COMPOSITIO MATHEMATICA

On the volume of isolated singularities

Yuchen Zhang

FOUNDATION
COMPOSITIO
MATHEMATICA

LONDON
MATHEMATICAL
SOCIETY

On the volume of isolated singularities

Yuchen Zhang

Abstract

We give an equivalent definition of the local volume of an isolated singularity $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$ given in [S. Boucksom, T. de Fernex, C. Favre, *The volume of an isolated singularity.* Duke Math. J. **161** (2012), 1455–1520] in the $\mathbb{Q}$-Gorenstein case and we generalize it to the non-$\mathbb{Q}$-Gorenstein case. We prove that there is a positive lower bound depending only on the dimension for the non-zero local volume of an isolated singularity if X is Gorenstein. We also give a non-$\mathbb{Q}$-Gorenstein example with $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$, which does not allow a boundary Δ such that the pair (X,Δ) is log canonical.

1. Introduction

Let X be a normal variety and let $f : X \to X$ be a finite endomorphism, i.e. a finite surjective morphism of degree greater than 1. If X is projective, an abundant literature shows that the existence of an endomorphism imposes strong restrictions on the global geometry of X. In a recent paper [BdFF12], Boucksom *et al.* introduce the volume $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$ of an isolated singularity and show that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ if there exists an endomorphism preserving the singularity. When X is $\mathbb{Q}$-Gorenstein, they show that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ if and only if X has log canonical singularities. For a better understanding of $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$, they suggest two problems.

Problem A. Does there exist a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume?

Problem B. Is it true that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X,Δ) is log canonical? (The converse is easily shown.)

In this paper, we will give an alternative definition of the volume in the $\mathbb{Q}$-Gorenstein case via log canonical modification (the existence of these modifications is shown in [OX12]). By using the DCC for the volume established in [HMX12], we will give a positive answer to Problem A.

THEOREM 1.1 (See Theorem 3.3). *There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

We will then generalize this new definition to the non-$\mathbb{Q}$-Gorenstein case. We will define the augmented volume $\mathrm{Vol}^+(X)$ as the liminf of the mth limiting volumes $\mathrm{Vol}_m(X)$. We will show that the augmented volume $\mathrm{Vol}^+(X)$ is greater than or equal to the volume $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$. When there exists a boundary Δ on X such that the pair (X,Δ) is log canonical, we have the following theorem.

THEOREM 1.2 (See Corollary 4.6). *The following statements are equivalent.*

(i) *There exists a boundary Δ on X such that (X,Δ) is log canonical.*
(ii) $\mathrm{Vol}_m(X) = 0$ *for some (hence any multiple of) integer $m \geqslant 1$.*

In §4.1, we will give a counterexample to Problem B.

THEOREM 1.3 (See Theorem 4.9). *There exists a polarized smooth variety (V, H) such that the affine cone $X = C(V, H)$ has positive $\mathrm{Vol}_m(X)$ for any positive integer m, but $\mathrm{Vol}_{BdFF}(X, 0) = 0$.*

A new idea is needed to investigate whether $\mathrm{Vol}^+(X) = \mathrm{Vol}_{BdFF}(X)$.

In [Ful12], Fulger introduces a different invariant $\mathrm{Vol}_F(X, 0)$ associated to an isolated singularity. It is shown that $\mathrm{Vol}_{BdFF}(X, 0) \geqslant \mathrm{Vol}_F(X, 0)$, with equality if X is $\mathbb{Q}$-Gorenstein. In [BdFF12, Example 5.4], an example is given where $\mathrm{Vol}_{BdFF}(X, 0) > \mathrm{Vol}_F(X, 0)$.

It is conjectured that for a normal variety X (perhaps not $\mathbb{Q}$-Gorenstein) which has only isolated singularities, there is a log canonical modification $f : Y \to X$ in the sense that $K_Y + E_f$ is f-ample and (Y, E_f) is log canonical. In [BH12, Proposition 2.4], it is proved that if such modification exists, then $\mathrm{Vol}_{BdFF}(X, 0) = 0$ if and only if f is an isomorphism in codimension 1.

2. Preliminaries

Throughout this paper, X is a normal variety over $\mathbb{C}$.

2.1 Valuations of $\mathbb{Q}$-divisors

Let X be a normal variety. A divisorial valuation v on X is a discrete valuation of the function field of X of the form $v = q\,\mathrm{val}_F$ where q is a positive integer and F is a prime divisor over X. Let $\mathcal{J} \subset \mathcal{K}$ be a finitely generated sub-$\mathcal{O}_X$-module of the constant sheaf of rational functions $\mathcal{K} = \mathcal{K}_X$ on X. We will refer to $\mathcal{J}$ as a fractional ideal sheaf on X, for short.

The valuation $v(\mathcal{J})$ of a non-zero fractional ideal sheaf $\mathcal{J} \subset \mathcal{K}$ along v is given by

$$v(\mathcal{J}) = \min\{v(\phi) \mid \phi \in \mathcal{J}(U), U \cap c_X(v) \neq \emptyset\}.$$

The valuation $v(I)$ of a formal linear combination $I = \sum a_k \cdot \mathcal{J}_k$ of fractional ideal sheaves $\mathcal{J}_k \subset \mathcal{K}$ along v is defined by $v(I) = \sum a_k \cdot v(\mathcal{J}_k)$, where a_k are real numbers.

The $\natural$-*valuation* (or *natural valuation*) along v of an $\mathbb{R}$-Weil divisor D on X is $v^\natural(D) = v(\mathcal{O}_X(-D)) = v(\mathcal{O}_X(\lfloor -D \rfloor))$. If C is Cartier, then we have that $v^\natural(C) = v(C)$ and $v^\natural(C + D) = v(C) + v^\natural(D)$. Note also that, as $\mathcal{O}_X(D) \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$, we have that $v^\natural(D) + v^\natural(-D) \geqslant 0$.

To any non-trivial fractional ideal sheaf $\mathcal{J}$ on X, we associate the divisor $\mathrm{div}(\mathcal{J}) = \sum \mathrm{val}_E^\natural(\mathcal{J}) \cdot E$, where the sum is taken over all prime divisors E on X. Consider now a birational morphism $f : Y \to X$ from a normal variety Y. For any divisor D on X, the $\natural$-*pullback* (or *natural pullback*) of D to Y is given by $f^\natural D = \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)$. In the other words, $f^\natural D = \sum \mathrm{val}_E^\natural(D) \cdot E$, where the sum is taken over all prime divisors E on Y. In particular, $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.

For every divisor D on X and every positive integer m, it is shown in [dFH09, Lemma 2.8] that $m \cdot v^\natural(D) \geqslant v^\natural(mD)$ and

$$\inf_{k \geqslant 1} \frac{v^\natural(kD)}{k} = \liminf_{k \to \infty} \frac{v^\natural(kD)}{k} = \lim_{k \to \infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

Let D be an $\mathbb{R}$-divisor on X. We define the *valuation along v of D* by

$$v(D) = \lim_{k \to \infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

Remark 2.1. It is not hard to see that actually

$$v(D) = \lim_{k \to \infty} \frac{v^\natural(kD)}{k} \in \mathbb{R}.$$

See [BdFF12, Proposition 2.1].

Remark 2.2. Even if D is a $\mathbb{Q}$-divisor on X, the valuation $v(D)$ may not be a rational number. See [Urb12, § 3].

If $f : Y \to X$ is a birational morphism from a normal variety Y, then the *pullback of D to Y* is defined by

$$f^*D = \sum \mathrm{val}_E(D) \cdot E,$$

where the sum is taken over all prime divisors E on Y. Notice that if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and m is a positive integer such that mD is Cartier, then

$$v(D) = \frac{v(mD)}{m} \quad \text{and} \quad f^*D = \frac{f^*(mD)}{m},$$

which coincide with the usual valuation and pullback of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If C is $\mathbb{Q}$-Cartier, then $v(C + D) = v(C) + v(D)$ and $f^*(C + D) = f^*C + f^*D$.

LEMMA 2.3. *Let $f : Y \to X$ and $g : V \to Y$ be two birational morphisms of normal varieties. Then, for every divisor D on X, the divisor $(fg)^\natural D - g^\natural(f^\natural D)$ is effective and g-exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, then $(fg)^\natural D = g^\natural(f^\natural D)$. A similar statement applies to f^* and g^*.*

Proof. See [dFH09, Lemma 2.7 and Remark 2.13]. □

2.2 Relative canonical divisors

We recall that a canonical divisor K_X on a normal variety X is, by definition, the (componentwise) closure of any canonical divisor of the regular locus of X. We also recall that X is said to be $\mathbb{Q}$-Gorenstein if some (equivalently, every) canonical divisor K_X is $\mathbb{Q}$-Cartier. For a proper birational morphism $f : Y \to X$ of normal varieties, we fix a canonical divisor K_Y on Y such that $f_*K_Y = K_X$. For any divisor D on X, we will write D_Y for the strict transform $f_*^{-1}D$ of D on Y.

For every $m \geqslant 1$, the *mth limiting relative canonical $\mathbb{Q}$-divisor $K_{m,Y/X}$ of Y over X* is

$$K_{m,Y/X} = K_Y - \frac{1}{m}f^\natural(mK_X).$$

The *relative canonical $\mathbb{R}$-divisor $K_{Y/X}$ of Y over X* is

$$K_{Y/X} = K_Y - f^*K_X.$$

Clearly, $K_{Y/X}$ is the limsup of the $\mathbb{Q}$-divisors $K_{m,Y/X}$. A $\mathbb{Q}$-divisor Δ on X is said to be a *boundary*, if $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The *log relative canonical $\mathbb{Q}$-divisor of (Y, Δ_Y) over (X, Δ)* is given by

$$K_{Y/X}^\Delta = K_Y + \Delta_Y - f^*(K_X + \Delta).$$

Remark 2.4. Our definition of the relative canonical $\mathbb{R}$-divisor is different from that in [dFH09]. In this paper, the relative canonical $\mathbb{R}$-divisor is defined as $K_{Y/X} = K_Y + f^*(-K_X)$. And $K_Y - f^*K_X$ is denoted by $K_{Y/X}^-$. It can be shown that, with this notation, $K_{Y/X} \geqslant K_{Y/X}^-$. But they are not equal in general. See [dFH09, Example 3.4].

For every integer $m \geqslant 1$, the *mth limiting log discrepancy $\mathbb{Q}$-divisor $A_{m,Y/X}$* of Y over X is

$$A_{m,Y/X} = K_Y + E_f - \frac{1}{m} f^{\natural}(mK_X),$$

where E_f is the reduced exceptional divisor of f. The *log discrepancy $\mathbb{R}$-divisor $A_{Y/X}$* of Y over X is

$$A_{Y/X} = K_Y + E_f - f^* K_X.$$

The *log discrepancy $\mathbb{Q}$-divisor* of (Y, Δ_Y) over (X, Δ) is given by

$$A^{\Delta}_{Y/X} = K_Y + \Delta_Y + E_f - f^*(K_X + \Delta).$$

Consider a pair $(X, I = \sum a_k \cdot \mathcal{J}_k)$ where $\mathcal{J}_k$ are non-zero fractional ideal sheaves on X and a_k are real numbers. A *log resolution* of (X, I) is a proper birational morphism $f : Y \to X$ from a smooth variety Y such that for every k the sheaf $\mathcal{J}_k \cdot \mathcal{O}_Y$ is the invertible sheaf corresponding to a divisor E_k on Y, the exceptional locus $\mathrm{Ex}(f)$ of f is also a divisor, and $\mathrm{Ex}(f) \cup E$ has simple normal crossing, where $E = \bigcup \mathrm{Supp}(E_k)$. If Δ is a boundary on X, then a log resolution of the log pair $((X, \Delta), I)$ is given by a log resolution $f : Y \to X$ of (X, I) such that $\mathrm{Ex}(f) \cup E \cup \mathrm{Supp}(f^*(K_X + \Delta))$ has simple normal crossings. The log resolution always exists (see [dFH09, Theorem 4.2]).

Let X be a normal variety, and fix an integer $m \geqslant 2$. Given a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, a boundary Δ on X is said to be *m-compatible* for X with respect to f if:

(i) $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$;

(ii) f is a log resolution for the log pair $((X, \Delta); \mathcal{O}_X(-mK_X))$; and

(iii) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

THEOREM 2.5. *For any normal variety X, any integer $m \geqslant 2$ and any log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, there exists an m-compatible boundary Δ for X with respect to f.*

Proof. See [dFH09, Theorem 5.4]. □

2.3 Shokurov's *b*-divisors

Let X be a normal variety. The set of all proper birational morphisms $\pi : X_{\pi} \to X$ from a normal variety X_{π} modulo isomorphism is (partially) ordered by $\pi' \geqslant \pi$ if and only if π' factors through π, and any two proper birational morphisms can be dominated by a third one. The *Riemann–Zariski space X* is defined as the projective limit, $\mathcal{X} = \varprojlim_{\pi} X_{\pi}$. The group of *Weil b-divisors* over X is defined as $\mathrm{Div}(\mathcal{X}) = \varprojlim_{\pi} \mathrm{Div}(X_{\pi})$, where $\mathrm{Div}(X_{\pi})$ denotes the group of Weil divisors on X_{π} and the limit is taken with respect to the push-forwards. The group of *Cartier b-divisors* over X is defined as $\mathrm{CDiv}(\mathcal{X}) = \varinjlim_{\pi} \mathrm{CDiv}(X_{\pi})$, where $\mathrm{CDiv}(X_{\pi})$ denotes the group of Cartier divisors on X_{π} and the limit is taken with respect to the pullbacks. An element in $\mathrm{Div}_{\mathbb{R}}(\mathcal{X}) = \mathrm{Div}(\mathcal{X}) \otimes \mathbb{R}$ (respectively, $\mathrm{CDiv}_{\mathbb{R}}(\mathcal{X}) = \mathrm{CDiv}(\mathcal{X}) \otimes \mathbb{R}$) will be called an $\mathbb{R}$-Weil *b*-divisor (respectively, $\mathbb{R}$-Cartier *b*-divisor), and similarly with $\mathbb{Q}$ in place of $\mathbb{R}$. Clearly, a Weil *b*-divisor W over X consists of a family of Weil divisors $W_{\pi} \in \mathrm{Div}(X_{\pi})$ that are compatible under push-forward. We say that W_{π} is the *trace* of W on the model X_{π}. Let C be an Cartier *b*-divisor. We say that $\pi : X_{\pi} \to X$ is a *determination* of C, if C can be obtained by pulling back C_{π} to models dominating π and pushing forward to other models, in which case we denote $C = \overline{C_{\pi}}$.

Let Z and W be two $\mathbb{R}$-Weil *b*-divisors over X. We say that $Z \leqslant W$, if for any model $\pi : X_{\pi} \to X$ we have $Z_{\pi} \leqslant W_{\pi}$. We say that an $\mathbb{R}$-Cartier *b*-divisor is *relatively nef* over X, if

its trace is relatively nef on one (hence any sufficiently high) determination. An $\mathbb{R}$-Weil b-divisor W is *relatively nef* over X if and only if there is a sequence of relatively nef $\mathbb{R}$-Cartier b-divisors over X such that the traces converge to the trace of W in the numerical class over X on each model. We will need the following variant of the negativity lemma in the future.

LEMMA 2.6. *Let W be a relatively nef $\mathbb{R}$-Weil b-divisor over X. Let $\pi : X_\pi \to X$ and $\pi' : X_{\pi'} \to X$ be two models over X such that π' factors through π via $\rho : X_{\pi'} \to X_\pi$. Then $W_{\pi'} \leqslant -\rho^*(-W_\pi)$.*

Proof. See [BdFF12, Proposition 2.12]. □

Let $C_1, \ldots, C_n$ be $\mathbb{R}$-Cartier b-divisors, where $n = \dim X$. Let f be a common determination. It is clear that the intersection number $C_{1,f} \cdot \ldots \cdot C_{n,f}$ is independent of the choice of f by the projection formula. We define $C_1 \cdot \ldots \cdot C_n$ to be the above intersection number. If $W_1, \ldots, W_n$ are relatively nef $\mathbb{R}$-Weil b-divisors over X, we define

$$W_1 \cdot \ldots \cdot W_n = \inf(C_1 \cdot \ldots \cdot C_n) \in [-\infty, \infty),$$

where the infimum is taken over all relatively nef $\mathbb{R}$-Cartier b-divisors C_i over X such that $C_i \geqslant W_i$ for each i. It is obvious that the intersection number is monotonic in the sense that, if $W_i \leqslant W_i'$ for each i, then $W_1 \cdot \ldots \cdot W_n \leqslant W_1' \cdot \ldots \cdot W_n'$. For further properties of the intersection number, we refer to [BdFF12, §4.3 and Appendix A].

Given a canonical divisor K_X on X, there is a unique canonical divisor K_{X_π} for each model $\pi : X_\pi \to X$ with the property that $\pi_* K_{X_\pi} = K_X$. Hence, a choice of K_X determines a *canonical b-divisor* $K_{\mathcal{X}}$ over X.

2.4 Nef envelopes

The *nef envelope* $\mathrm{Env}_X(D)$ of an $\mathbb{R}$-*divisor* D on X is an $\mathbb{R}$-Weil b-divisor over X whose trace on a model $\pi : X_\pi \to X$ is $-\pi^*(-D)$. For more details, see [BdFF12, §2]. If D is $\mathbb{Q}$-Cartier, then $\mathrm{Env}_X(D)$ is the $\mathbb{Q}$-Cartier b-divisor $\overline{D}$.

The *nef envelope* $\mathrm{Env}_{\mathcal{X}}(W)$ of an $\mathbb{R}$-*Weil b-divisor* W over X is the largest relatively nef $\mathbb{R}$-Weil b-divisor Z over X such that $Z \leqslant W$. It is well defined by [BdFF12, Proposition 2.15]. It is clear that if $W_1 \leqslant W_2$, then $\mathrm{Env}_{\mathcal{X}}(W_1) \leqslant \mathrm{Env}_{\mathcal{X}}(W_2)$.

The *log discrepancy b-divisor* is defined as

$$A_{\mathcal{X}/X} = K_{\mathcal{X}} + E_{\mathcal{X}/X} + \mathrm{Env}_X(-K_X),$$

where the trace of $E_{\mathcal{X}/X}$ in any model π is equal to the reduced exceptional divisor E_π over X. It is clear that the trace of $A_{\mathcal{X}/X}$ on a model $\pi : X_\pi \to X$ is $A_{X_\pi/X}$. Similarly, for every integer $m \geqslant 1$, we define the *mth limiting log discrepancy b-divisor* $A_{m,\mathcal{X}/X}$ to be a $\mathbb{Q}$-Weil b-divisor whose trace on a model $\pi : X_\pi \to X$ is $A_{m,X_\pi/X}$. It is easy to check that $A_{m,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}$ and $A_{\mathcal{X}/X}$ is the limsup of $A_{m,\mathcal{X}/X}$.

The *volume of the singularity* on X is defined by

$$\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = -\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})^n.$$

It is shown in [BdFF12] that if X has isolated singularity, then $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$ is a well-defined non-negative finite real number.

2.5 Log canonical modification

Suppose that (X, Δ) is a pair such that X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A birational projective morphism $f : Y \to X$ is called a log canonical

modification of (X, Δ) if:

(i) $(Y, \Delta_Y + E_f)$ is log canonical;

(ii) $K_Y + \Delta_Y + E_f$ is f-ample.

Here Δ_Y is the strict transform of Δ and E_f is the reduced exceptional divisor of f. It is shown in [OX12] that the log canonical modification exists uniquely up to isomorphism for any log pair (X, Δ). Clearly, if $f' : Y' \to X$ is a log resolution of the pair (X, Δ), then

$$Y \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_* \mathcal{O}_{Y'}(m(K_{Y'} + \Delta_{Y'} + E_{f'})).$$

LEMMA 2.7. *Let (X, Δ) be a pair as above which is not log canonical. Let $f : Y \to X$ be the log canonical model. Write $f^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Y + \Delta_Y + B$, and $B = \sum b_i B_i$ as the sum of distinct prime divisors such that $f_*(B) = \Delta$. We let $B^{>1}$ be the non-zero divisor $\sum_{b_i > 1} b_i B_i$; then $\mathrm{Supp}(B^{>1}) = \mathrm{Ex}(f)$. In particular, $\mathrm{Ex}(f) \subset Y$ is of pure codimension 1.*

Proof. See [OX12, Lemma 2.4]. □

3. $\mathbb{Q}$-Gorenstein case

Assume that X is a $\mathbb{Q}$-Gorenstein normal variety with isolated singularities. We pick $\Delta = 0$ and suppose that $f : Y \to X$ is the log canonical modification of X. Let $F = f^*K_X - K_Y - E_f$. We define $\mathrm{Vol}(X) = -(K_Y + E_f - f^*K_X)^n$. By the negativity lemma (see [KM98, Lemma 3.39]), $F \geqslant 0$. Since $K_Y + E_f$ is f-ample and F is f-exceptional, we have that

$$\mathrm{Vol}(X) = -(K_Y + E_f - f^*K_X)^n = F \cdot (K_Y + E_f)^{n-1} \geqslant 0.$$

Remark 3.1. This definition can be extended to the case where X has isolated non-log-canonical locus.

THEOREM 3.2. *If X is a $\mathbb{Q}$-Gorenstein normal variety which has isolated singularities, then $\mathrm{Vol}_{BdFF}(X, 0) = \mathrm{Vol}(X)$.*

Proof. We show that $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor and equals $\overline{A_{Y/X}}$, where $f : Y \to X$ is the log canonical modification of $(X, 0)$. Then the theorem follows immediately.

We only need to show that on a sufficiently high model $f' : Y' \to X$ which factors through $f : Y \to X$ via $g : Y' \to Y$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_{f'}$ equals to $g^*A_{Y/X}$.

First, we show that $g^*A_{Y/X} \leqslant D$. Since (Y, E_f) is log canonical, we have that

$$\begin{aligned}
A_{Y'/X} - g^*A_{Y/X} &= K_{Y'} + E_{f'} - f'^*K_X - g^*(K_Y + E_f - f^*K_X) \\
&= K_{Y'} + (E_f)_{Y'} + E_g - g^*(K_Y + E_f) \\
&\geqslant 0,
\end{aligned}$$

where $(E_f)_{Y'}$ is the strict transform of E_f on Y'. As $A_{Y/X}$ is f-ample, we have that $g^*A_{Y/X}$ is f'-nef. We can conclude that $\overline{A_{Y/X}}$ is a relatively nef $\mathbb{Q}$-Cartier b-divisor such that $\overline{A_{Y/X}} \leqslant A_{\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A_{Y/X}} \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. In particular, $g^*A_{Y/X} \leqslant D$.

On the other hand, by the definition of nef envelope, we have that $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is relatively nef over X. We may apply Lemma 2.6. Thus, $D \leqslant -g^*(-\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_f)$. By definition, $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_f \leqslant A_{Y/X}$. Hence, $D \leqslant -g^*(-A_{Y/X}) = g^*A_{Y/X}$, since $A_{Y/X}$ is $\mathbb{Q}$-Cartier. Therefore, $D = g^*A_{Y/X}$. □

THEOREM 3.3. *There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

Proof. Suppose that X is a Gorenstein normal variety with isolated singularities and $f : Y \to X$ is its log canonical modification. Let $F = f^*K_X - K_Y - E_f = \sum a_i \cdot E_i$, where E_i are f-exceptional divisors. Since X is Gorenstein, we have that a_i are positive integers by Lemma 2.7. If $\mathrm{Vol}(X) > 0$, then $F \neq 0$, hence $F \geqslant E_f$. We have that

$$\mathrm{Vol}(X) \geqslant E_f \cdot (K_Y + E_f)^{n-1} = ((K_Y + E_f)|_{E_f})^{n-1} = (K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}.$$

Since (Y, E_f) is log canonical, by [Kol92, 16.6], the coefficients of $\mathrm{Diff}_{E_f}(0)$ lie in $\{0, 1\} \cup \{1 - 1/m \mid m \geqslant 2\}$, which is a DCC set. By [HMX12, Theorem 1.3], we have that $(K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}$ lies in a DCC set. The theorem follows. $\square$
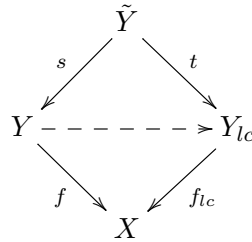
4. Non-$\mathbb{Q}$-Gorenstein case

Let X be a normal variety which has only isolated singularities. For any integer $m \geqslant 2$, fix a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$. By Theorem 2.5, we can find a boundary Δ such that $K^\Delta_{Y/X} = K_{m,Y/X}$. Let $f_{lc} : Y_{lc} \to X$ be the log canonical modification of the pair (X, Δ). Then

$$Y_{lc} \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_*\mathcal{O}_Y(m(K_Y + \Delta_Y + E_f)).$$

Assuming that Δ' is another m-compatible boundary for X with respect to f and $f'_{lc} : Y'_{lc} \to X$ is the corresponding log canonical modification, we have that $K^\Delta_{Y/X} = K^{\Delta'}_{Y/X}$. Hence, $\Delta_Y - \Delta'_Y = f^*(\Delta - \Delta')$. Now

$$f_*\mathcal{O}_Y(m(K_Y + \Delta'_Y)) = f_*\mathcal{O}_Y(m(K_Y + \Delta_Y - f^*(\Delta - \Delta')))$$
$$= f_*\mathcal{O}_Y(m(K_Y + \Delta_Y)) \otimes \mathcal{O}_X(m(\Delta' - \Delta))$$

for sufficiently divisible m, as $\Delta - \Delta'$ is $\mathbb{Q}$-Cartier. Thus, there is a natural X-isomorphism $\sigma : Y_{lc} \to Y'_{lc}$ such that $f_{lc} = f'_{lc} \circ \sigma$. Fix a common resolution of Y and Y_{lc}, $\tilde{f} : \tilde{Y} \to X$, as in the following diagram:

$$
\begin{array}{ccc}
 & \tilde{Y} & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\
Y \dashrightarrow & & Y_{lc} \\
{\scriptstyle f}\searrow & & \swarrow{\scriptstyle f_{lc}} \\
 & X &
\end{array}
$$

Noticing that $\tilde{Y}$ is also a common resolution of Y and Y'_{lc}, we have that the morphism $s : \tilde{Y} \to Y$ is independent of the choice of Δ.

THEOREM 4.1. *The $\mathbb{R}$-Weil b-divisor $\mathrm{Env}_\mathcal{X}(A_{m,\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor. If Δ is m-compatible for X with respect to $\tilde{f} : \tilde{Y} \to X$, then*

$$\mathrm{Env}_\mathcal{X}(A_{m,\mathcal{X}/X}) = \overline{A^\Delta_{Y_{lc}/X}}.$$

Proof. We will mimic the proof of Theorem 3.2. We only need to show that on a sufficiently high model $\rho : Z \to X$ which factors through $\tilde{f} : \tilde{Y} \to X$ by $\pi : Z \to \tilde{Y}$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_{\rho}$ is equal to $(t \circ \pi)^* A_{Y_{lc}/X}^{\Delta}$.

First, we show that $(t \circ \pi)^* A_{Y_{lc}/X}^{\Delta} \leqslant D$. Since Δ is m-compatible for X with respect to $\tilde{f}$, we have that

$$K_{\tilde{Y}} + \Delta_{\tilde{Y}} - \tilde{f}^*(K_X + \Delta) = K_{\tilde{Y}} - \frac{1}{m}\tilde{f}^{\natural}(mK_X).$$

Hence,

$$-\frac{1}{m}\rho^{\natural}(mK_X) = \pi^*\left(-\frac{1}{m}\tilde{f}^{\natural}(mK_X)\right) = \pi^*\Delta_{\tilde{Y}} - \rho^*(K_X + \Delta),$$

by Lemma 2.3 and the fact that $\tilde{Y}$ is smooth. Now the difference

$$
\begin{aligned}
&A_{m,Z/X} - (t \circ \pi)^* A_{Y_{lc}/X}^{\Delta} \\
&= K_Z + E_\rho - \frac{1}{m}\rho^{\natural}(mK_X) - (t \circ \pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) + \rho^*(K_X + \Delta) \\
&= K_Z + E_\rho + \pi^*\Delta_{\tilde{Y}} - (t \circ \pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\
&= (K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_{t \circ \pi} - (t \circ \pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})) + (\pi^*\Delta_{\tilde{Y}} - \Delta_Z).
\end{aligned}
$$

Since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical, we have that the first term is effective and $(t \circ \pi)$-exceptional. As Δ_Z is the strict transform of the effective divisor $\Delta_{\tilde{Y}}$, the second term is effective and π-exceptional, hence $(t \circ \pi)$-exceptional. We conclude that $A_{m,Z/X} - (t \circ \pi)^* A_{Y_{lc}/X}^{\Delta}$ is effective and $(t \circ \pi)$-exceptional. Since $A_{Y_{lc}/X}^{\Delta}$ is f-ample, we have that $(t \circ \pi)^* A_{Y_{lc}/X}^{\Delta}$ is ρ-nef. We can conclude that $\overline{A_{Y_{lc}/X}^{\Delta}}$ is a relatively nef $\mathbb{Q}$-Cartier b-divisor such that $\overline{A_{Y_{lc}/X}^{\Delta}} \leqslant A_{m,\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A_{Y_{lc}/X}^{\Delta}} \leqslant \mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$. In particular, $(t \circ \pi)^* A_{Y_{lc}/X}^{\Delta} \leqslant D$.

On the other hand, by the definition of nef envelope, we have that $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ is relatively nef over X. We may apply Lemma 2.6. Thus, $D \leqslant -(t \circ \pi)^*(-\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_{f_{lc}})$. By definition, $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_{f_{lc}} \leqslant A_{m,Y_{lc}/X}$. Hence, $D \leqslant -(t \circ \pi)^*(-A_{m,Y_{lc}/X}) = (t \circ \pi)^* A_{Y_{lc}/X}^{\Delta}$, since $A_{Y_{lc}/X}^{\Delta}$ is $\mathbb{Q}$-Cartier. Therefore, $D = (t \circ \pi)^* A_{Y_{lc}/X}^{\Delta}$. □

We can define the volume of singularities of X as follows.

DEFINITION 4.2. The mth limiting volume of singularity of X is

$$\mathrm{Vol}_m(X) = -\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})^n.$$

COROLLARY 4.3. *In the setting of Theorem 4.1, if Δ is m-compatible for X with respect to $\tilde{f}$, then*

$$\mathrm{Vol}_m(X) = -(A_{Y_{lc}/X}^{\Delta})^n = -A_{Y_{lc}/X}^{\Delta} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} \geqslant 0.$$

Proof. The first equation is straightforward by Theorem 4.1 and the definition of intersection number. The second equation is valid since $A_{Y_{lc}/X}^{\Delta}$ is f_{lc}-exceptional. By the negativity lemma, we have that $A_{Y_{lc}/X}^{\Delta} \leqslant 0$. Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, we have the inequality in the corollary. □

For an arbitrary boundary Δ on X, we have the following inequalities.

PROPOSITION 4.4. *Suppose that Δ is a boundary on X, m is the index of $K_X + \Delta$ and $f : Y \to X$ is the log canonical modification of (X, Δ). Then:*

(i) $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$;

(ii) $\mathrm{Vol}_m(X) \leqslant -(A^{\Delta}_{Y/X})^n$.

Proof. For any model $\pi : X_\pi \to X$, we have that

$$\pi^*(m(K_X + \Delta)) + \pi^{\natural}(-m\Delta) = \pi^{\natural}(mK_X).$$

Hence,

$$\begin{aligned}
A_{m,X_\pi/X} &= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^{\natural}(mK_X) \\
&= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^{\natural}(-m\Delta) - \frac{1}{m}\pi^*(m(K_X + \Delta)) \\
&\geqslant K_{X_\pi} + E_\pi + \Delta_{X_\pi} - \pi^*(K_X + \Delta) = A^{\Delta}_{X_\pi/X}.
\end{aligned}$$

Thus, as b-divisors, $A_{m,\mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X}$, hence,

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}).$$

On the other hand, for any model $f' : Y' \to Y$ factoring through f via $g : Y' \to Y$, we have that $A^{\Delta}_{Y'/X} \geqslant g^* A^{\Delta}_{Y/X}$, since $(Y, \Delta_Y + E_f)$ is log canonical. As b-divisors, $A^{\Delta}_{\mathcal{X}/X} \geqslant \overline{A^{\Delta}_{Y/X}}$. As $K_Y + \Delta_Y + E_f$ is f-ample, we have that $\overline{A^{\Delta}_{Y/X}}$ is relatively nef over X. Thus, $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$. We have proved (i).

Since both $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ and $\overline{A^{\Delta}_{Y/X}}$ are relatively nef and exceptional over X, (ii) follows from the inequality between intersection numbers. $\quad\square$

Remark 4.5. In Proposition 4.4, one can show that $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) = \overline{A^{\Delta}_{Y/X}}$.

For any two positive integers m and l and any model $f : Y \to X$, since

$$\frac{1}{m}f^{\natural}(mK_X) \geqslant \frac{1}{lm}f^{\natural}(lmK_X) \geqslant f^*K_X,$$

we have that

$$A_{m,Y/X} \leqslant A_{lm,Y/X} \leqslant A_{Y/X},$$

hence

$$A_{m,\mathcal{X}/X} \leqslant A_{lm,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}.$$

By the definition of nef envelope, we have that

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{lm,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}).$$

Since they are both exceptional over X by Theorem 4.1, we have the following inequality of volumes:

$$\mathrm{Vol}_m(X) \geqslant \mathrm{Vol}_{lm}(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X, 0).$$

Corollary 4.6. *The following statements are equivalent.*

    (i) *There exists a boundary Δ on X such that (X, Δ) is log canonical.*

    (ii) $\mathrm{Vol}_m(X) = 0$ *for some (hence any multiple of) integer $m \geqslant 1$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $m(K_X + \Delta)$ is Cartier. By Proposition 4.4,

$$A_{m, \mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X} \geqslant 0,$$

since (X, Δ) is log canonical. As 0 is a relatively nef b-divisor over X, we have that $\mathrm{Env}_{\mathcal{X}}(A_{m, \mathcal{X}/X}) \geqslant 0$. On the other hand, by Theorem 4.1 and the negativity lemma, $\mathrm{Env}_{\mathcal{X}}(A_{m, \mathcal{X}/X}) \leqslant 0$. Hence, $\mathrm{Env}_{\mathcal{X}}(A_{m, \mathcal{X}/X}) = 0$. We can conclude that $\mathrm{Vol}_m(X) = 0$.

$(ii) \Rightarrow (i)$. Let Δ be an m-compatible boundary for X with respect to $\tilde{f}$ in the setting of Theorem 4.1. By Theorem 2.5, such a boundary always exists. Since $\mathrm{Vol}_m(X) = 0$, by Corollary 4.3, we have that

$$-A^{\Delta}_{Y_{lc}/X} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} = 0.$$

Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, this is equivalent to $A^{\Delta}_{Y_{lc}/X} = 0$. Thus, we have that

$$f_{lc}^*(K_X + \Delta) = K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}.$$

For any model $\rho : Z \to X$ factoring through f_{lc} via $\pi : Z \to Y_{lc}$, we have that

$$\begin{aligned}
A^{\Delta}_{Z/X} &= K_Z + \Delta_Z + E_{\rho} - \rho^*(K_X + \Delta) \\
&= K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_{\pi} - \pi^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\
&\geqslant 0,
\end{aligned}$$

since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical. Therefore, (X, Δ) is log canonical. $\qquad\square$

Definition 4.7. The augmented volume of singularities on X is

$$\mathrm{Vol}^+(X) = \liminf_m \mathrm{Vol}_m(X) = \lim_{k \to \infty} \mathrm{Vol}_{k!}(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X, 0).$$

Remark 4.8. While it is proved in the appendix of [BdFF12] that the intersection number is continuous, it is not clear that $\mathrm{Env}_{\mathcal{X}}(A_{m, \mathcal{X}/X})$ converges to $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. It would be interesting to have an example with $\mathrm{Vol}^+(X) > \mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$.

4.1 Cone singularities

We will give a counterexample to Problem B in this section.

Let (V, H) be a non-singular projective polarized variety of dimension $n - 1$. The vertex 0 is the isolated singularity of the variety

$$X = \mathrm{Spec} \bigoplus_{m \geqslant 0} H^0(V, \mathcal{O}_V(mH)).$$

We assume that H is sufficiently ample so that X is normal. Blowing up 0 gives a resolution of singularities for X which we denote by Y. The induced map $f : Y \to X$ is isomorphic to the contraction of the zero section E of the total space of the vector bundle $\mathcal{O}_V(H)$. Let $\pi : Y \to V$ be the bundle map. We have that $E \cong V$. The co-normal bundle of E in Y is

$$\mathcal{O}_E(-E) \cong \mathcal{O}_V(H).$$

1422

Let us slightly change our notation from previous sections. Let Γ be a boundary on X, Γ_Y be the strict transform of Γ on Y and $\Delta = \Gamma_Y|_E$. Since $A^\Gamma_{Y/X}$ is exceptional, we may assume that $A^\Gamma_{Y/X} = -aE$ for some rational number a. Restricting to E, we have that

$$K_V + \Delta \sim_{\mathbb{Q}} aH$$

by the adjunction formula. On the other hand, assuming that Δ is an effective $\mathbb{Q}$-Cartier divisor on V such that $\Delta \sim_{\mathbb{Q}} -K_V + aH$, we may set $\Gamma = C_\Delta$ and get that $A^{C_\Delta}_{Y/X} = -aE$, where C_Δ is the cone over Δ in X.

Let C be an elliptic curve, U be a semi-stable vector bundle on C of rank 2 and degree 0 and $V = \mathbb{P}(U)$ be the ruled surface over C. The nef cone $\mathrm{Nef}(V)$ and pseudo-effective cone $\overline{\mathrm{NE}}(V)$ are the same. They are spanned by the section C_0 corresponding to the tautological bundle $\mathcal{O}_{\mathbb{P}(U)}(1)$ and a fiber F of the ruling (for details, see [Laz04, § 1.5.A]). Moreover, as in [Sho00, Example 1.1], if C' is an effective curve on V such that $C' \equiv mC_0$ for some positive integer m, then $C' = mC_0$.

THEOREM 4.9. *Let V be the ruled surface as above. Fix an ample divisor H on V. Let X be the affine cone over (V, H). Suppose that H is sufficiently ample so that X is normal. Then $\mathrm{Vol}^+(X) = 0$, hence $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$. But there is no effective $\mathbb{Q}$-divisor Γ such that (X, Γ) is log canonical.*

Proof. Fix an ample divisor H on V. Since $K_V \sim -2C_0$, we have that $-K_V + aH$ is ample for any rational number $a > 0$. Let D be a smooth curve in $|n(-K_V + aH)|$ for some sufficiently large positive integer n, and set $\Delta = (1/n)D$. Then $(Y, \pi^*\Delta + E)$ is the log canonical modification of (X, C_Δ), since $(K_Y + \pi^*\Delta + E)|_E \sim_{\mathbb{Q}} aH$ is ample and (X, C_Δ) is not log canonical. Suppose that m is the index of $K_X + C_\Delta$. By Proposition 4.4(ii),

$$\mathrm{Vol}_m(X) \leqslant -(A^{C_\Delta}_{Y/X})^3 = (aE)^3 = a^3 H^2.$$

As $a \to 0$, we conclude that $\mathrm{Vol}^+(X) = 0$, hence $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$.

If (X, Γ) is log canonical for some effective $\mathbb{Q}$-divisor Γ on X, then $A^\Gamma_{Y/X} = -aE \geqslant 0$, hence $a \leqslant 0$. On the other hand, let $\Delta = \Gamma_Y|_E \in \overline{\mathrm{NE}}(V)$. Then

$$\Delta \sim_{\mathbb{Q}} -K_V + aH \sim_{\mathbb{Q}} 2C_0 + aH.$$

Since $\Delta \geqslant 0$, we have $a \geqslant 0$ and hence $a = 0$. Thus $A^\Gamma_{Y/X} = 0$, and $(Y, \Gamma_Y + E)$ is log canonical. But $\Delta = \Gamma_Y|_E$ is an effective $\mathbb{Q}$-divisor linearly equivalent to $2C_0$, and hence $\Delta = 2C_0$, a contradiction. □

In [dFH09, Definition 7.1], a normal variety X is defined to be log canonical if for one (hence any sufficiently divisible) positive integer m, the mth limiting log discrepancy b-divisor $A_{m,\mathcal{X}/X} \geqslant 0$. And in [dFH09, Proposition 7.2], it was proved that X is log canonical if and only if there is a boundary Δ such that the pair (X, Δ) is log canonical. It is natural to ask whether this definition is equivalent to the one requiring that the log discrepancy b-divisor $A_{\mathcal{X}/X} \geqslant 0$.

COROLLARY 4.10. *Let X be the affine cone over (V, H) as in Theorem 4.9. Then $A_{\mathcal{X}/X} \geqslant 0$, but X is not log canonical.*

Proof. The corollary follows immediately from the fact that $A_{\mathcal{X}/X} \geqslant 0$ is equivalent to $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ (see [BdFF12, Proposition 4.19] or Corollary 4.6). □

References

BdFF12   S. Boucksom, T. de Fernex and C. Favre, *The volume of an isolated singularity*, Duke Math. J. **161** (2012), 1455–1520.

BH12   A. Broustet and A. Höring, *Singularities of varieties admitting an endomorphism*, Preprint (2012), arXiv:1210.6254.

dFH09   T. de Fernex and C. Hacon, *Singularites on normal varieties*, Compositio. Math. **145** (2009), 393–414.

Ful12   M. Fulger, *Local volumes*, PhD thesis, University of Michigan (2012).

HMX12   C. Hacon, J. McKernan and C. Xu, *ACC for log canonical thresholds*, Preprint (2012), arXiv:1208.4150.

KM98   J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134 (Cambridge University Press, Cambridge, 1998).

Kol92   J. Kollár *et al.*, *Flips and abundance for algebraic threefolds. Papers from the Second Summer Seminar on Algebraic Geometry held at University of Utah, Salt Lake City, Utah, August 1991*, Astérisque **211** (Société Mathématique de France, Paris, 1992).

Laz04   R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series], vol. 48 (Springer, Berlin, 2004).

OX12   F. Odaka and C. Xu, *Log-canonical models of singular pairs and its applications*, Math. Res. Lett. **19** (2012), 325–334.

Sho00   V. Shokurov, *Complements on surfaces*, J. Math. Sci. (N. Y.) **102** (2000), 3876–3932.

Urb12   S. Urbinati, *Discrepancies of non-$\mathbb{Q}$-Gorenstein varieties*, Michigan Math. J. **61** (2012), 265–277.

Yuchen Zhang   yzhang@math.utah.edu

Department of Mathematics, University of Utah, 155 S 1400 E Room 233, Salt Lake City, UT 84102, USA