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OPPENHEIM'S INEQUALITY FOR THE SECOND IMMANANT

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ABSTRACT. Denote by d_2 the immanant afforded by S_n and the character corresponding to the partition $(2, 1^{n-2})$. If $n \geq 4$, the following analog of Oppenheim's inequality is proved:

$$d_2(A \circ B) \geq \left(\prod_{t=1}^{n} a_{tt}\right) d_2(B),$$

for all n-by-n positive semidefinite hermitian A and B.

Let χ be an irreducible character of the symmetric group S_n. The *immanant* afforded by χ is the complex valued function of the n-by-n (complex) matrices $A = (a_{ij})$ defined by

$$d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

The irreducible characters of S_n correspond in a natural way to the partitions of n. For example, ϵ, the alternating character, corresponds to the partition (1^n) and d_ϵ is the determinant. In 1930, A. Oppenheim proved the following inequality for the Hadamard product, $A \circ B$, of positive semidefinite hermitian matrices (write $A, B \geq 0$):

$$\det(A \circ B) \geq \left(\prod_{t=1}^{n} a_{tt}\right) \det(B). \tag{1}$$

In this note, we prove an analogous result for the "second immanant."

Denote by χ_2 the character of S_n corresponding to the partition $(2, 1^{n-2})$. Then

$$\chi_2(\sigma) = \epsilon(\sigma)(F(\sigma) - 1), \qquad \sigma \in S_n,$$

where $F(\sigma)$ is the number of fixed points of σ. We will write d_2 for the immanant afforded by χ_2. This *second immanant* has been the object of several recent studies. (See, e.g., [3], [4], and [6].)

THEOREM. *If* $n \geq 4$, *then*

$$d_2(A \circ B) \geq \left(\prod_{t=1}^{n} a_{tt}\right) d_2(B), \tag{2}$$

for all $A, B \geq 0$.

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Before proving the theorem, we give an application and show that the assumption $n \geq 4$ is necessary. If we denote by J_k the k-by-k matrix each of whose entries is one, and let $A = J_k \oplus J_{n-k}$, then

$$A \circ B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix},$$

where B_{11} is the leading k-by-k principal submatrix of B, and B_{22} is the complementary principal submatrix. With this choice for A, Inequality (2) becomes

$$(3) \qquad d_2\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \geq d_2(B), \qquad B \geq 0.$$

This analog of the Fischer Inequality was obtained previously in [4].

In case $n = 2$, d_2 is the permanent, and (2) is actually reversed: Indeed, $\operatorname{per}(A \circ B) \leq a_{11}a_{22} \operatorname{per}(B)$ is equivalent to $0 \leq |b_{12}|^2 \det(A)$ in the 2-by-2 case. If $n = 3$, then

$$d_2(A) = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

Letting

$$A = \frac{1}{3}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{3}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

we find that

$$d_2(A \circ B) = \frac{1460}{729} < \frac{56}{27} = \left(\prod_{t=1}^{3} a_{tt}\right) d_2(B).$$

In particular, (2) is invalid. In this case, however, the reversed inequality is also invalid: If $B = J_3$, then $d_2(B) = 0$. Taking $A = I_3$, the identity matrix, we see that

$$d_2(A \circ B) = d_2(A) = 2 > 0 = \left(\prod_{t=1}^{3} a_{tt}\right) d_2(B).$$

PROOF. If either A or B has a zero on the main diagonal, then both sides of (2) are zero and we are finished. Otherwise, we may write $A = C \circ \hat{A}$, where the (i, j)-entry of C is $c_{ij} = (a_{ii}a_{jj})^{1/2}$ and the (i, j)-entry of $\hat{A}$ is a_{ij}/c_{ij}. Denote by $\mathscr{C}_n$ the set of n-by-n *correlation matrices*, i.e.,

$$\mathscr{C}_n = \{X = (x_{ij}) | X \geq 0 \quad \text{and} \quad x_{ii} = 1 \text{ for all } i\}.$$

Then $\hat{A} \in \mathscr{C}_n$. Moreover, since d_2 is a multilinear function of its rows (or columns),

$$d_2(A \circ B) = \left(\prod_{t=1}^{n} a_{tt}\right) d_2(\hat{A} \circ B).$$

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Therefore, our desired inequality can be replaced by $d_2(\hat{A} \circ B) \geq d_2(B)$, for all $B \geq 0$ and $\hat{A} \in \mathscr{C}_n$. Modifying B in the same way, we find that it suffices to prove

$$d_2(\hat{A} \circ \hat{B}) \geq d_2(\hat{B}), \tag{4}$$

for all $\hat{A}, \hat{B} \in \mathscr{C}_n$. It is proved in [1, Corollary 2] that the spectrum of B majorizes the spectrum of $\hat{A} \circ B$ when B is hermitian and $\hat{A} \in \mathscr{C}_n$. (See [5] for an outstanding treatment of majorization.) On the other hand, it was shown in [4] that if $n \geq 4$, the restriction of d_2 to $\mathscr{C}_n$ is a Schur-concave function of the spectrum, i.e., if $X, Y \in \mathscr{C}_n$, and if the spectrum of Y majorizes the spectrum of X, then $d_2(X) \geq d_2(Y)$. Thus, (4) is immediate from these two previous results.

Denote by χ_k the character of S_n corresponding to the partition $(k, 1^{n-k})$ and by d_k (rather than d_{χ_k}) the corresponding immanant.

CONJECTURE. *If $2 < k \leq n/2$, then*

$$d_k(A \circ B) \geq \left(\prod_{t=1}^{n} a_{tt}\right) d_k(B),$$

for all $A, B \geq 0$.

In this notation, d_n is the permanant. It was conjectured in [1] (also see [2]) that $\mathrm{per}(A \circ B) \leq (\Pi\, a_{tt})\mathrm{per}\,(B)$ for all $A, B \geq 0$ and for all n.

REFERENCES

1. R. B. Bapat and V. S. Sunder, *On majorization and Schur products*, Linear Algebra Appl. **72** (1985), pp. 107–117.

2. J. Chollet, *Is there a permanental analogue to Oppenheim's inequality?* Amer. Math. Monthly **89** (1982), pp. 57–58.

3. R. Grone, *An inequality for the second immanant*, Linear and Multilinear Algebra **18** (1985), pp. 147–152.

4. R. Grone and R. Merris, *A Fischer inequality for the second immanant*, Linear Algebra Appl., **87** (1987), 77–83.

5. A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

6. R. Merris, *The second immanantal polynomial and the centroid of a graph*, SIAM J. Algebraic and Discrete Methods, **7** (1986), 484–503.

7. A. Oppenheim, *Inequalities connected with definite hermitian forms*, J. London Math. Soc. **5** (1930), pp. 114–119.

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