathbb{A}$ and $S' \subseteq G$ such that

$$\Big(\bigvee_{j \le i-1} U(s_j)pU(s_j)^*\Big) \wedge U(s_i)pU(s_i)^* = 0 \quad \text{for all } s_i \in S'.$$

Suppose that p is not a minimal projection. It will be enough to show that there is a projection $p' \in \mathbb{A}$ and $t \in G - S'$ such that

$$\Big(\bigvee_{j \le i-1} U(s_j)p'U(s_j)^*\Big) \wedge U(s_i)p'U(s_i)^* = 0 \quad \text{for all } s_i \in S,$$

where $S = S' \cup \{t\}$. Since G is finite we will eventually obtain a minimal projection.

For each projection $q \in \mathbb{A}$, we have $\sum_G U(s)qU(s)^* \in \mathbb{A}$. Moreover,

$$U(t)\Big(\sum_G U(s)qU(s)^*\Big)U(t)^* = \sum_G U(s)qU(s)^*$$

for all $t \in G$. Since the group action is ergodic, $\sum_G U(s)qU(s)^* = c1_{\mathcal{H}}$ for some complex number c. It follows that

$$\bigvee_G U(s)qU(s)^* = 1_{\mathcal{H}} \tag{3.1}$$

for all nonzero projections $q \in \mathbb{A}$. Assume, without loss of generality, that $1_G \in S'$. Moreover, by replacing p with a proper, nonzero subprojection we can assume

https://doi.org/10.1017/S0004972712001049 Published online by Cambridge University Press

that $\bigvee_{s \in S'} U(s)pU(s)^* < 1_{\mathcal{H}}$. By (3.1), there exists $t \in G$ such that $U(t)pU(t)^* \nleq \bigvee_{s \in S'} U(s)pU(s)^*$. Note that $t \notin S'$. Let

$$q = U(t)pU(t)^* - \Big(U(t)pU(t)^* \wedge \Big(\bigvee_{s \in S'} U(s)pU(s)^*\Big)\Big).$$

By Lemma 3.1, $q \wedge (\bigvee_{s \in S'} U(s)pU(s)^*) = 0$. Then $p' = U(t)^* q U(t)$ is the desired projection.

We will now describe how to transform the set of minimal projections $\{U(s_i)pU(s_i)^*\}_{s_i \in S}$ obtained above into a set of orthogonal minimal projections. Let $q_i = U(s_i)pU(s_i)^*$ for all $s_i \in S$. For each $i \geq 2$, define

$$p_i = \bigvee_{1 \leq j \leq i} q_j - \bigvee_{1 \leq j \leq i-1} q_j$$

and $p_1 = q_1$. Then $p_i \in \mathbb{A}$ for all i, and $p_i \perp p_j$ for all $i \neq j$. Moreover, by the second part of Lemma 3.1, each p_i is a minimal projection. □

Suppose that (π, U) is an irreducible representation of (A, G, σ). Then the action of G on $\pi(A)'$ is ergodic. Applying Proposition 3.2 to the algebra $\pi(A)'$, we get that π decomposes as a direct sum of finitely many irreducible representations. Let ρ be an irreducible subrepresentation of π. It follows from [1, Theorem 3.4] that there exists an irreducible ω-representation of G_ρ such that (π, U) is unitarily equivalent to $(\overline{\rho_\omega}, \overline{U_\omega})$. It follows that the map Φ, as defined in (2.2), is surjective. We are now in position to state our main theorem.

Theorem 3.3. *Suppose that $A \rtimes_\sigma G$ is a crossed product C^*-algebra, where G is a finite group. Let $\widetilde{\Gamma}\backslash G$ be the set of orbits in $\widetilde{\Gamma}$ under the group action. Then there exists a bijective correspondence between $\widetilde{\Gamma}\backslash G$ and the dual space $\widehat{A \rtimes_\sigma G}$.*

Proof. Recall that there is a canonical correspondence between the irreducible representations of $A \rtimes_\sigma G$ and (A, G, σ). By the preceding discussion the map $\Phi : \widetilde{\Gamma} \mapsto \widehat{A \rtimes_\sigma G}$ is surjective. Moreover, by Corollary 2.3, $\Phi(\pi_1, W_{\omega_1}) = \Phi(\pi_2, W_{\omega_2})$ if and only if (π_2, W_{ω_2}) is in the orbit of (π_1, W_{ω_1}). □

References

[1] A. Arias and F. Latremoliere, 'Irreducible representations of C^*-crossed products by finite groups', *J. Ramanujan Math. Soc.* **25**(2) (2010), 193–231.
[2] S. Echterhoff and D. Williams, 'Inducing primitive ideals', *Trans. Amer. Math. Soc.* **360**(11) (2008), 6113–6129.
[3] R. Hoegh-Krohn, M. Landstad and E. Stormer, 'Compact ergodic groups of automorphisms', *Ann. of Math. (2)* **114**(1) (1981), 75–86.
[4] G. Mackey, 'Unitary representations of group extensions, I', *Acta Math.* **99** (1958), 265–311.
[5] M. Rieffel, 'Actions of finite groups on C^*-algebras', *Math. Scand.* **47** (1980), 176–257.

[6] M. Takesaki, 'Covariant representations of C^*-algebras and their locally compact automorphism groups', *Acta Math.* **119** (1967), 273–303.
[7] D. Williams, 'The topology on the primitive ideal space of transformation group C^*-algebras and CCR transformation group C^*-algebras', *Trans. Amer. Math. Soc.* **266** (1981), 335–359.

FIRUZ KAMALOV, Mathematics Department,
Canadian University of Dubai, Dubai, UAE
e-mail: firuz@cud.ac.ae

https://doi.org/10.1017/S0004972712001049 Published online by Cambridge University Press