BULL. AUSTRAL. MATH. SOC. 20D10
VOL. 65 (2002) [271–275]

ON $\mathcal{F}$-RESIDUALS OF FINITE GROUPS

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We prove that there exists a soluble, saturated and s-closed formation $\mathcal{F}$ of groups such that the class $(G^{\mathcal{F}} \mid G$ is a group, $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of $G)$ is not closed under subdirect products. This result gives a negative answer to an open problem recently proposed by L.A. Shemetkov in 1998.

1. PRELIMINARIES

In this paper, all groups considered are finite groups. By a formation $\mathcal{F}$ of groups, we mean a class of groups which is closed under homomorphic images and also every group G has a smallest normal subgroup with quotient in $\mathcal{F}$. This normal subgroup of G is called the $\mathcal{F}$-residual of G and is denoted by $G^{\mathcal{F}}$. In fact, $G^{\mathcal{F}}$ is the set $\cap\{N \mid N \lhd G,\ G/N \in \mathcal{F}\}$. We also call a formation $\mathcal{F}$ a saturated formation if $G \in \mathcal{F}$ whenever the Frattini factor group $G/\Phi(G)$ is in $\mathcal{F}$. A formation $\mathcal{F}$ is called s-closed if it contains every subgroup of each group $G \in \mathcal{F}$. If every group of $\mathcal{F}$ is soluble, then $\mathcal{F}$ is called a soluble formation. Let π be a set of primes. Then, the group G is called a π-group if every prime divisor of the order of G is in π.

For the $\mathcal{F}$-residuals of finite groups, it is well known that if $\mathcal{F}$ is a s-closed formation, then $(A \times B)^{\mathcal{F}} = A^{\mathcal{F}} \times B^{\mathcal{F}}$, for any groups A and B. It has also been proved by Doerk and Hawkes [1] that if a formation $\mathcal{F}$ is soluble or saturated, then $(A \times B)^{\mathcal{F}} = A^{\mathcal{F}} \times B^{\mathcal{F}}$ for any groups A and B. They also constructed an example to show that this result is generally not true for non-soluble formations $\mathcal{F}$. Thus, if a formation $\mathcal{F}$ is s-closed, or soluble, or saturated, then the class $(G^{\mathcal{F}} \mid G$ is a group$)$ is closed under direct products.

In this connection, the following question naturally arises: Let $\mathcal{F}$ be a soluble, or saturated, or s-closed formation of groups. Is it true that the class of $\mathcal{F}$-residuals $(G^{\mathcal{F}} \mid G$ is a group$)$ is still closed under subdirect products? This question was proposed by Shemetkov in [5]. If this question of Shemetkov can be answered affirmatively, then we can further improve the theorem of Doerk and Hawkes in [1]. In this paper, we prove that there exists a soluble, saturated and s-closed formation $\mathcal{F}$ of groups such that the

Received 4th September, 2001.
Research for the first author was jointly supported by a grant of NSF, China and 333GCF grant of Jiangsu Province, China and a Croucher Fellowship of Hong Kong. Research for the second author was partially supported by a CUHK direct grant #2060196 (2000/2001).

https://doi.org/10.1017/S0004972700020311 Published online by Cambridge University Press

class $(G^{\mathcal{F}} \mid G$ is a group $)$ is not closed under subdirect products. Thus, a negative answer to the problem of Shemetkov is obtained. This means that the theorem of Doerk and Hawkes for $\mathcal{F}$-residual of groups can not be further sharpened to subdirect products. This result has some significance and its proof is not trivial.

Throughout this paper, all concepts and notations are standard (see [**2**, **3**]). In addition, we use $[A]B$ to denote the semidirect product of A with B, where A, B are groups.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two formations of groups, then we define $\mathcal{F}_1\mathcal{F}_2 = \phi$, when $\mathcal{F}_2 = \phi$, otherwise, we define $\mathcal{F}_1\mathcal{F}_2 = \{G \mid G$ is a group and $G^{\mathcal{F}_2} \in \mathcal{F}_1\}$. The class $\mathcal{F}_1\mathcal{F}_2$ of groups is said to be the product of $\mathcal{F}_1$ and $\mathcal{F}_2$.

In order to prove our main results, the following lemma on $\mathcal{F}$-residuals is needed.

Lemma 1.1. (See [**3**, Lemma 2.1.3].) *Let $\varphi : A \to B$ be an epimorphism of a group A onto a group B. Then $(A^{\mathcal{F}})^{\varphi} = B^{\mathcal{F}}$ for every formation $\mathcal{F}$.*

We also need some results on modules. Let F be a field, G a group and M a FG-module. Denote the radical of M by $\mathrm{Rad}(M)$, that is, $\mathrm{Rad}(M)$ is the intersection of all maximal submodules of M. By the socle $\mathrm{soc}(M)$ of M, we mean the sum of all minimal submodules of M. If q is a prime, then we use F_q to denote the field with q elements. A FG-module V is called a trivial FG-module if $C_G(V) = G$. We now cite the following useful results from the literature.

Lemma 1.2. (See [**2**, Lemma B, 3.14].) *Let H be a group and M an F_pH-module for some prime p. Then $\mathrm{Rad}(M) = \Phi([M]H) \cap M$.*

Lemma 1.3. (See [**2**, Theorem B, 4.6].) *Every simple KG-module V is isomorphic with $P/\mathrm{Rad}(P)$ for some indecomposable projective KG-module P.*

Lemma 1.4. (See [**2**, Theorem B. 4.10].) *Every indecomposable projective FG-module V has an unique minimal submodule which is isomorphic to $V/\mathrm{Rad}(V)$.*

Lemma 1.5. (See [**6**], or [**3**, Theorem 5.3.7].) *Let G be a group, p be a prime. If V is a trivial simple F_pG-module and P_V is the projective envelope of V, then*

$$C_G(P_V) = O_{p'}(G).$$

Lemma 1.6. (See [**3**, Lemma 3.5.13].) *If a group G has only one minimal normal subgroup and $O_p(G) = 1$, where p is a prime, then there exists a simple F_pG-module L such that $C_G(L) = 1$.*

2. Results on subdirect products

We now use the lemmas in section 1 to prove our main results.

THEOREM 2.1. *Let $\mathcal{F} = \mathcal{F}\mathcal{F}$ be a soluble non-identity formation. Then the class $(G^{\mathcal{F}} \mid G$ is a group) is not closed under subdirect products.*

PROOF: Let Z_p be a group of prime order p in $\mathcal{F}$ and B a simple group such that $B \notin \mathcal{F}$. Then, by Lemma 1.6, there exists a simple F_pB-module M_1 such that $C_B(M_1) = 1$. Let $A = [M_1]B$. Then M_1 is the only minimal normal subgroup of A and hence $O_{p'}(A) = 1$. If V is a trivial simple F_pA-module, then by Lemma 1.3, there exists an indecomposable projective F_pA-module P such that $P/\operatorname{Rad}(P)$ is isomorphic to V. Let $R = \operatorname{Rad}(P)$ and $G = [P]A$. Then, by Lemma 1.2, $R = \Phi(G) \cap P$. Assume that $R = 1$. Then P is a trivial F_pA-module, that is, $C_A(P) = A$. However, by Lemma 1.5, we have $C_A(P) = O_{p'}(A) = 1$. This is a contradiction, and hence $R \neq 1$. Let $L = \operatorname{soc}(P)$. We shall show that L is the only minimal normal subgroup of G. In fact, by Lemma 1.4, L is the unique minimal submodule of P. Because $C_A(P) = A$, we know that L is the only minimal normal subgroup of G contained in P. Assume that there exists a minimal normal subgroup D of G such that $D \not\subseteq P$. Then $D \subseteq C_G(P)$. Since P is an elementary Abelian group, $P \subseteq C_G(P)$ and by Dedekind's modular law (see [4, 1.3.14]), we derive that $C_G(P) = C_G(P) \cap PA = P(C_G(P) \cap A) = PC_A(P) = PA = P$. This contradiction shows that L is the only minimal normal subgroup of G.

Now, let $\mathcal{N}_p$ be the class of all p-groups. We claim that $\mathcal{N}_p\mathcal{F} = \mathcal{F}$. To prove our claim, we first suppose that $\mathcal{N}_p\mathcal{F} \not\subseteq \mathcal{F}$. Then, we can let E be a group of minimal order in $\mathcal{N}_p\mathcal{F}\backslash\mathcal{F}$. Since $\mathcal{F}$ is a formation, E has an unique minimal normal subgroup $X = E^{\mathcal{F}}$. Because $E \in \mathcal{N}_p\mathcal{F}, X \in \mathcal{N}_p$. Hence $X = P_1 \times P_2 \times \cdots \times P_t$ where $P_1 \simeq P_2 \simeq \cdots \simeq P_t \simeq Z_p \in \mathcal{F}$. It follows that $X \in \mathcal{F}$. But $\mathcal{F} = \mathcal{F}\mathcal{F}$ and $X = E^{\mathcal{F}} \in \mathcal{F}$, we have $E \in \mathcal{F}$. This contradiction shows that $\mathcal{N}_p\mathcal{F} \subseteq \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{N}_p\mathcal{F}$ always, we establish our claim $\mathcal{F} = \mathcal{N}_p\mathcal{F}$. Now, let M be a group and $T = (M^{\mathcal{F}})^{\mathcal{N}_p}$. In particular $T \subsetneq M^{\mathcal{F}}$. Since T is a characteristic subgroup of $M^{\mathcal{F}}$ and $M^{\mathcal{F}} \trianglelefteq M$, we can easily derive that T is normal in M. However, by Lemma 1.1, we have $M^{\mathcal{F}}/T = (M/T)^{\mathcal{F}} \in \mathcal{N}_p$. This leads to $M/T \in \mathcal{N}_p\mathcal{F} = \mathcal{F}$ and hence $T = M^{\mathcal{F}}$, a contradiction. Thus, $(M^{\mathcal{F}})^{\mathcal{N}_p} = M^{\mathcal{F}}$ for every group M and hence $M^{\mathcal{F}} \subseteq T$, so $T = M^{\mathcal{F}}$.

Let $T = G_1 \times G_2$, where $G_1 \simeq G_2 \simeq G$, and let $\varphi : G_1 \to G_2$ be an isomorphism of the groups G_1 onto G_2. Then, by the theorem of Doerk–Hawkes in [1], we have $T^{\mathcal{F}} = G_1^{\mathcal{F}} \times G_2^{\mathcal{F}}$. Let H be the diagonal of $\{(a, a^{\phi}) : a \in G_1^{\mathcal{F}}\}$ of $G_1^{\mathcal{F}} \times G_2^{\mathcal{F}}$. Then, H is clearly a subdirect product of the groups $G_1^{\mathcal{F}}$ and $G_2^{\mathcal{F}}$. Since $A \simeq G/P$ and $A/M_1 \simeq B \notin \mathcal{F}$, we have $G/P \notin \mathcal{F}$ and $G^{\mathcal{F}} \neq 1$. It follows that $L \subseteq G^{\mathcal{F}}$. Since $L \simeq P/R$, by Lemma 1.4 and $P/R \simeq V$, we know that L is a trivial F_pA-module. Hence $L \subseteq Z(G)$. Let L_1 be a subgroup of G_1 such that $L_1 \subseteq Z(G_1)$ and $L_1 \simeq L$. Then, we can easily see that $L_1 \subseteq Z(G_1 \times G_2)$. Let $F = L_1H$. Since $L_1 \cap H = 1$, we have $F = L_1 \times H$. It is clear that F is a subdirect product of the groups $G_1^{\mathcal{F}}$ and $G_2^{\mathcal{F}}$. Assume that there exists a group X such that $F = X^{\mathcal{F}}$. Then, by using the above result, we have $F^{\mathcal{N}_p} = F$. Since $F/H \simeq L \in \mathcal{N}_p$, we have $F^{\mathcal{N}_p} \subseteq H$ and hence $H = F$. This contradiction shows that

$F \neq X^{\mathcal{F}}$ for every group X. In other words, the class $(G^{\mathcal{F}} \mid G$ is a group) is not closed under subdirect products. ∎

THEOREM 2.2. *Let $\mathcal{F} = \mathcal{F}\mathcal{F}$ be a non-identity soluble formation. If $\mathcal{F}$ is not the class of all soluble groups, then the class $(G^{\mathcal{F}} \mid G$ is a soluble group) is not closed under subdirect products.*

PROOF: Assume $Z_p \in \mathcal{F}$ for every prime p, where Z_p is a group of order p. Then $\mathcal{N}_p\mathcal{F} = \mathcal{F}$ for all primes p (see the proof of Theorem 2.1). We now claim that $\mathcal{F}$ is the class of all soluble groups. In fact, if it is not, then we can let G be a group of minimal order in $\mathcal{S}\backslash\mathcal{F}$, where $\mathcal{S}$ is the class of all soluble groups. In this case, $G^{\mathcal{F}}$ is the unique minimal normal subgroup of G and clearly, $G^{\mathcal{F}} \in \mathcal{N}_p$ for some prime p. It follows that $G \in \mathcal{N}_p\mathcal{F} = \mathcal{F}$, a contradiction. This proves our claim. However, this contradicts our assumption that $\mathcal{F}$ is not the class of all soluble groups. Therefore there exist groups Z_p and Z_q of prime orders p and q (where $p \neq q$) respectively such that $Z_p \in \mathcal{F}$ and $Z_q \notin \mathcal{F}$. Now, the theorem can be proved by using the same arguments as Theorem 3.1. ∎

In summing up our Theorem 2.1 and Theorem 2.2, we obtain the following corollary.

COROLLARY 2.3. *There exists a saturated, soluble and s-closed formation $\mathcal{F}$ such that the class $(G^{\mathcal{F}} \mid G$ is a group) is not closed under subdirect products.*

PROOF: Let π be a set of primes and $\mathcal{F}$ the class of all soluble π-groups. Then it is obvious that the class $\mathcal{F}$ a s-closed, soluble and saturated formation such that $\mathcal{F} = \mathcal{F}\mathcal{F}$. Hence, by our Theorem 2.1, the class $(G^{\mathcal{F}} \mid G$ is a group) is not closed under subdirect products. ∎

Corollary 2.3 provides a negative answer to Shemetkov's question in [**5**].

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