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OPERATORS ON $L_2(I) \oplus C^r$

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Several authors have considered eigenvalue problems for differential equations where the eigenvalue parameter also appears in the boundary conditions. Such problems do not appear to arise from any spectral problem associated with a linear operator on a Hilbert space. However, it is possible to reset such problems in this context. This has been done for certain second order cases by Walter [**4**] using a special measure on the interval in question, and by Fulton [**1**, **2**] using the type of space indicated in the title of this article.

It is our purpose here to consider a general class of operators on $L_2(I) \oplus C^r$, which are based on a differential expression τ of order n on I. We shall first investigate adjoints, boundary conditions, and self-adjointness for such operators. We shall then show that all eigenvalue problems of the form $\tau y = \lambda y$, with boundary conditions which involve λ in a linear fashion, can be reset in the context of such operators.

1. A general class of operators on $L_2(I) \oplus C^r$. Let τ be a differential expression of order n

$$(1.1) \qquad \tau y = \sum_{j=0}^{n} p_j y^{(n-j)}$$

where $p_j \in C^{n-j}(I)$ and $p_0 \neq 0$ on I. We shall denote by $D_1(\tau)$ the domain of the maximal operator in $L_2(I)$ associated with τ, and by $D_0(\tau)$ the domain of the minimal operator. If $V_1, \ldots, V_N$ is a basis for the boundary functionals associated with τ on I then

$$D_0(\tau) = \{y \in D_1(\tau) \,|\, V_j(y) = 0, 1 \leqq j \leqq N\}.$$

The operators we wish to consider on $L_2(I) \oplus C^r$ will be based on τ for the L_2 to L_2 part, but since they naturally involve mappings from L_2 to $\oplus C^r$ and vice versa it seems natural to allow a finite dimensional perturbation of τ (see [**3**]).

Let $\chi(\tilde{\chi})$ denote a $m \times 1(\tilde{m} \times 1)$ column vector with linearly independent entries in $L_2(I)$. Let A, B, C, D, E, F be matrices with complex entries of dimensions $r \times r$, $m \times r$, $m \times \tilde{m}$, $m \times N$, $r \times \tilde{m}$,

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$r \times N$ respectively. Denote by $V(y)$ the $N \times 1$ column with entries $V_j(y)$. Let

$$(1.2) \qquad L\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \tau y + \chi^T\{C(y|\widetilde{\chi}) + DV(y) + Bz\} \\ Az + E(y|\widetilde{\chi}) + FV(y) \end{bmatrix}.$$

This expression L defines a mapping from $D_1(\tau) \oplus C^r$ into $L_2(I) \oplus C^r$, and thus can be used to define operators on the latter as soon as it is endowed with an inner product.

If for a positive definite $r \times r$ matrix P we impose the inner product

$$((y_1, z_1) \,|\, (y_2, z_2))_p = (y_1|y_2) + z_2^* P z_1,$$

$L_2(I) \oplus C^r$ becomes a Hilbert space $\mathscr{H}_p$. However, we may find N such that $N^*PN = I$ and $U:\mathscr{H}_I \rightarrow \mathscr{H}_p$ by $U(y, z) = (y, Nz)$ is unitary. Thus we may always replace P by I, and shall assume that this has been done.

2. Adjoints and boundary conditions. Let $\widetilde{V}_1, \ldots, \widetilde{V}_N$ denote a basis for the boundary functionals associated with τ^* on I. Thus the Green's formula for τ is

$$(2.1) \qquad (\tau y|u) - (y|\tau^* u) = \widetilde{V}(u)^* Q(\tau) V(y),$$

where $Q(\tau)$ is $N \times N$ non-singular. Note that the Green's formula for τ^* will involve $Q(\tau^*) = -Q(\tau^*)$.

We denote by $T_1(L)$ the operator on $L_2(I) \oplus C^r$ with domain $D_1(\tau) \oplus C^r$, defined by (1.2). Let

$$(2.2) \qquad \widetilde{W}(u, w) = \widetilde{V}(u) - Q(\tau^*)^{-1} D^*(u|\chi) - Q(\tau^*)^{-1} F^* w.$$

THEOREM 2.1 *The operator $T_1(L)$ has an adjoint with domain*

$$D_0^* = \{(u, w) \in D_1(\tau^*) \oplus C^r | \widetilde{W}(u, w) = 0\},$$

and is defined by

$$(2.3) \qquad T_1(L)^* \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \tau^* u + \widetilde{\chi}^T\{C^*(u|\chi) + E^* w\} \\ A^* w + B^*(u|\chi) \end{bmatrix}.$$

Proof. Suppose

$$(u, w) \in D(T_1(L)^*) = D_0^* \quad \text{and}$$

$$T_1(L)^*(u, w) = (\hat{u}, \hat{w}).$$

Then for all $y \in D_1(\tau)$ and $z \in \quad r$

$$\begin{aligned} 0 &= (T_1(L)(y, z) \,|\, (u, w)) - ((y, z) \,|\, (\hat{u}, \hat{w})) \\ (2.4) \qquad &= (\tau y + \chi^T\{C(y|\widetilde{\chi}) + DV(y) + Bz\} \,|u) \\ &\quad + w^*\{Az + E(y|\widetilde{\chi}) + FV(y)\} - (y|\hat{u}) - \hat{w}^* z. \end{aligned}$$

In particular (2.4) holds if $z = 0$ and $y \in D_0(\tau)$, so

$$0 = (\tau y|u) - (y|\hat{u} - \tilde{\chi}^T\{C^*(u|\chi) + E^*w\})$$

for all such y, and it follows that $u \in D_1(\tau^*)$ and

$$\hat{u} = \tau^*u + \tilde{\chi}^T\{C^*(u|\chi) + E^*w\}.$$

On the other hand (2.4) must hold for $y = 0$, so

$$0 = (u|\chi)^*Bz + w^*Az - \hat{w}^*z.$$

Since z is arbitrary

$$\hat{w} = A^*w + B^*(u|\chi),$$

and the proof of (2.3) is complete.

Now using (2.3) in (2.4) we obtain the further condition that

$$\begin{aligned} 0 &= \{\tilde{V}(u)^*Q(\tau) + (u|\chi)^*D + w^*F\}V(y) \\ &= \tilde{W}(u, w)^*Q(\tau)V(y). \end{aligned}$$

Since $V(y)$ is arbitrary and $Q(\tau)$ non-singular this implies that $(u, w) \in D_1(\tau^*) \oplus C^r$ is in D_0^* if and only if $\tilde{W}(u, w) = 0$, which completes the proof.

Definition 2.1. An expression $\tilde{L}$ of the form (1.2) is *adjoint to* L if and only if the restriction of $T_1(\tilde{L})$ to D_0^* coincides with $T_1(L)^*$.

It is clear that we do not have a unique adjoint expression here. Since $\tilde{L}$ can differ from (2.3) only in terms which vanish when $(u, w) \in D_0^*$ there may be additions to $\tilde{\chi}$. However, these can be regarded as in L already, with appropriate zero entries in C and E.

THEOREM 2.2. *The expression* $\tilde{L}$ *defined by*

$$(2.5) \qquad \tilde{L}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \tau^*u + \tilde{\chi}^T\{\tilde{C}(u|\chi) + \tilde{D}\tilde{V}(u) + \tilde{B}w\} \\ \tilde{A}w + \tilde{E}(u|\chi) + \tilde{F}\tilde{V}(u) \end{bmatrix}$$

is adjoint to L *if and only if*

$$(2.6) \qquad \begin{aligned} \tilde{C} &= C^* + \tilde{D}Q(\tau)^{*-1}D^* \\ \tilde{B} &= E^* + \tilde{D}Q(\tau)^{*-1}F^* \\ \tilde{A} &= A^* + \tilde{F}Q(\tau)^{*-1}F^* \\ \tilde{E} &= B^* + \tilde{F}Q(\tau)^{*-1}D^*. \end{aligned}$$

Proof. If $(u, w) \in D_0^*$ then $\tilde{W}(u, w) = 0$ so

$$\tilde{V}(u) = -Q(\tau)^{*-1}D^*(u|\chi) - Q(\tau)^{*-1}F^*w$$

and

$$\tilde{L}\begin{bmatrix} u \\ w \end{bmatrix}$$

$$= \begin{bmatrix} \tau^* u + \tilde{\chi}^T\{ (\tilde{C} - \tilde{D}Q(\tau)^{*^{-1}}D^*)(u|\chi) + (\tilde{B} - \tilde{D}Q(\tau)^{*^{-1}}F^*)w \} \\ (\tilde{A} - \tilde{F}Q(\tau)^{*^{-1}}F^*)w + (\tilde{E} - \tilde{F}Q(\tau)^{*^{-1}}D^*)(u|\chi) \end{bmatrix}.$$

Since the entries in χ are linearly independent $(u|\chi)$ is arbitrary in C^m, so this coincides with (2.3) if and only if (2.6) holds.

It is immediate from Theorem 2.2 that if $\tilde{L}$ is adjoint to L then L is adjoint to $\tilde{L}$. The adjoint to $T_1(\tilde{L})$ is a restriction of $T_1(L)$ and is a minimal operator associated with L. Since $\tilde{L}$ is not unique, neither is this minimal operator. We thus modify our notation. Let

$$\begin{aligned} W(y, z) &= V(y) - Q(\tau)^{-1}\tilde{D}^*(y|\tilde{\chi}) - Q(\tau)^{-1}\tilde{F}^* z, \\ D_0(L; \tilde{L}) &= \{ (y, z) \in D_1(L) \,|\, W(y, z) = 0\}, \end{aligned} \tag{2.7}$$

and $T_0(L; \tilde{L}) = T_1(\tilde{L})^*$ is the restriction of $T_1(L)$ to $D_0(L; \tilde{L})$. Also, we now denote D_0^* by $D_0(\tilde{L}; L)$ and $T_1(L)^*$ by $T_0(L; \tilde{L})$. At this point it is a straightforward calculation to verify the Green's formula for $T_1(L)$ and $T_1(\tilde{L})$:

$$\begin{aligned} &(T_1(L)(y, z) \,|\, (u, w)) - ((y, z) \,|\, T_1(\tau)(u, w)) \\ &= \tilde{W}(u, w)^* Q(\tau) W(y, z). \end{aligned} \tag{2.8}$$

It is clear that $D_1(L)$ is dense in $L_2(I) \oplus C^r$, and the denseness of $D_0(L; \tilde{L})$ follows, for example, from Lemma 2.2 of [**3**]. Thus $T_0(L; \tilde{L})$ has an adjoint, which will be the closure of $T_1(\tilde{L})$. We shall show that $T_1(\tilde{L})$ is closed by showing that the domain of $T_0(L; \tilde{L})^*$ is contained in $D_1(\tilde{L})$.

THEOREM 2.3 *For adjoint expressions L and $\tilde{L}$ the associated maximal operators are closed and*

$$T_0(L; \tilde{L})^* = T_1(\tilde{L}), \quad T_0(\tilde{L}; L)^* = T_1(L).$$

Proof. It is sufficient to prove the first equality, and as noted above, we need only show that if $(u, w) \in D(T_0(L; \tilde{L})^*)$ then $(u, w) \in D_1(\tilde{L})$.

If $T_0(L; \tilde{L})^*(u, w) = (\hat{u}, \hat{w})$ then for all $(y, z) \in D_0(L; \tilde{L})$

$$0 = (T_0(L; \tilde{L})(y, z) \,|\, (u, w)) - ((y, z) \,|\, (\hat{u}, \hat{w})). \tag{2.9}$$

In particular this must hold if y vanishes outside a compact subinterval $[a, b]$ of I and $W(y, z) = 0$. If $[a, b]$ is contained in the interior of I then $V(y) = 0$, and $W(y, z) = 0$ is

$$(y|\psi_j) + \xi_j^* z = 0$$

where

$$\psi = -\overline{Q(\tau)}^{-1}\tilde{D}^T\tilde{\chi}$$

and ξ_j is the j^{th} column of $\chi = \tilde{F}Q(\tau)^{*^{-1}}$. Let $h = y^{(n)}$ so that for $1 \leqq k \leqq n$

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$$y^{(n-k)}(x) = \begin{cases} \displaystyle\int_a^x \frac{(x-s)^{k-1}}{(k-1)!} h(s)ds & x \in [a, b] \\ 0 & x \notin [a,b] \end{cases}$$

and

$$\tau y =$$

$$\begin{cases} p_0(x)h(x) + \displaystyle\sum_{k=1}^{n} p_k(x) \int_a^x \frac{(x-s)^{k-1}}{(k-1)!} h(s)ds & x \in [a, b] \\ 0 & x \notin [a, b]. \end{cases}$$

Now continuity requirements force h to be orthogonal on $[a, b]$ to all polynomials of degree $< n$, and since (y, z) must be orthogonal to (ψ_j, ξ_j) $1 \leqq j \leqq N$ it is easy to see that (h, z) is orthogonal to

$$\left(\int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \psi_j(s)ds, \xi_j \right) \quad 1 \leqq j \leqq N$$

as well as to $(q, 0)$ for every polynomial q of degree $< n$.

If we now rewrite (2.9) we have

$$0 = \int_a^b h(x) \left\{ p_0(x)\overline{u(x)} + \int_x^b \sum_{k=1}^{n} p_k(s) \frac{(s-x)^{k-1}}{(k-1)!} \overline{u(s)}ds \right.$$

$$+ \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \widetilde{\chi}^*(s)\bar{\bar{C}}ds \int_a^b \overline{u(t)}\chi(t)dt$$

$$\left. + w^*\widetilde{B}^* \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \overline{\widetilde{\chi}(s)}ds - \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \overline{\hat{u}(s)}ds \right\} dx$$

$$+ [w^*\widetilde{A}^* - \hat{w}^*]z.$$

Thus we have a vector in $L_2([a, b]) \oplus C^r$ which is orthogonal to all (h, z). Thus there exists a polynomial q of degree $< n$, and constants c_j, $1 \leqq j \leqq N$ such that

$$\widetilde{A}w - \hat{w} = \sum_1^N c_j\xi_j$$

and

$$\overline{p_0(x)}u(x) + \int_x^b \sum_{k=1}^{n} \overline{p_k(s)} \frac{(s-x)^{k-1}}{(k-1)!} u(s)ds$$

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$$+ \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \widetilde{\chi}^T(s)ds\widetilde{C}(u|\chi)_{[a,b]}$$

$$+ \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \widetilde{\chi}^T(s)ds\widetilde{B}w - \int_x^b \frac{(s-x)^{n-1}}{(n-1)!} \hat{u}(s)ds$$

$$= q(x) + \sum_1^N c_j \int_x^b \frac{(s-x)^{n-1}}{(u-1)!} \psi_j(s)ds \text{ a.e. on } [a, b].$$

By altering u on a null set this becomes an equality everywhere on $[a, b]$. Thus u is absolutely continuous on $[a, b]$, and on differentiation we obtain an a.e. equality between the derivative of an absolutely continuous function and an absolutely continuous function. This new equality must thus hold everywhere on $[a, b]$. Repeating this argument we find that $u^{(k)}$ is absolutely continuous on $[a, b]$ $0 \leqq k \leqq n-1$, and

$$\tau^*u + \widetilde{\chi}^T[\widetilde{C}(u|\chi)_{[a,b]} + \widetilde{B}w] - \hat{u} = \sum_1^N c_j\psi_j \quad \text{a.e. on } [a, b].$$

Recalling the definition of the ψ_j we see that

$$\tau^*u + \widetilde{\chi}^T[\widetilde{C}(u|\chi) + \widetilde{B}w] - \hat{u} = \sum_1^N d_j\psi_j \quad \text{a.e. on } [a, b].$$

If $[a, b]$ is not contained in the interior of I the same argument works if y is restricted by

$$y^{(k)}(a) = y^{(k)}(b) = 0 \quad 0 \leqq k \leqq n - 1.$$

Now if $\phi_1, \dots, \phi_M$ form an orthonormal basis for the subspace of $L_2(I)$ spanned by $\psi_1, \dots, \psi_N$ it follows that for any compact subinterval $J = [a, b]$ of I there are constants c_k^J such that

$$\tau^*u + \widetilde{\chi}^T[\widetilde{C}(u|\chi) + \widetilde{B}w] - \hat{u} = \sum_1^M c_k^J\phi_k \quad \text{a.e. on } J.$$

If $\Delta(J) = \det[(\phi_j|\phi_k)_J]$ it is clear that as J expands to I, $\Delta(J) \rightarrow 1$. Thus there is a compact interval J_0 such that $J \supset J_0$ implies $\Delta(J) > \frac{1}{2}$ so $\phi_1, \dots, \phi_M$ are linearly independent on such J. Thus if J_1 and J_2 both include J_0

$$\sum_1^M c_k^{J_1}\phi_k = \sum_1^M c_k^{J_2}\phi_k \quad \text{a.e. on } J_0$$

and by linear independence $c_k^{J_1} = c_k^{J_2}$ $1 \leqq k \leqq M$. Thus the c_k^J do not depend on J. It follows that $\tau^*u \in L_2(I)$ and so $(u, w) \in D_1(\tau)$.

The operators we wish to consider are closed densely defined operators

T with $T_0(L; \tilde{L}) \subset T \subset T_1(L)$. This does, in fact, include all closed, densely defined operators $T \subset T_1(L)$ with

$$D(T) = \{ (y, z) \in D_1(L) | HV(y) + J(y|\tilde{\chi}) + Kz = 0\}$$

where H, J, K are $k \times N$, $k \times \tilde{m}$, $k \times r$ respectively. In order that $D(T)$ be dense the rank of H must be greater than or equal to that of $[JK]$ for otherwise the conditions include ones of the form

$$((y, z) |(\phi, \xi)) = 0 \quad \text{where } (\phi, \xi) \neq 0.$$

Let us drop extraneous conditions and let rank $H = k$. There is then a $N \times k$ right inverse $\hat{H}$ to H and if we set

$$\tilde{D} = -J^*\hat{H}Q(\tau)^* \quad \text{and} \quad \tilde{F} = -K^*\hat{H}Q(\tau)^*$$

we determine $\tilde{L}$ adjoint to L, and thus $W(y, z)$ so that

$$HW(y, z) = HV(y) + J(y|\tilde{\chi}) + Kz.$$

In a precisely analogous fashion to the classical differential operator case we have

THEOREM 2.4. *If T is a closed operator with*

$$T_0(L; \tilde{L}) \subset T \subset T_1(L)$$

then there is an integer k $(0 \leqq k \leqq N)$ and a $k \times N$ matrix M of rank k such that

$$D(T) = \{ (y, z) \in D_1(L) |MW(y, z) = 0\}.$$

Furthermore if $\tilde{M}$ is a $(N - K) \times N$ matrix of rank $N - k$ such that

$$(2.10) \quad MQ(\tau)^{-1}\tilde{M}^* = 0,$$

then

$$D(T^*) = \{ (u, w) \in D_1(\tilde{L}) |\tilde{M}\tilde{W}(u, w) = 0\},$$

and T^* is the restriction of $T_1(\tilde{L})$ to $D(T^*)$.

Proof. Since T is a closed, densely defined operator $T = T^{**}$ and $(y, z) \in D(T)$ if and only if $(y, z) \in D_1(L)$ and

$$\tilde{W}(u, w)^*Q(\tau)W(y, z) = 0 \quad \text{for all } (u, w) \in D(T^*).$$

Now $\tilde{W}$ is a linear map of $D(T^*)$ into C^N and we can choose (u_j, w_j) $1 \leqq j \leqq k$ such that $\tilde{W}(u_j, w_j)$ form a basis for the range. It follows immediately that if M is the matrix with rows $\tilde{W}(u_j, u_j)^*Q(\tau)$ then M has rank k and $D(T)$ is described as above.

Now let U be a $N \times N$ unitary matrix with first k rows (denoted by U_1) spanning the row space of M. Let U_2 denote the $(N - k) \times N$ matrix consisting of the last $N - k$ rows of U. Then $(y, z) \in D(T)$ if and only if

$U_1W(y, z) = 0$ and $U_2W(y, z) = c$ is arbitrary. Thus $(u, w) \in D(T^*)$ if and only if

$$\widetilde{W}(u, w)^*Q(\tau)U_2^*c = 0$$

for all c, or equivalently

$$U_2Q(\tau)^*\widetilde{W}(u, w) = 0.$$

Now suppose $\widetilde{M}$ is $(N - k) \times N$ of rank $N - k$, and (2.10) holds. Then the rows of $\widetilde{M}Q(\tau)^{*-1}$ are orthogonal to the vows of M, and thus to the rows of U_1. It follows that the rows of $\widetilde{M}Q(\tau)^{*-1}$ are all in the row space of U_2, so (noting the identity of ranks) there is a non-singular K such that

$$\widetilde{M}Q(\tau)^{*-1} = KU_2 \quad \text{and} \quad \widetilde{M}\widetilde{W}(u, w) = KU_2Q(\tau)^*\widetilde{W}(u, w),$$

which completes the proof.

Due to the fact that the boundary conditions involved in the definitions of $D(T)$ and $D(T^*)$ can be used to change the form of L and $\widetilde{L}$ on those domains, it is clear that T and T^* can also be defined using different pairs of adjoint expressions. We shall now exhibit, with respect to a given T and T^* arising from L and L^*, a canonical pair L_0 and $\widetilde{L}_0$, also an adjoint pair, which can be used to define T and T^*, but being more simply related to each other than L and $\widetilde{L}$. We denote the matrices involved in L_0 and $\widetilde{L}_0$ by using zero subscripts.

THEOREM 2.5. *Let T and T^* be adjoint operators arising from the adjoint pair L and $\widetilde{L}$ with*

$$D(T) = \{(y, z) \in D_1(L) \,|\, MW(y, z) = 0\}$$

$$D(T^*) = \{(u, w) \in D_1(\widetilde{L}) \,|\, \widetilde{M}\widetilde{W}(u, w) = 0\}$$

where $MQ(\tau)^{-1}\widetilde{M}^* = 0$ *and* $\operatorname{rank} M + \operatorname{rank} \widetilde{M} = N$. *Then there exists an adjoint pair L_0 and $\widetilde{L}_0$ such that*

$$L_0(y, z) = L(y, z) \quad \text{for all } (y, z) \in D(T),$$

$$\widetilde{L}_0(u, w) = \widetilde{L}(u, w) \quad \text{for all } (u, w) \in D(T^*),$$

$$D(T) = \{(y, z) \in D_1(L_0) \,|\, MW_0(y, z) = 0\}$$

$$D(T^*) = \{(u, w) \in D_1(\widetilde{L}_0) \,|\, \widetilde{M}\widetilde{W}_0(u, w) = 0\}.$$

Furthermore the coefficient matrices of L_0 and $\widetilde{L}_0$ satisfy

$$(2.11) \quad \widetilde{C}_0 = C_0^*,\ \widetilde{B}_0 = E_0^*,\ \widetilde{A}_0 = A_0^*,\ \widetilde{E}_0 = B_0^*$$

and

$$(2.12) \quad D_0Q(\tau)^{-1}\widetilde{D}_0^*,\ D_0Q(\tau)^{-1}\widetilde{F}_0^*,\ F_0Q(\tau)^{-1}\widetilde{F}_0^*,\ F_0Q(\tau)^{-1}\widetilde{D}_0^* \quad \textit{all} = 0.$$

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Proof. We define L_0 and $\widetilde{L}_0$ by

$$L_0\begin{bmatrix} y \\ z \end{bmatrix} = L\begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} \chi^T\Lambda \\ \Omega \end{bmatrix} MW(y, z)$$

$$\widetilde{L}_0\begin{bmatrix} u \\ w \end{bmatrix} = \widetilde{L}\begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} \widetilde{\chi}^T\widetilde{\Lambda} \\ \widetilde{\Omega} \end{bmatrix} \widetilde{M}\widetilde{W}(u, w),$$

where Λ, Ω, $\widetilde{\Lambda}$, $\widetilde{\Omega}$ are $m \times k$, $r \times k$, $\widetilde{m} \times (N - k)$, $r \times (N - k)$ respectively and $k = \text{rank } M$. It is clear that L_0 and L will coincide on $D(T)$ and $\widetilde{L}_0$ and $\widetilde{L}$ will coincide on $D(T^*)$.

The verification that L_0 and $\widetilde{L}_0$ form an adjoint pair is straightforward. For illustration we will demonstrate the validity of the first condition of (2.6)

$$\begin{aligned} &\widetilde{C}_0 - C_0^* - \widetilde{D}_0 Q(\tau)^{*-1} D_0^* \\ &= (\widetilde{C} + \widetilde{\Lambda} M Q(\tau)^{*-1} D^*) - (C - \Lambda M Q(\tau)^{-1} \widetilde{D}^*)^* \\ &- (\widetilde{D} + \widetilde{\Lambda}\widetilde{M}) Q(\tau)^{*-1} (D + \Lambda M)^* \\ &= \widetilde{C} - C^* - \widetilde{D} Q(\tau)^{*-1} D^*, \end{aligned}$$

since

$$\widetilde{M} Q(\tau)^{*-1} M^* = (M Q(\tau)^{-1} \widetilde{M}^*)^* = 0.$$

Thus if the first condition of (2.6) is satisfied for L and $\widetilde{L}$ it is also satisfied for L_0 and $\widetilde{L}_0$. The others are similar.

It follows from (2.6) that (2.11) will follow from (2.12) so we must show that it is possible to choose Λ, Ω, $\widetilde{\Lambda}$, $\widetilde{\Omega}$ so that (2.12) is satisfied. Recall from the proof of Theorem 2.4 the unitary matrix U with first k rows U_1 and last $N - k$ rows U_2 such that $M = RU_1$ and $\widetilde{M} = \widetilde{R} U_2 Q(\tau)^*$ where R and $\widetilde{R}$ are non-singular. If we set

$$S = -U_1^* R^{-1} \quad \text{and} \quad \widetilde{S} = -Q(\tau)^{*-1} U_2^* \widetilde{R}^{-1}$$

and then $\Lambda = DS$, $\Omega = FS$, $\widetilde{\Lambda} = \widetilde{D}\widetilde{S}$, $\widetilde{\Omega} = \widetilde{F}\widetilde{S}$ we find that the conditions of (2.12) amount to the unitarity of U. In particular

$$\begin{aligned} D_0 Q(\tau)^{-1} \widetilde{F}_0^* &= (D + \Lambda M) Q(\tau)^{-1} (\widetilde{F} + \widetilde{\Omega}\widetilde{M})^* \\ &= D[I + SM] Q(\tau)^{-1} [I + \widetilde{M}^* \widetilde{S}^*] \widetilde{F}^* \\ &= D[I - U_1^* U_1] Q(\tau)^{-1} [I - Q(\tau) U_2^* U_2 Q(\tau)^{-1}] \widetilde{F}^* \\ &= D[I - U_1^* U_1][I - U_2^* U_2] Q(\tau)^{-1} \widetilde{F}^* \\ &= D[I - U_1^* U_1 - U_2^* U_2] Q(\tau)^{-1} \widetilde{F}^* \end{aligned}$$

since $U_1 U_2^* = 0$. The fact that $I = U_1^* U_1 + U_2^* U_2$ is just the unitary nature of U. All the others follow in the same way.

Finally

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$$MW_0(y, z) = MW(y, z) - MQ(\tau)^{-1}\widetilde{M}^*[\widetilde{\Lambda}^*(y|\widetilde{\chi}) + \widetilde{\Omega}^*z]$$
$$= MW(y, z).$$

Similarly $\widetilde{M}\widetilde{W}_0(u, w) = \widetilde{M}\widetilde{W}(u, w)$ and the proof is complete.

3. Self-adjointness. In order that an expression L have the possibility of generating self-adjoint operators it is necessary that $T_1(L)^*$ be a restriction of $T_1(L)$. This implies that for $(u, w) \in D(T_1(L)^*)$ we also have $(u, w) \in D_1(L)$ and

$$\tau u + \chi^T[C(u|\widetilde{\chi}) + DV(u) + Bw]$$
$$= \tau^*u + \widetilde{\chi}^T[C^*(u|\chi) + E^*w].$$

This means that for all u such that there exists a w with $(u, w) \in D(T_1(L)^*)$, $(\tau - \tau^*)u$ belongs to the joint span of the entries in χ and $\widetilde{\chi}$. Since $D(T_1(L)^*)$ is dense in $L_2(I) \oplus C^r$ it follows that the manifold of such u's is infinite dimensional. Thus $\tau^* = \tau$.

With $\tau^* = \tau$ it is natural to use $\widetilde{V} = V$ (so that $Q(\tau)^* = -Q(\tau)$) and replace χ and $\widetilde{\chi}$ with a column which spans both of them. We shall denote this possibly enlarged column by χ.

THEOREM 3.1. *If $\tau^* = \tau$, $\widetilde{V} = V$, and $\widetilde{\chi} = \chi$; $T_1(L)^*$ is a restriction of $T_1(L)$ if and only if L is adjoint to itself, i.e.,*

$$\begin{aligned} C &= C^* - DQ(\tau)^{-1}D^* \\ B &= E^* - DQ(\tau)^{-1}F^* \\ A &= A^* - FQ(\tau)^{-1}F^* \\ E &= B^* - FQ(\tau)^{-1}D^*. \end{aligned} \tag{3.1}$$

Proof. Here

$$L\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \tau y + \chi^T\{C(y|\chi) + DV(y) + Bz\} \\ Az + E(y|\chi) + FV(y) \end{bmatrix},$$

and by Theorem 2.1

$$D(T_1(L)^*) = \{ (u, w) \in D_1(\tau) \oplus C^r | V(u)$$
$$- Q(\tau)^{-1}D^*(u|\chi) - Q(\tau)^{-1}F^*w = 0\}$$

with

$$T_1(L)^*\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \tau u + \chi^T\{C^*(u|\chi) + E^*w\} \\ A^*w + B^*(u|\chi) \end{bmatrix},$$

and this is the restriction of $T_1(L)$ to $D(T_1(L)^*)$ if and only if (3.1) holds.

From Theorem 2.4 and 3.1 we have

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COROLLARY 3.1. *If L is self-adjoint then a closed operator T, $T_0(L; \tilde{L}) \subset T \subset T_1(L)$ is self-adjoint if and only if $N = 2l$ is even and*

$$D(T) = \{ (y, z) \in D_1(L) \,|\, MW(y, z) = 0\}$$

where M is an $l \times 2l$ matrix of rank l such that

$$MQ(\tau)^{-1}M^* = 0.$$

Here we also obtain a canonical form analogous to that obtained in Theorem 2.5.

THEOREM 3.2. *If L is a self-adjoint expression and T is a self-adjoint restriction of $T_1(L)$ with*

$$D(T) = \{ (y, z) \in D_1(L) \,|\, MW(y, z) = 0\},$$

then there is a self-adjoint expression L_0 such that $L(y, z) = L_0(y, z)$ for all $(y, z) \in D(T)$, and

$$D(T) = \{ (y, z) \in D_1(L_0) \,|\, MW_0(y, z) = 0\}.$$

Furthermore, the coefficient matrices of L_0 satisfy

$$(3.2) \qquad C_0 = C_0^*, \ B_0 = E_0^*, \ A_0 = A_0^*$$

and

$$(3.3) \quad D_0Q(\tau)^{-1}D_0^*, \ D_0Q(\tau)^{-1}F_0^*, \ F_0Q(\tau)^{-1}F_0^* \text{ all } = 0.$$

Proof. We define L_0 by

$$L_0\begin{bmatrix} y \\ z \end{bmatrix} = L\begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} \chi^T\Lambda \\ \Omega \end{bmatrix} MW(y, z).$$

It is clear that L_0 and L coincide on $D(T)$, and the self-adjointness of L_0 follows from that of L and the fact that $MQ(\tau)^{-1}M^* = 0$.

We must show that Λ and Ω can be chosen so that (3.3) is satisfied, for this combined with (3.1) for L_0 will imply (3.2).

Since $Q(\tau)$ is non-singular and $Q(\tau)^* = -Q(\tau)$ it follows that $iQ(\tau)$ is hermitian and there exists a non-singular matrix X such that $Q_1 = X^*(Iq(\tau))X$ is diagonal with diagonal entries $+1$ or -1. Thus $Q_1 = Q_1^* = Q_1^{-1}$. Let $M_1 = MX$ and choose R non-singular so that RM_1 has orthonormal rows. Thus RM_1Q_1 has orthonormal rows. Now

$$0 = MQ(\tau)^{-1}M^* = -iM_1Q_1M_1^*,$$

so

$$RM_1(RM_1Q_1)^* = R(M_1Q_1M_1^*)R^* = 0,$$

and the matrix U with first l rows RM_1 and last l rows RM_1Q_1 is unitary.

Since

$$I = U^*U = M_1^*R^*RM_1 + Q_1M_1^*R^*RM_1Q_1,$$

if we set $\Lambda = -DXM_1^*R^*R^*$ and $\Omega = -FXM_1^*R^*R$, the conditions (3.3) are all equivalent to the unitarity of U and the proof is complete.

4. Application to problems for τ with λ in boundary conditions. Here we consider problems $\tau y = \lambda y$ with boundary conditions which involve λ in a linear fashion. Thus let H and G be $r \times N$ matrices, K a $(p - r) \times N$ matrix and consider

$$\begin{aligned} &\tau u - \lambda u = 0 \\ (4.1)\qquad &HV(y) = \lambda GV(y) \\ &KV(y) = 0. \end{aligned}$$

Since we always drop extraneous conditions we shall assume that the $p \times N$ matrix

$$J(\lambda) = \begin{bmatrix} H - \lambda G \\ K \end{bmatrix}$$

has rank p except possibly for isolated values of λ.

If we define the expression $L(y, z) = (\tau y, HV(y))$, and define an operator T on $L_2(I) \oplus C^r$ using L with

$$D(T) = \{(y, z) \in D_1(L) \,|\, GV(y) = z, KV(y) = 0\},$$

we obtain an operator for which the spectral problem gives (4.1). However this does not fit with the discussion of Section 2 unless the matrix M involved, $M = [G^TK^T]^T$, is of rank p. Thus it is necessary to replace the boundary conditions of (4.1) by an equivalent set for which this condition is fulfilled.

PROPOSITION 4.1. *The boundary conditions of* (4.1) *are equivalent to a set of conditions in which $J(\lambda)$ has rank p for all λ and M also has rank p* $(M = [G^TK^T]^T)$.

Proof. If $J(\lambda_0)$ has rank $< p$ there is a non-zero row vector $c = [c_1c_2]$ (c_1 the first r entries of c) such that $cJ(\lambda_0) = 0$. Note that if $c_1 = 0$ then $c_2K = 0$ and $J(\lambda)$ will have rank $< p$ for all λ. We now delete from $H - \lambda G$ a row corresponding to a non-zero entry in c_1 and adjoin the row c_1G to K. This is possible since

$$\begin{aligned} 0 &= c_1[HV(y) - \lambda GV(y)] + c_2KV(y) \\ &= (\lambda_0 - \lambda)c_1GV(y), \end{aligned}$$

so the modified set of conditions is equivalent to the original set. Repeating this argument we arrive at a stage where the new $J(\lambda)$ has rank p for all λ.

Now if the row spaces of G and K have a non-zero intersection (or if G has rank $< r$) there are row vectors c and d such that $cG = dK$ and $c \neq 0$. Then the row cH can be adjoined to K and a row of $H - \lambda G$ corresponding to a non-zero component of c deleted. This follows from

$$0 = c[HV(y) - \lambda GV(y)] + \lambda dKV(y) = cHV(y).$$

Repeating if necessary the proof is complete.

THEOREM 4.1. *The eigenvalue problem* (4.1) *where* $J(\lambda)$ *and* M *have rank* p *for all* λ *arises from the spectral problem for the operator* T *on* $L_2(I) \oplus C^r$ *defined by the expression*

$$L(y, z) = (\tau y, HV(y))$$

with

$$D(T) = \{(y, z) \in D_1(L)GV(y) = z, KV(y) = 0\}.$$

Furthermore, the adjoint operator T^* *arises from an eigenvalue problem of the form* (4.1) *if and only if* $[H^TG^TK^T]^T$ *has rank* $p + r$ (*so, in particular* $r < N - p$).

Proof. The spectral problem for T is $\tau y = \lambda y$, $HV(y) = \lambda z$, $GV(y) = z$, and $KV(y) = 0$; which are precisely (4.1) with z as an auxiliary set of variables.

From the remarks following Theorem 2.3, since $M = \{G^TK^T]^T$ is of rank p we may choose $\widetilde{F}$ (thus determining $\widetilde{L}$ and $W(y, z)$) so that $MW(y, z) = 0$ has first r conditions $GV(y) = z$ and last $p - r$ conditions $KV(y) = 0$. This requires

$$GQ(\tau)^{-1}\widetilde{F}^* = I \quad \text{and} \quad KQ(\tau)^{-1}\widetilde{F}^* = 0.$$

Note that in $\widetilde{L}$,

$$\widetilde{A} = \widetilde{F}Q(\tau)^{*-1}H^*.$$

The operator T^* is determined by $\widetilde{L}$ and conditions $\widetilde{M}\widetilde{W}(u, w) = 0$ where $\widetilde{M}$ is $(N - p) \times N$ of rank $N - p$ and

$$MQ(\tau)^{-1}\widetilde{M}^* = 0.$$

Modifying the form of L and $\widetilde{L}$ using the boundary conditions we see that it is impossible to obtain an expression $\widetilde{L}_1$ defining T^* which has $\widetilde{A}_1 = 0$ (necessary in order that T^* give rise to a problem of the form (4.1)) unless it is possible initially to choose $\widetilde{F}$ so that

$$HQ(\tau)^{-1}\widetilde{F}^* = 0.$$

Now if $[H^TG^TK^T]^T$ has rank $< p + r$ the row spaces of H and G must have a non-zero intersection and there are non-zero row vectors c and d such that $cH = dG$. It follows that

$$cHQ(\tau)^{-1}\tilde{F}^* = dGQ(\tau)^{-1}\tilde{F}^* = d \neq 0,$$

so it is impossible to have $HQ(\tau)^{-1}\tilde{F}^* = 0$. On the other hand if $J_1 = [H^TG^TK^T]^T$ has rank $p + r$ (so $N \geqq P + r$) then there is an $N \times (p + r)$ right inverse $\hat{J}_1$ to J_1 and we may choose

$$\tilde{F} = [0I0]\hat{J}_1^* Q(\tau)^*,$$

where the partitioned matrix consists of $r \times r$, $r \times r$, and $r \times (p - r)$ blocks. Then

$$GQ(\tau)^{-1}\tilde{F}^* = I,\ HQ(\tau)^{-1}\tilde{F}^* = 0, \quad \text{and } KQ(\tau)^{-1}\tilde{F}^* = 0.$$

Let us now examine the question of when two eigenvalue problems of the form (4.1) arise from adjoint operators. Consider

(A) $\quad \tau y - \lambda y = 0$

$$HV(y) - \lambda GV(y) = 0$$

$$KV(y) = 0$$

(B) $\quad \tau^* u - \lambda u = 0$

$$\tilde{H}\tilde{V}(u) - \lambda \tilde{G}\tilde{V}(u) = 0$$

$$\tilde{K}\tilde{V}(u) = 0$$

where H, $\tilde{H}$, G, $\tilde{G}$ are $r \times N$, K is $(p - r) \times N$, and $\tilde{K}$ is $(N - p - r) \times N$. Let the $(p + r) \times N$ matrix $[H^TG^TK^T]^T$ be of rank $p + r$, and the $(N - p + r) \times N$ matrix $[\tilde{H}^T\tilde{G}^T\tilde{K}^T]^T$ be of rank $N - p + r$.

THEOREM 4.2. *The eigenvalue problems* (A) *and* (B) *are adjoint to each other in the sense that they arise from operators T and T^* respectively if and only if*

$$\begin{bmatrix} H \\ G \\ K \end{bmatrix} Q(\tau)^{-1}[\tilde{H}^*\tilde{G}^*\tilde{K}^*] = \begin{bmatrix} 0 & -S & 0 \\ S & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

where the $r \times r$ matrix S is non-singular. Furthermore problem (A) *always has an adjoint of the form* (B).

Proof. Let

$$L(y, z) = (\tau y, FV(y)) \quad \text{and} \quad \tilde{L}(u, w) = (\tau^* u, \tilde{F}\tilde{V}(u))$$

be adjoint expressions. Then

$$W(y, z) = V(y) - Q(\tau)^{-1}\tilde{F}^* z$$

$$\tilde{W}(u, w) = \tilde{V}(u) - Q(\tau^*)^{-1}F^* w.$$

In order that T defined by L with

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$$D(T) = \{ (y, z) \in D_1(L) \,|MW(y, z) = 0\}$$

give rise to (A) we must choose F, $\widetilde{F}$, and M in such a way that $MW(y, z) = 0$ and $FV(y) = \lambda z$ are equivalent to

$$HV(y) - \lambda GV(y) = 0 \quad \text{and} \quad KV(y) = 0$$

on elimination of z. Thus

$$0 = \lambda MV(y) - MQ(\tau)^{-1}\widetilde{F}^*FV(y)$$

must include

$$HV(y) - \lambda GV(y) = 0 \quad \text{and} \quad KV(y) = 0.$$

Since M is of rank p, $MQ(\tau)^{-1}\widetilde{F}^*F$ must be of lower rank, in fact of rank r. We may thus arrange that the last $p - r$ rows of $MQ(\tau)^{-1}\widetilde{F}^*F$ vanish and the last $p - r$ conditions reduce to the last $p - r$ entries of $\lambda MV(y)$ must vanish. Thus choose $M = [G^TK^T]^T$ and require that

(4.3) $$\begin{bmatrix} G \\ K \end{bmatrix} Q(\tau)^{-1}\widetilde{F}^*F = \begin{bmatrix} H \\ 0 \end{bmatrix}.$$

Similarly for (B) we can choose $\widetilde{M} = [\widetilde{G}^T\widetilde{K}^T]^T$ and require that

(4.4) $$\begin{bmatrix} \widetilde{G} \\ \widetilde{K} \end{bmatrix} Q(\tau^*)^{-1}F^*\widetilde{F} = \begin{bmatrix} \widetilde{H} \\ 0 \end{bmatrix}.$$

Now in order to have adjointness we must have:

(4.5) $$FQ(\tau)^{-1}\widetilde{F}^* = 0$$

and

(4.6) $$MQ(\tau)^{-1}\widetilde{M}^* = 0.$$

Now since H is of rank r and from (4.3)

$$GQ(\tau)^{-1}\widetilde{F}^*F = H$$

it follows that $GQ(\tau)^{-1}\widetilde{F}^*$ and F must both be of rank r. Thus the former is invertible and $F = CH$ where C is non-singular. Similarly $\widetilde{F} = \widetilde{C}\widetilde{H}$ where $\widetilde{C}$ is non-singular. Thus $FQ(\tau)^{-1}\widetilde{F}^* = 0$ implies $HQ(\tau)^{-1}\widetilde{H}^* = 0$, and substitution in (4.3) and (4.4) yield

$$\begin{bmatrix} G \\ K \end{bmatrix} Q(\tau)^{-1}\widetilde{H}^*\widetilde{C}^*CH = \begin{bmatrix} H \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \widetilde{G} \\ \widetilde{K} \end{bmatrix} Q(\tau^*)^{-1}H^*C^*\widetilde{C}\widetilde{H} = \begin{bmatrix} \widetilde{H} \\ 0 \end{bmatrix}.$$

The rank argument yields

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$$GQ(\tau)^{-1}\tilde{H}^*\tilde{C}^*C = I$$
$$\tilde{G}Q(\tau^*)^{-1}H^*C^*\tilde{C} = I.$$

Since C and $\tilde{C}$ are non-singular so is

$$S = GQ(\tau)^{-1}\tilde{H}^* = (\tilde{C}^*C)^{-1} = [(C^*\tilde{C})^{-1}]^*$$
$$= (\tilde{G}Q(\tau^*)^{-1}H^*)^* = -HQ(\tau)^{-1}\tilde{G}^*.$$

Note that (4.6) implies the lower right 2×2 block of (4.2) and we have verified the (1, 1) entry and the statements about the (1, 2) and (2, 1) entries of (4.2) so it remains to verify the vanishing of $KQ(\tau)^{-1}\tilde{H}^*$ and $HQ(\tau)^{-1}\tilde{K}^*$. Now

$$0 = KQ(\tau)^{-1}\tilde{F}^*F = KQ(\tau)^{-1}\tilde{H}^*\tilde{C}^*CH,$$

and $\tilde{C}^*CH$, being of rank r, has a right inverse, thus

$$KQ(\tau)^{-1}\tilde{H}^* = 0.$$

Similarly

$$HQ(\tau)^{-1}\tilde{K}^* = -[\tilde{K}Q(\tau^*)^{-1}H^*]^* = 0$$

and the proof of (4.2) is complete.

Thus if (4.2) is satisfied and we choose $r \times r$ non-singular matrices C and $\tilde{C}$ such that $\tilde{C}^*C = S^{-1}$ and define $F = CH$, $\tilde{F} = \tilde{C}\tilde{H}$, $M = [G^TK^T]^T$, and $\tilde{M} = [\tilde{G}^T\tilde{K}^T]^T$ we determine expressions L and $\tilde{L}$ adjoint to each other, and boundary conditions determining adjoint operators T and T^* which yield problems (A) and (B) respectively.

Now if we are given only (A), the matrix $[H^TG^TK^T]^T$ is of rank $p + r$ and we may adjoin $N - p - r$ rows J so that $R = [H^TG^TK^TJ^T]^T$ is non-singular. Let

$$P = \begin{bmatrix} 0 & I_r & 0 & 0 \\ -I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{N-p-r} \end{bmatrix}$$

be $(N - p + r) \times N$ and define

$$\begin{bmatrix} \tilde{H} \\ \tilde{G} \\ \tilde{K} \end{bmatrix} = PR^{*-1}Q(\tau)^*.$$

Then

$$\begin{bmatrix} H \\ G \\ K \end{bmatrix} Q(\tau)^{-1}[\tilde{H}^*\tilde{G}^*\tilde{K}^*] = \begin{bmatrix} H \\ G \\ K \end{bmatrix} R^{-1}P^* = \begin{bmatrix} 0 & -Ir & 0 \\ Ir & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $\tilde{H}$, $\tilde{G}$, $\tilde{K}$ determine a problem (B) which is adjoint to (A).

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An important remark must be inserted here. Even if $\tau^* = \tau$ and (4.2) holds with $\tilde{H} = H$, $\tilde{G} = G$, and $\tilde{K} = K$, it does not follow that problem (A) (which now coincides with problem (B)) is really self-adjoint. The difficulty is that we may not be able to make the operator T on $L_2(I) \oplus C^r$ self-adjoint.

COROLLARY 4.1. *If* $\tau^* = \tau$ *and* (4.2) *holds for* $\tilde{H} = H$, $\tilde{G} = G$, $\tilde{K} = K$; *with* $S = GQ(\tau)^{-1}H^*$ *positive definite, then* (A) *is self-adjoint in the sense that it arises from a self-adjoint operator* T *on* $L_2(I) \oplus C^r$.

Proof. In order to have L self-adjoint it is necessary and sufficient to have $\tilde{F} = F$, which requires $\tilde{C} = C$, so

$$GQ(\tau)^{-1}H^*C^*C = I.$$

Since C^*C is positive definite its inverse $S = GQ(\tau)^{-1}H^*$ must also have that property. We can then choose C to be the positive square root of S^{-1}.

5. Regular problems. Let τ be regular on $I = [a, b]$ of order n. Thus $N = 2n$ and we may use

$$V(y) = [y(a), y'(a), \dots, y^{(n-1)}(a), y(b), y'(b), \dots, y^{(n-1)}(b)].$$

Let L be an expression (1.2), $\tilde{L}$ be an adjoint expression and T an operator generated by L with

$$D(T) = \{(y, z) \in D_1(L) \mid MW(y, z) = 0\}$$

for some $n \times 2n$ matrix M of rank n.

Let $u_1, \dots, u_n$ form the basis for solutions of $\tau y - \lambda y = 0$ on $[a, b]$ determined by the initial conditions

$$u_j^{(k-1)}(a) = \delta_{jk}.$$

We shall denote by $\mathscr{K}(f)$ the solution to $\tau y - \lambda y = f$ which satisfies zero initial conditions at a. Note that

$$\mathscr{K}(f) = \int_a^b K(t, s\ \lambda) f(s) ds,$$

where the kernel K can be easily expressed in terms of $u_1, u_2, \dots, u_n$.

We now solve $(T - \lambda I)(y, z) = (f, w)$. Let P denote the $m \times 1$ column vector

$$C(y|\tilde{\chi}) + DV(y) + Bz$$

and note that a solution to

$$\tau y + \chi^T P - \lambda y = f$$

must be of the form

(5.1) $$y = \sum_1^n c_j u_j - \mathscr{K}(\chi)^T P + \mathscr{K}(f).$$

To determine y and z we must determine n c_j's, r z_j's, and m P_j's. In the process of doing this we include $2n$ $V_j(y)$'s and $\tilde{m}$ $(y|\tilde{\chi}_j)$'s as auxiliary unknowns. We obtain the necessary $3n + m + \tilde{m} + r$ equations by using

$$(A - \lambda I)z + E(y|\tilde{\chi}) + FV(y) = w \quad (r \text{ equations}),$$

the boundary conditions (n equations), the definition of P (m equations), applying V to (5.1) ($2n$ equations), and taking the inner product of (5.1) with $\tilde{\chi}$ ($\tilde{m}$ equations). This yields the system

(5.2) $$\begin{aligned} &V(y) - V(u^T)C + V(\mathscr{K}(\chi)^T)P = V(\mathscr{K}(f)) \\ &(y|\tilde{\chi}) - (u|\tilde{\chi}^T)^T C + (\mathscr{K}(\chi)\,|\tilde{\chi}^T)^T P = (\mathscr{K}(f)\,|\tilde{\chi}) \\ &- DV(y) - C(y|\tilde{\chi}) + P - Bz = 0 \\ &FV(y) + E(y|\tilde{\chi}) + (A - \lambda I)z = w \\ &MV(y) - MQ(\tau)^{-1}\tilde{D}^*(y|\tilde{\chi}) - MQ(\tau)^{-1}\tilde{F}^* z = 0. \end{aligned}$$

The various matrices denoted compactly in this system are:

$V(u^T)$ $2n \times n$ with j, k entry $V_j(u_k)$

$V(\mathscr{K}(\chi)^T)$ $2n \times m$ with j, k entry $V_j(\mathscr{K}(\chi_k))$

$(u|\tilde{\chi}^T)^T$ $\tilde{m} \times n$ with j, k entry $(u_k|\tilde{\chi}_j)$

$(\mathscr{K}(\chi)\,|\tilde{\chi}^T)^T$ $\tilde{m} \times m$ with j, k entry $(\mathscr{K}(\chi_k)\,|\tilde{\chi}_j)$.

While it is easy enough to eliminate $V(y)$ and $(y|\tilde{\chi})$ to obtain a system for c, P, and z, the coefficients become rather complex. It is clear from the properties of solutions to $\tau y - \lambda y = 0$ that the determinant of coefficients in (5.2), denoted by $\Delta(\lambda)$, is entire. If $\Delta(\lambda) = 0$ then with $f = w = 0$ there will be a non-trivial solution for c, P, and z and thus λ is an eigenvalue of the operator T. Since $\Delta(\lambda)$ is entire, the eigenvalues of T will form an at most countable set with ∞ as the only possible limit point, unless $\Delta(\lambda)$ is identically 0.

If $\Delta(\lambda) \neq 0$ it is clear that c, P, and z can be found so that the solution for y and z will be of the form

(5.3) $$\begin{aligned} y(t, \lambda) &= \psi(t, \lambda)^T w + \int_a^b G(t, s, \lambda) f(s) ds \\ z(\lambda) &= \Theta(\lambda) w + (f|\tilde{\psi}(\cdot, \bar{\lambda})). \end{aligned}$$

This formula for $(T - \lambda I)^{-1}$ makes it clear that this operator is completely continuous, so that if T is self-adjoint we may obtain an expansion in eigenvectors precisely as is done for self-adjoint regular differential operators.

6. Examples. (i) Let

$$L\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} iy' + \left\{ (y|1) - \dfrac{1-i}{2}(y(0) + iy(1)) - iz \right\} \\ z + i(y|1) - \dfrac{1+i}{2}(y(0) + iy(1)) \end{bmatrix}$$

define an operator T on $L_2([0, 1]) \oplus C$ with

$$D(T) = \{ (y, z) \in D_1(L)$$
$$| y(0) - iy(1) - (1 - i)(y|1) + (1 + i)z = 0\}.$$

This operator is self-adjoint with eigenvalues $\mu = 0$ and $\lambda_n = \pi/2 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$ and eigenvectors

$$\psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \chi_n = \begin{bmatrix} \Omega_n \\ \omega_n \end{bmatrix}$$

respectively, where

$$\Omega_n = e^{-i\lambda_n t} - \frac{1-i}{\lambda_n}, \quad \omega_n = -\frac{1+i}{\lambda_n}.$$

The eigenvectors form an orthonormal basis for $L_2([0, 1])$.

In $(T - \lambda I)^{-1}$ $\tilde{\psi} = \psi$ and

$$\psi(t, \lambda) = \frac{i\left[\lambda e^{-i\lambda(t-\frac{1}{2})} - 2 \sin \dfrac{\lambda}{2}\right]}{\lambda^2\left[\cos \dfrac{\lambda}{2} - \sin \dfrac{\lambda}{2}\right]} = \sum_{-\infty}^{\infty} \frac{\tilde{\omega}_n \Omega_n(t)}{\lambda_n - \lambda}$$

$$\Theta(\lambda) = \frac{-\lambda \cos \dfrac{\lambda}{2} + (\lambda + 2) \sin \dfrac{\lambda}{2}}{\lambda^2\left[\cos \dfrac{\lambda}{2} - \sin \dfrac{\lambda}{2}\right]} = \sum_{-\infty}^{\infty} \frac{\tilde{\omega}_n \omega_n}{\lambda_n - \lambda}$$

$$G(t, s, \lambda) = \frac{1}{\lambda^2\left[\cos \dfrac{\lambda}{2} - \sin \dfrac{\lambda}{2}\right]}\left[2 \sin \frac{\lambda}{2} - \lambda e^{-i\lambda(t-\frac{1}{2})} - \lambda e^{i\lambda(s-\frac{1}{2})} + J\right]$$

$$= -\frac{1}{\lambda} + \sum_{-\infty}^{\infty} \frac{\Omega_n(t)\overline{\Omega_n(s)}}{\lambda_n - \lambda},$$

where

$$J = \begin{cases} \dfrac{(1-i)}{2}\lambda^2 e^{-i\lambda(t-1/2)} \cdot e^{i\lambda s} & 0 \leqq s \leqq t \leqq 1 \\ \dfrac{(1+i)}{2}\lambda^2 e^{-i\lambda t} \cdot e^{i\lambda(s-1/2)} & 0 \leqq t \leqq s \leqq 1. \end{cases}$$

(ii) $iy' - \lambda y = 0$ with

$$ay(0) + by(1) = \lambda(cy(0) + dy(1)).$$

Here $H = [ab]$, $G = [cd]$ and in order to have λ really in the boundary condition we must have $ad - bc \neq 0$.

$$\begin{bmatrix} H \\ G \end{bmatrix} Q(\tau)^{-1}[H^*G^*] = \begin{bmatrix} i(|a|^2 - |b|^2) i(a\bar{c} - b\bar{d}) \\ i(\bar{a}c - \bar{b}d) i(|c|^2 - |d|^2) \end{bmatrix}.$$

Thus for self-adjointness $|b| = |a|$, $|d| = |c|$, $i(\bar{a}c - \bar{b}d) = -i(a\bar{c} - b\bar{d}) > 0$. Thus we may take $a = 1$, $b = e^{i\alpha}$, $d = ce^{i\beta}$ with α, β real. Now

$$-i(a\bar{c} - b\bar{d}) = -2\bar{c}e^{i\frac{(\alpha-\beta)}{2}} \sin \frac{(\alpha - \beta)}{2} \quad \text{and}$$

$$i(\bar{a}c - \bar{b}d) = -2ce^{-i\frac{(\alpha-\beta)}{2}} \sin \frac{(\alpha - \beta)}{2}.$$

Thus

$$\sin \frac{(\alpha - \beta)}{2} \neq 0 \quad \text{and} \quad c = -ke^{i\frac{(\alpha-\beta)}{2}} \operatorname{sign}\left(\sin \frac{(\alpha - \beta)}{2}\right)$$

where $k > 0$ for self-adjointness. If $k < 0$ the eigenvalue problem (ii) is its own adjoint, but does not arise from a self-adjoint operator.

(iii) $\tau y = -(py')' + qy$ on $I = [0, 1]$ where p is non-negative and C^1 and q is continuous. Here

$$Q(\tau) = \begin{bmatrix} 0 & p(0) & 0 & 0 \\ -p(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & -p(1) \\ 0 & 0 & p(1) & 0 \end{bmatrix}$$

(a) Separated boundary conditions

$$a_{11}y(0) + a_{12}y'(0) = \lambda[b_{11}y(0) + b_{12}y'(0)]$$

$$a_{21}y(1) + a_{22}y'(1) = \lambda[b_{21}y(1) + b_{22}y'(1)].$$

For self-adjointness the conditions of Corollary 4.1 require that all a's and b's be real and

$$a_{11}b_{12} - a_{12}b_{11} > 0, \quad a_{21}b_{22} - a_{22}b_{21} < 0.$$

These conditions correspond to those of Fulton [**1**, **2**] and Walter [**4**]. Note that if either or both signs are wrong we have an apparently self-adjoint problem which does not arise from a self-adjoint operator.

(b) Non-separated boundary conditions. We give two examples of non-separated conditions which arise from self-adjoint operators:

(i) $$y(0) = \lambda[ap(0)y'(0) - bp(1)y'(1)]$$
$$y(1) = \lambda[\bar{b}p(0)y'(0) - cp(1)y'(1)].$$

If $\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$ is positive definite this arises from a self-adjoint operator.

(ii) $$y(0) - p(0)y'(0) + y(1) = \lambda[y(0) + y(1) - p(1)y'(1)]$$
$$p(0)y'(0) + p(1)y'(1) = 0$$

arises from a self-adjoint operator.

References

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