

The Face Semigroup Algebra of a Hyperplane Arrangement

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Abstract. This article presents a study of an algebra spanned by the faces of a hyperplane arrangement. The quiver with relations of the algebra is computed and the algebra is shown to be a Koszul algebra. It is shown that the algebra depends only on the intersection lattice of the hyperplane arrangement. A complete system of primitive orthogonal idempotents for the algebra is constructed and other algebraic structure is determined including: a description of the projective indecomposable modules, the Cartan invariants, projective resolutions of the simple modules, the Hochschild homology and cohomology, and the Koszul dual algebra. A new cohomology construction on posets is introduced, and it is shown that the face semigroup algebra is isomorphic to the cohomology algebra when this construction is applied to the intersection lattice of the hyperplane arrangement.

1 Introduction

Let \mathcal{A} denote a finite collection of linear hyperplanes in \mathbb{R}^d . Then \mathcal{A} dissects \mathbb{R}^d into open subsets called *chambers*. The closures of the chambers are polyhedral cones whose relatively open faces are called the *faces* of the hyperplane arrangement \mathcal{A} . The set \mathcal{F} of faces of \mathcal{A} can be endowed with a semigroup structure. Geometrically, the product xy of faces x and y is the face entered by moving a small positive distance along a straight line from x towards y . The k -algebra spanned by the faces of \mathcal{A} with this multiplication is the *face semigroup algebra* of the hyperplane arrangement \mathcal{A} . Here k denotes a field.

The face semigroup algebra $k\mathcal{F}$ has enjoyed recent attention mainly due to two interesting results. The first result is that a large class of seemingly unrelated Markov chains can be studied in a unified setting via the semigroup structure on the faces of a hyperplane arrangement. The Markov chains are encoded as random walks on the chambers of a hyperplane arrangement [5]. A step in this random walk moves from a chamber to the product of a face with the chamber according to some probability distribution on the faces of the arrangement. This identification associates the transition matrix of the Markov chain with the matrix of a linear transformation on the face semigroup algebra of the hyperplane arrangement. Questions about the Markov chain can then be answered using algebraic techniques [10]. For example, a combinatorial description of the eigenvalues with multiplicities of the transition matrix is given and the transition matrix is shown to be diagonalizable.

The second interesting result concerns the descent algebra of a finite Coxeter group, a subalgebra of the group algebra of the Coxeter group. To any finite Coxeter group is associated a hyperplane arrangement and the Coxeter group acts on

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the faces of this arrangement. This gives an action of the Coxeter group on the face semigroup algebra of the arrangement. The subalgebra of elements invariant under the action of the Coxeter group is anti-isomorphic to the descent algebra of the Coxeter group [6, 10]. The descent algebra was introduced in [26] and the proof that it is indeed an algebra is rather involved. This approach via hyperplane arrangements provides a new and somewhat simpler setting for studying the descent algebra. See [25] and [23].

This article presents a study of the algebraic structure of the face semigroup algebra $k\mathcal{F}$ of an arbitrary central hyperplane arrangement in \mathbb{R}^d . Throughout this article k will denote a field of arbitrary characteristic and \mathcal{A} a finite collection of hyperplanes passing through the origin in \mathbb{R}^d . The *intersection lattice* of \mathcal{A} is the set \mathcal{L} of intersections of subsets of hyperplanes in \mathcal{A} ordered by inclusion. (Note that some authors order the intersection lattice by reverse inclusion rather than inclusion.)

The structure of the article is as follows. Sections 2 and 3 recall notions from the theory of posets and hyperplane arrangements, respectively. Section 4 defines the face semigroup algebra of a hyperplane arrangement and describes its irreducible representations. In Section 5 a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ is constructed. This leads to a description of the projective indecomposable $k\mathcal{F}$ -modules (Section 6) and a computation of the Cartan invariants of $k\mathcal{F}$ (see 6.4). The projective indecomposable modules are used to construct projective resolutions of the simple $k\mathcal{F}$ -modules in Section 7. The quiver with relations of $k\mathcal{F}$ is computed in Section 8. Section 9 proves that $k\mathcal{F}$ is a Koszul algebra and computes the Ext-algebra (or Koszul dual) of $k\mathcal{F}$. This is used in Section 9.4 to compute the Hochschild homology and cohomology of $k\mathcal{F}$. Section 10 explores connections with poset cohomology. A new cohomology construction is introduced, and it is shown that the cohomology algebra, with its cohomology cup product, is isomorphic to $k\mathcal{F}$. Finally, connections with the Whitney cohomology of the geometric lattice \mathcal{L}^* are explored.

2 Posets

This section collects some background from the theory of posets for the convenience of the reader. An excellent reference is [27, Chapter 3].

A *poset* is a finite set P together with a partial order \leq . The *opposite poset* P^* of a poset P is the set P with partial order defined by $x \leq y$ in P^* if and only if $x \geq y$ in P . For $x, y \in P$, write $x \triangleleft y$ and say y *covers* x or x is *covered by* y if $x < y$ and there does not exist $z \in P$ with $x < z < y$. The *Hasse diagram* of P is the graph with exactly one vertex for each x in P and exactly one edge between x and y if and only if $x \triangleleft y$ or $y \triangleleft x$. An edge of the Hasse diagram is called a *cover relation*.

A *chain* in P is a sequence of elements $x_0 < x_1 < \cdots < x_r$ in P . A chain $x_0 < x_1 < \cdots < x_r$ is *unrefinable* if $x_{i-1} \triangleleft x_i$ for all $1 \leq i \leq r$. The *length* of the chain $x_0 < x_1 < \cdots < x_r$ is r . The *length* or *rank* of a poset is the length of the longest chain in P . For $x \leq y$ in P , the *interval* between x and y is the set $[x, y] = \{z \in P \mid x \leq z \leq y\}$. The interval $[x, y]$ is a poset and its rank is denoted by $\ell([x, y])$.

A (finite) poset L is a *lattice* if every pair of elements x, y in L has a least upper bound (called *join*) $x \vee y$ and a greatest lower bound (called *meet*) $x \wedge y$ (with respect to the relation \leq). There exists an element $\hat{0}$ called the *bottom* of L satisfying $\hat{0} \leq x$

for all $x \in L$. Similarly, there exists an element $\hat{1}$ in L called the *top* of L satisfying $x \leq \hat{1}$ for all $x \in L$.

The *Möbius function* μ of a finite poset P is defined recursively by the equations

$$\mu(x, x) = 1 \quad \text{and} \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z),$$

for all $x < y$ in P . If $x \not\leq y$, then set $\mu(x, y) = 0$. The *Möbius inversion formula* [27, Section 3.7] states that $g(x) = \sum_{y \leq x} f(y)$ if and only if $f(x) = \sum_{y \leq x} g(y)\mu(y, x)$, where $f, g: P \rightarrow \mathbb{R}$.

3 Hyperplane Arrangements

This section recalls some background from the theory of hyperplane arrangements (see [21]).

3.1 Hyperplane Arrangements

A *hyperplane arrangement* \mathcal{A} in \mathbb{R}^d is a finite set of hyperplanes in \mathbb{R}^d . We restrict our attention to *central* hyperplane arrangements where all the hyperplanes contain $0 \in \mathbb{R}^d$. Each hyperplane $H \in \mathcal{A}$ determines two open half-spaces of \mathbb{R}^d denoted H^+ and H^- . The choice of which half-space to label $+$ or $-$ is arbitrary, but fixed.

3.2 The Face Poset

A *face* of \mathcal{A} is a nonempty intersection of the form $x = \bigcap_{H \in \mathcal{A}} H^{\sigma_H}$, where $\sigma_H \in \{+, -, 0\}$ and $H^0 = H$. The sequence $\sigma(x) = (\sigma_H)_{H \in \mathcal{A}}$ is the *sign sequence* of x . A *chamber* c is a face such that $\sigma_H(c) \neq 0$ for all $H \in \mathcal{A}$.

The *face poset* \mathcal{F} of \mathcal{A} is the set of faces of \mathcal{A} partially ordered by

$$x \leq y \iff \text{for each } H \in \mathcal{A} \text{ either } \sigma_H(x) = 0 \text{ or } \sigma_H(x) = \sigma_H(y).$$

Equivalently, $x \leq y \iff x \subset \bar{y}$. If $x \leq y$, then we say x is a *face* of y . Note that the chambers are the maximal elements in this partial order.

3.3 The Support Map and the Intersection Lattice

The *support* of a face $x \in \mathcal{F}$ is the the intersection of the hyperplanes in \mathcal{A} containing x ,

$$\text{supp}(x) = \bigcap_{\substack{H \in \mathcal{A} \\ \sigma_H(x)=0}} H.$$

The set $\mathcal{L} = \text{supp}(\mathcal{F})$ of supports of faces of \mathcal{A} is a graded lattice ordered by inclusion called the *intersection lattice* of \mathcal{A} . (Some authors order the intersection lattice by *reverse* inclusion, so some care is needed while reading the literature.) The rank of $X \in \mathcal{L}$ is the dimension of the subspace $X \subset \mathbb{R}^d$ if the intersection of all the hyperplanes in the arrangement is trivial. For $X, Y \in \mathcal{L}$ the *meet* $X \wedge Y$ of X and Y is

the intersection $X \cap Y$ and the join $X \vee Y$ of X and Y is $X + Y$, the smallest subspace of \mathbb{R}^d containing X and Y . The opposite poset \mathcal{L}^* of \mathcal{L} is a geometric lattice. The top element $\widehat{1}$ of \mathcal{L} is the ambient vector space \mathbb{R}^d and the bottom element $\widehat{0}$ is the intersection of all hyperplanes in the arrangement $\bigcap_{H \in \mathcal{A}} H$. The chambers are the faces of support $\widehat{1}$. Since $\text{supp}(x) \leq \text{supp}(y)$ if $x \leq y$, the support map $\text{supp}: \mathcal{F} \rightarrow \mathcal{L}$ is an order-preserving poset surjection.

3.4 Deletion and Restriction

Fix $X \in \mathcal{L}$. The faces y of \mathcal{A} with $\text{supp}(y) \leq X$ are the faces of the arrangement $\mathcal{A}_X = \{H \cap X \mid X \not\leq H \in \mathcal{A}\}$. \mathcal{A}_X is the restriction to X and the face poset of \mathcal{A}_X is denoted by $\mathcal{F}_{\leq X}$. The intersection lattice $\mathcal{L}_{\leq X}$ of \mathcal{A}_X is the interval $[\widehat{0}, X]$ of \mathcal{L} .

Given $X \in \mathcal{L}$ let $\mathcal{A}^X = \{H \in \mathcal{A} \mid X \subset H\}$ denote the set of hyperplanes in \mathcal{A} containing X . \mathcal{A}^X is a deletion of \mathcal{A} . If $x \in \mathcal{F}$ with $\text{supp}(x) = X$, then the face poset \mathcal{F}^X of \mathcal{A}^X is isomorphic to the subposet of \mathcal{F} of all faces having x as a face: $\mathcal{F}^X \cong \{y \in \mathcal{F} \mid x \leq y\}$. The intersection lattice of \mathcal{A}^X is the interval $[X, \widehat{1}] \subset \mathcal{L}$.

4 The Face Semigroup Algebra

This section recalls the semigroup structure on the faces of a hyperplane arrangement and the irreducible representations of the resulting semigroup algebra. See [10] for details.

4.1 The Face Semigroup

For $x, y \in \mathcal{F}$ the product xy is the face of \mathcal{A} with sign sequence

$$\sigma_H(xy) = \begin{cases} \sigma_H(x) & \text{if } \sigma_H(x) \neq 0, \\ \sigma_H(y) & \text{if } \sigma_H(x) = 0. \end{cases}$$

This product is associative and noncommutative, with identity element the intersection of all the hyperplanes in the arrangement $1 = \bigcap_{H \in \mathcal{A}} H$. Note that the support of the identity element 1 is $\widehat{0}$ (and not $\widehat{1}$). The support map $\text{supp}: \mathcal{F} \rightarrow \mathcal{L}$ satisfies $\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y)$ for all $x, y \in \mathcal{F}$. Therefore supp is a semigroup surjection, where \mathcal{L} is considered a semigroup with product given by join \vee , as well as an ordering-preserving poset surjection.

Remark 4.1. There is a nice geometric interpretation of this product. The face xy is the face that one enters by moving a small positive distance along any straight line from x to y .

Proposition 4.2 For all $x, y \in \mathcal{F}$,

- (1) $x^2 = x$,
- (2) $xyx = xy$,
- (3) $xy = y$ if and only if $x \leq y$,

- (4) $xy = x$ if and only if $\text{supp}(y) \leq \text{supp}(x)$,
- (5) $\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y)$,

Remark 4.3. Conditions (1) and (2) of the proposition say that \mathcal{F} is a *left regular band*.

4.2 The Face Semigroup Algebra

The *face semigroup algebra* of \mathcal{A} with coefficients in the field k is the semigroup algebra $k\mathcal{F}$ of the face semigroup \mathcal{F} of \mathcal{A} . Explicitly, it consists of linear combinations of elements of \mathcal{F} with multiplication induced by the product of \mathcal{F} . The face semigroup algebra $k\mathcal{F}$ is a finite dimensional associative algebra with identity $1 = \bigcap_{H \in \mathcal{A}} H$.

Unless explicitly stated otherwise, no assumptions will be made on the characteristic of the field k .

4.3 Irreducible Representations

This section summarizes [10, Section 7.2] constructing the irreducible representations of $k\mathcal{F}$.

Since \mathcal{F} and \mathcal{L} are semigroups, the support map $\text{supp}: \mathcal{F} \rightarrow \mathcal{L}$ extends linearly to a surjection of algebras $\text{supp}: k\mathcal{F} \rightarrow k\mathcal{L}$. The kernel of this map is nilpotent, and the semigroup algebra $k\mathcal{L}$ is isomorphic to a product of copies of the field k , one copy for each element of \mathcal{L} . This implies that $\ker(\text{supp})$ is the Jacobson radical of $k\mathcal{F}$ and that the irreducible representations of $k\mathcal{F}$ are given by the components of the composition $k\mathcal{F} \xrightarrow{\text{supp}} k\mathcal{L} \xrightarrow{\cong} \prod_{X \in \mathcal{L}} k$. This last map sends $X \in \mathcal{L}$ to the vector with 1 in the X -position if $Y \geq X$ and 0 otherwise. The X -component of this surjection is the map $\chi_X: k\mathcal{F} \rightarrow k$ defined on the faces $y \in \mathcal{F}$ by

$$\chi_X(y) = \begin{cases} 1 & \text{if } \text{supp}(y) \leq X, \\ 0 & \text{otherwise.} \end{cases}$$

The elements

$$(4.1) \quad E_X = \sum_{Y \geq X} \mu(X, Y)Y,$$

one for each $X \in \mathcal{L}$, correspond to the standard basis vectors of $\prod_{X \in \mathcal{L}} k$ under the isomorphism $k\mathcal{L} \cong \prod_{X \in \mathcal{L}} k$ above. They form a basis of $k\mathcal{L}$ and also form a *complete system of primitive orthogonal idempotents* (see Section 5).

5 Primitive Idempotents

Let A be a k -algebra. An element $e \in A$ is *idempotent* if $e^2 = e$. It is a *primitive idempotent* if e is idempotent and we cannot write $e = e_1 + e_2$ where e_1 and e_2 are nonzero idempotents in A with $e_1e_2 = 0 = e_2e_1$. Equivalently, e is primitive if and only if Ae is an indecomposable A -module. A set of elements $\{e_i\}_{i \in I} \subset A$ is a *complete system of primitive orthogonal idempotents* for A if e_i is a primitive idempotent for

every i , if $e_i e_j = 0$ for $i \neq j$, and if $\sum_i e_i = 1$. If $\{e_i\}_{i \in I}$ is a complete system of primitive orthogonal idempotents for A , then $A \cong \bigoplus_{i \in I} Ae_i$ as left A -modules and $A \cong \bigoplus_{i,j \in I} e_i A e_j$ as k -vector spaces.

5.1 Complete System of Primitive Orthogonal Idempotents

For each $X \in \mathcal{L}$, fix an $x \in \mathcal{F}$ with $\text{supp}(x) = X$ and define elements in $k\mathcal{F}$ recursively by the formula,

$$(5.1) \quad e_X = x - \sum_{Y>X} x e_Y.$$

Note that $e_{\hat{1}}$ is an arbitrarily chosen chamber.

Lemma 5.1 *Let $w \in \mathcal{F}$ and $X \in \mathcal{L}$. If $\text{supp}(w) \not\leq X$, then $w e_X = 0$.*

Proof We proceed by induction on X . This is vacuously true if $X = \hat{1}$. Suppose the result holds for all $Y \in \mathcal{L}$ with $Y > X$. Suppose $w \in \mathcal{F}$ and $W = \text{supp}(w) \not\leq X$. Using the definition of e_X and the identity $w x w = w x$ (Proposition 4.2 (2)),

$$w e_X = w x - \sum_{Y>X} w x e_Y = w x - \sum_{Y>X} w x (w e_Y).$$

By induction, $w e_Y = 0$ if $W \not\leq Y$. Therefore, the summation runs over Y with $W \leq Y$. But $Y > X$ and $Y \geq W$ if and only if $Y \geq W \vee X$, so the summation runs over Y with $Y \geq W \vee X$.

$$w e_X = w x - \sum_{Y>X} w x (w e_Y) = w x - \sum_{Y \geq X \vee W} w x e_Y.$$

Now let z be the element of support $X \vee W$ chosen in defining $e_{X \vee W}$. So $e_{X \vee W} = z - \sum_{Y>X \vee W} z e_Y$. Note that $z e_{X \vee W} = e_{X \vee W}$ since $z = z^2$. Therefore, $z = \sum_{Y \geq X \vee W} z e_Y$. Since $\text{supp}(w x) = W \vee X = \text{supp}(z)$, it follows from Proposition 4.2 (4) that $w x = w x z$. Combining the last two statements,

$$w e_X = w x - \sum_{Y \geq X \vee W} w x e_Y = w x \left(z - \sum_{Y \geq X \vee W} z e_Y \right) = 0. \quad \blacksquare$$

Theorem 5.2 *The elements $\{e_X\}_{X \in \mathcal{L}}$ form a complete system of primitive orthogonal idempotents in $k\mathcal{F}$.*

Proof Complete. $1 = \bigcap_{H \in \mathcal{A}} H$ is the only element of support $\hat{0}$. Hence, $e_{\hat{0}} = 1 - \sum_{Y>\hat{0}} e_Y$. Equivalently, $1 = \sum_{Y \in \mathcal{L}} e_Y$.

Idempotent. Since e_Y is a linear combination of elements of support at least Y , $e_Y z = e_Y$ for any z with $\text{supp}(z) \leq Y$ (Proposition 4.2 (4)). Using the definition of e_X , the facts $e_X = x e_X$ and $e_Y = e_Y y$, and Lemma 5.1,

$$e_X^2 = \left(x - \sum_{Y>X} x e_Y \right) e_X = x e_X - \sum_{Y>X} x e_Y (y e_X) = x e_X = e_X.$$

Orthogonal. We show that for every $X \in \mathcal{L}$, $e_X e_Y = 0$ for $Y \neq X$. If $X = \widehat{1}$, then $e_X e_Y = e_X x e_Y = 0$ for every $Y \neq X$ by Lemma 5.1 since $X = \widehat{1}$ implies $X \not\leq Y$. Now suppose the result holds for $Z > X$. That is, $e_Z e_Y = 0$ for all $Y \neq Z$. If $X \not\leq Y$, then $e_X e_Y = 0$ by Lemma 5.1. If $X < Y$, then

$$e_X e_Y = x e_Y - \sum_{Z > X} x(e_Z e_Y) = x e_Y - x e_Y^2 = 0.$$

Primitive. We will show that e_X lifts $E_X = \sum_{Y \geq X} \mu(X, Y)Y$ (see equation (4.1)) for all $X \in \mathcal{L}$, a primitive idempotent in $k\mathcal{L}$. If $X = \widehat{1}$, then $\text{supp}(e_{\widehat{1}}) = \widehat{1} = E_{\widehat{1}}$. Suppose the result holds for $Y > X$. Then $\text{supp}(e_X) = \text{supp}(x - \sum_{Y > X} x e_Y) = X - \sum_{Y > X} (X \vee E_Y)$. Since E_Y is a linear combination of elements $Z \geq Y$, it follows that $X \vee E_Y = E_Y$ if $Y > X$. Therefore, $\text{supp}(e_X) = X - \sum_{Y > X} E_Y$. The Möbius inversion formula applied to $E_X = \sum_{Y \geq X} \mu(X, Y)Y$ gives $X = \sum_{Y \geq X} E_X$. Hence, $\text{supp}(e_X) = X - \sum_{Y > X} E_Y = E_X$.

To see that this is sufficient, suppose E is a primitive idempotent in $k\mathcal{L}$ and that e is an idempotent lifting E . Suppose $e = e_1 + e_2$ with e_i orthogonal idempotents. Then $E = \text{supp}(e) = \text{supp}(e_1) + \text{supp}(e_2)$. Since E is primitive and $\text{supp}(e_1)$ and $\text{supp}(e_2)$ are orthogonal idempotents, $\text{supp}(e_1) = 0$ or $\text{supp}(e_2) = 0$. Say $\text{supp}(e_1) = 0$. Then e_1 is in the kernel of supp . This kernel is nilpotent, so $e_1^n = 0$ for some $n \geq 0$. Hence $e_1 = e_1^n = 0$. Therefore, e is a primitive idempotent. ■

Remark 5.3. We can replace $x \in \mathcal{F}$ in equation (5.1) with any linear combination $\tilde{x} = \sum_{\text{supp}(x)=X} \lambda_x x$ of elements of support X whose coefficients λ_x sum to 1. The proofs still hold since the element \tilde{x} is idempotent and satisfies $\text{supp}(\tilde{x}) = X$ and $\tilde{x}y = \tilde{x}$ if $\text{supp}(y) \leq X$. Unless explicitly stated we will use the idempotents constructed above.

5.2 A Basis of Primitive Idempotents

Proposition 5.4 *The set $\{x e_{\text{supp}(x)} \mid x \in \mathcal{F}\}$ is a basis of $k\mathcal{F}$ of primitive idempotents.*

Proof Let $y \in \mathcal{F}$. Then by Corollary 5.2 and Lemma 5.1,

$$y = y1 = y \sum_X e_X = \sum_{X \geq \text{supp}(y)} y e_X = \sum_{X \geq \text{supp}(y)} (yx) e_X.$$

Since $\text{supp}(yx) = \text{supp}(y) \vee \text{supp}(x) = X$, the face y is a linear combination of the elements of the form $x e_{\text{supp}(x)}$. So these elements span $k\mathcal{F}$. Since the number of these elements is the cardinality of \mathcal{F} , which is the dimension of $k\mathcal{F}$, the set forms a basis of $k\mathcal{F}$. The elements are idempotent since $(x e_X)^2 = (x e_X)(x e_X) = x e_X^2 = x e_X$ (since $x y x = x y$ for all $x, y \in \mathcal{F}$). Since $x e_X$ also lifts the primitive idempotent $E_X = \sum_{Y \geq X} \mu(X, Y)Y \in k\mathcal{L}$, it is also a primitive idempotent (see the end of the proof of Corollary 5.2). ■

6 Projective Indecomposable Modules

This section describes the projective indecomposable $k\mathcal{F}$ -modules and computes the Cartan invariants of $k\mathcal{F}$.

6.1 Projective Indecomposable Modules

For $X \in \mathcal{L}$, let $\mathcal{F}_X \subset \mathcal{F}$ denote the set of faces of support X . For $y \in \mathcal{F}$ and $x \in \mathcal{F}_X$ let

$$y \cdot x = \begin{cases} yx & \text{supp}(y) \leq \text{supp}(x), \\ 0 & \text{supp}(y) \not\leq \text{supp}(x). \end{cases}$$

Then $k\mathcal{F}_X$ is a $k\mathcal{F}$ -module.

Lemma 6.1 *Let $X \in \mathcal{L}$. Then $\{ye_X \mid \text{supp}(y) = X\}$ is a basis for $k\mathcal{F}e_X$.*

Proof Suppose $\sum_{w \in \mathcal{F}} \lambda_w we_X \in k\mathcal{F}e_X$. If $\text{supp}(w) \not\leq X$, then $we_X = 0$. So suppose $\text{supp}(w) \leq X$. Then $\text{supp}(wx) = \text{supp}(w) \vee X = X$. Therefore,

$$\sum_{w \in \mathcal{F}} \lambda_w we_X = \sum_{w \in \mathcal{F}} \lambda_w (wx)e_X \in \text{span}_k\{ye_X \mid \text{supp}(y) = X\},$$

where x is the element chosen in the construction of e_X (recall that $e_X = xe_X$ since $x^2 = x$). So the elements span $k\mathcal{F}e_X$. These elements are linearly independent, being a subset of a basis of $k\mathcal{F}$ (Proposition 5.4). ■

Proposition 6.2 *The $k\mathcal{F}$ -modules $k\mathcal{F}_X$ are all the projective indecomposable $k\mathcal{F}$ -modules. The radical of $k\mathcal{F}_X$ is $\text{span}_k\{y - y' \mid y, y' \in \mathcal{F}_X\}$.*

Proof Define a map $\phi: k\mathcal{F}_X \rightarrow k\mathcal{F}e_X$ by $w \mapsto we_X$. Then ϕ is surjective since $\phi(y) = ye_X$ for $y \in \mathcal{F}_X$ and since $\{ye_X \mid \text{supp}(y) = X\}$ is a basis for $k\mathcal{F}e_X$ (Lemma 6.1). Since $\dim k\mathcal{F}_X = \#\mathcal{F}_X = \dim k\mathcal{F}e_X$, the map ϕ is an isomorphism of k -vector spaces.

ϕ is a $k\mathcal{F}$ -module map. Let $y \in \mathcal{F}$ and let $x \in \mathcal{F}_X$. If $\text{supp}(y) \leq X$, then $\phi(y \cdot x) = \phi(yx) = yxe_X = y\phi(x)$. If $\text{supp}(y) \not\leq X$, then $y \cdot x = 0$. Hence, $\phi(x \cdot y) = 0$. Also, since $\text{supp}(y) \not\leq X$, it follows that $ye_X = 0$. Therefore, $y\phi(x) = yxe_X = yx(ye_X) = yx0 = 0$. So $\phi(y \cdot x) = y\phi(x)$. Hence ϕ is an isomorphism of $k\mathcal{F}$ -modules. Since $k\mathcal{F}e_X$ are all the projective indecomposable $k\mathcal{F}$ -modules, so are the $k\mathcal{F}_X$. ■

6.2 Cartan Invariants

Let $\{e_X\}_{X \in I}$ be a complete system of primitive orthogonal idempotents for a finite dimensional k -algebra A . The *Cartan invariants* of A are defined to be the numbers

$$c_{X,Y} = \dim \text{Hom}_A(Ae_X, Ae_Y),$$

where $X, Y \in I$. The invariant $c_{X,Y}$ is the multiplicity of the simple module $S_X = (A/\text{rad}A)e_X$ as a composition factor of the left A -module Ae_Y . The *Cartan matrix* of A is the matrix $[c_{X,Y}]$.

The following is Theorem 1.7.3 of [4].

Theorem 6.3 (Idempotent Refinement Theorem) *Let N be a nilpotent ideal in a ring R and let e be an idempotent in R/N . Then any two idempotents in R lifting e are conjugate in R .*

Proposition 6.4 For $X, Y \in \mathcal{L}$,

$$\dim_k \text{Hom}_{k\mathcal{F}}(k\mathcal{F}e_X, k\mathcal{F}e_Y) = |\mu(X, Y)|,$$

where μ is the Möbius function of \mathcal{L} . Therefore the Cartan invariants of $k\mathcal{F}$ are $c_{X,Y} = |\mu(X, Y)|$ and the Cartan matrix is triangular of determinant 1.

Proof Since $\text{Hom}_{k\mathcal{F}}(k\mathcal{F}e_X, k\mathcal{F}e_Y) \cong e_X k\mathcal{F}e_Y$, it follows that $c_{X,Y} = \dim e_X k\mathcal{F}e_Y$. We will use Zaslavsky’s Theorem [30]: The number of chambers in a hyperplane arrangement is $\sum_{X \in \mathcal{L}} |\mu(X, \mathbb{R}^d)|$.

For each $W \in \mathcal{L}$, let w denote an element of support W . If $W \geq X$, then $\text{supp}(xw) = W$, so replace w with xw and construct idempotents e_W as in Subsection 5.1. (By the Idempotent Refinement Theorem above, it does not matter which lifts of the idempotents in $k\mathcal{L}$ we use to compute the Cartan invariants: $e_X k\mathcal{F}e_Y \cong \tilde{e}_X k\mathcal{F}\tilde{e}_Y$ if e_X and \tilde{e}_X are conjugate and if e_Y and \tilde{e}_Y are conjugate.) Then for each $W \geq X$ we have $xe_W = e_W$, so $x = x \sum_W e_W = x \sum_{W \geq X} e_W = \sum_{W \geq X} e_W$. This gives the equality

$$(6.1) \quad k(x\mathcal{F}) = xk\mathcal{F} = \sum_{W \geq X} e_W k\mathcal{F}.$$

Note that $x\mathcal{F}$ is the face poset of the hyperplane arrangement $\mathcal{A}^X = \{H \in \mathcal{A} \mid X \subset H\}$ and that the faces of support Y in \mathcal{A}^X are the chambers in the restricted arrangement $(\mathcal{A}^X)_Y$ (see Section 3.4). Zaslavsky’s Theorem applied to $(\mathcal{A}^X)_Y$ gives the number of faces of support Y in \mathcal{A}^X is $\sum_{W \in [X, Y]} |\mu(W, Y)|$ since the intersection lattice of $(\mathcal{A}^X)_Y$ is the interval $[X, Y]$ in \mathcal{L} . But the number of faces of support Y in $(\mathcal{A}^X)_Y$ is the cardinality of the set $x\mathcal{F}_Y$, which is the dimension of

$$k(x\mathcal{F}_Y) \cong xk\mathcal{F}_Y \cong xk\mathcal{F}e_Y \cong \bigoplus_{X \leq W \leq Y} e_W k\mathcal{F}e_Y$$

by (6.1) and Lemma 5.1. Therefore for each $X, Y \in \mathcal{L}$,

$$\sum_{X \leq W \leq Y} \dim e_W k\mathcal{F}e_Y = \sum_{X \leq W \leq Y} |\mu(W, Y)|.$$

The result now follows by induction. If $X = Y$, then $\dim e_X k\mathcal{F}e_X = |\mu(X, X)|$. Suppose the result holds for all W with $X < W \leq Y$. Then

$$\begin{aligned} \dim e_X k\mathcal{F}e_Y &= \sum_{X \leq W \leq Y} |\mu(W, Y)| - \sum_{X < W \leq Y} \dim e_W k\mathcal{F}e_Y \\ &= \sum_{X \leq W \leq Y} |\mu(W, Y)| - \sum_{X < W \leq Y} |\mu(W, Y)| \\ &= |\mu(X, Y)|. \quad \blacksquare \end{aligned}$$

7 Projective Resolutions of the Simple Modules

7.1 A Projective Resolution of the Simple Module Corresponding to $\widehat{1}$

In Section 5C of [11], an exact sequence of $k\mathcal{F}$ -modules is constructed to compute the multiplicities of the eigenvalues of random walks on the chambers of a hyperplane arrangement. This construction in combination with the above description of the projective indecomposable $k\mathcal{F}$ -modules yields a projective resolution of the simple $k\mathcal{F}$ -modules.

Let $\mathcal{F}_p \subset \mathcal{F}$ denote the set of faces of codimension p . For $x \in \mathcal{F}$ and $y \in \mathcal{F}_p$, let

$$x \cdot y = \begin{cases} xy & \text{supp}(x) \leq \text{supp}(y), \\ 0 & \text{supp}(x) \not\leq \text{supp}(y). \end{cases}$$

Fix an orientation ϵ_X for every subspace $X \in \mathcal{L}$. If x is a codimension one face of y , then pick a positively oriented basis $\{e_1, \dots, e_i\}$ of $X = \text{supp}(x)$ and a vector v in y and put $[x:y] = \epsilon_Y(e_1, \dots, e_i, v)$, where $Y = \text{supp}(y)$. Since X is a codimension one subspace of Y , the mapping $v \mapsto \epsilon_Y(e_1, \dots, e_i, v)$ is constant on the open halfspaces of Y determined by X . This implies the identity,

$$(7.1) \quad [x:y] = [\tilde{x}:\tilde{x}y], \text{ if } \text{supp}(\tilde{x}) = \text{supp}(x).$$

Lemma 7.1 ([11, Section 5, Lemma 2]) *Let $x, y \in \mathcal{F}$ with x of codimension two in y . Then there are exactly two faces w and z in the open interval (x, y) and we have*

$$[x:w][w:y] = -[x:z][z:y].$$

Proposition 7.2 ([11, Section 5, Lemma 4]) *The following is an exact sequence of $k\mathcal{F}$ -modules.*

$$\dots \longrightarrow k\mathcal{F}_p \xrightarrow{\partial_p} \dots \longrightarrow k\mathcal{F}_1 \xrightarrow{\partial_1} k\mathcal{F}_0 \xrightarrow{\partial_0} k \longrightarrow 0,$$

where the action of $k\mathcal{F}$ on k is given by $w \cdot \lambda = \lambda$ for all $w \in \mathcal{F}$ and $\lambda \in k$. The differential ∂_i is given by $\partial_0(c) = 1$ for all $c \in \mathcal{F}_0$ and for $x \in \mathcal{F}_p$,

$$\partial_p(x) = \sum_{y \gg x} [x:y]y.$$

Sketch of the proof It is easy to check that the complex consists of $k\mathcal{F}$ -modules and that ∂_i is a $k\mathcal{F}$ -module map. It remains to explain why the complex is exact. Suppose that the intersection of all the hyperplanes is a point; otherwise quotient out by that subspace. Intersecting the hyperplane arrangement with a sphere centered at the origin induces a regular cell decomposition Σ of the $(d - 1)$ -sphere whose cells correspond to the faces $x \neq 1$ of \mathcal{A} . The dual of Σ is the boundary of a polytope (a zonotope, actually) Z . Therefore, the poset of nonempty faces of Z is anti-isomorphic to the face poset \mathcal{F} of \mathcal{A} . Since Z is contractible, any augmented cellular chain complex will be an exact sequence of k -vector spaces. The above complex is precisely the augmented cellular chain complex with incidence numbers given by $[x:y]$. (See [14].) Therefore, it is exact. ■

Note that $k\mathcal{F}_p \cong \bigoplus_{\text{codim}(X)=p} k\mathcal{F}_X$ as $k\mathcal{F}$ -modules and that $k\mathcal{F}_X$ is projective by Proposition 6.2, where $\text{codim}(X)$ is the codimension of the subspace X . So the $k\mathcal{F}$ -modules $k\mathcal{F}_p$ are projective. Also note that in order for ∂_0 to be a $k\mathcal{F}$ -module morphism, the action of $k\mathcal{F}$ on k must be given by $\chi_{\hat{\gamma}}$. That is, k is the simple module afforded by the irreducible representation $\chi_{\hat{\gamma}}$. This proves the following result.

Corollary 7.3 *The exact sequence*

$$\dots \longrightarrow k\mathcal{F}_p \xrightarrow{\partial_p} \dots \longrightarrow k\mathcal{F}_1 \xrightarrow{\partial_1} k\mathcal{F}_0 \xrightarrow{\partial_0} k \longrightarrow 0$$

is a projective resolution of the simple $k\mathcal{F}$ -module afforded by the irreducible representation $\chi_{\hat{\gamma}}: k\mathcal{F} \rightarrow k$.

7.2 Projective Resolutions of the Simple Modules

Recall that the simple $k\mathcal{F}$ -modules are indexed by $X \in \mathcal{L}$, afforded by the representations $\chi_X: k\mathcal{F} \rightarrow k$,

$$\chi_X(y) = \begin{cases} 1 & \text{if } \text{supp}(y) \leq X, \\ 0 & \text{otherwise.} \end{cases}$$

Also recall that $\mathcal{F}_{\leq X}$ denotes the face semigroup of \mathcal{A}_X , consisting of the set of faces in \mathcal{F} of support contained in X (Subsection 3.4). Let $(\mathcal{F}_{\leq X})_p$ denote the set of faces in \mathcal{A}_X of codimension p in X . Applying the previous result to the hyperplane arrangement \mathcal{A}_X gives a projective resolution

$$\dots \longrightarrow k(\mathcal{F}_{\leq X})_p \xrightarrow{\partial} \dots \xrightarrow{\partial} k(\mathcal{F}_{\leq X})_1 \xrightarrow{\partial} k\mathcal{F}_X \longrightarrow k_X \longrightarrow 0$$

of the simple $k\mathcal{F}_{\leq X}$ -module k_X with action given by $w \cdot \lambda = \lambda$ for all $w \in \mathcal{F}_{\leq X}$ and $\lambda \in k$. The algebra surjection $k\mathcal{F} \rightarrow k\mathcal{F}_{\leq X}$ given by $w \mapsto \chi_X(w)w$ for $w \in \mathcal{F}$ puts a $k\mathcal{F}$ -module structure on each $k(\mathcal{F}_{\leq X})_p$ and on k . The $k\mathcal{F}$ -module structure on k is precisely that given by $\chi_X: k\mathcal{F} \rightarrow k$. Each $k(\mathcal{F}_{\leq X})_p$ is a projective $k\mathcal{F}$ -module since the $k\mathcal{F}$ -module structure on $k(\mathcal{F}_{\leq X})_p$ decomposes as

$$k(\mathcal{F}_{\leq X})_p \cong \bigoplus_{\substack{Y \leq X, \\ \text{codim}_X(Y)=p}} k\mathcal{F}_Y,$$

where $\text{codim}_X(Y)$ denotes the codimension of Y in X . This establishes the following.

Proposition 7.4 *Let $X \in \mathcal{L}$. Then*

$$\dots \longrightarrow \left(\bigoplus_{\substack{Y \in \mathcal{L} \\ \text{codim}_X(Y)=p}} k\mathcal{F}_Y \right) \xrightarrow{\partial} \dots \xrightarrow{\partial} k\mathcal{F}_X \longrightarrow k_X \longrightarrow 0$$

is a projective resolution of the simple $k\mathcal{F}$ -module k_X afforded by $\chi_X: k\mathcal{F} \rightarrow k$, where $\partial(w) = \sum_{y \succ w} [w:y] \chi_X(y)y$ and $\text{codim}_X(Y)$ denotes the codimension of Y in X .

8 The Quiver of the Face Semigroup Algebra

8.1 The Quiver of a Split Basic Algebra

A finite dimensional k -algebra A is a (split) basic algebra if every simple module of A has dimension one. The Ext-quiver or just quiver Q of a split basic algebra A is a directed graph with one vertex for each isomorphism class of simple modules of A . The number of arrows $x \rightarrow y$ is $\dim \text{Ext}_A^1(S_x, S_y)$, where S_x and S_y are simple modules corresponding to the vertices x and y .

A path p in Q is a sequence of arrows $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_r$. The path starts at $s(p) = x_0$ and terminates at $t(p) = x_r$. The length of p is r . Two paths p and q are parallel if they start and terminate at the same vertices: $s(p) = s(q)$ and $t(p) = t(q)$. The path algebra kQ of a quiver Q is the k -vector space spanned by the paths in Q with the product of two paths defined by path composition: if $p = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_r$ and $q = y_0 \rightarrow y_1 \rightarrow \dots \rightarrow y_s$, then

$$p \cdot q = \begin{cases} y_0 \rightarrow \dots \rightarrow y_s \rightarrow x_1 \rightarrow \dots \rightarrow x_r & \text{if } x_0 = s(p) = t(q) = y_s, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P \subset kQ$ be the ideal of kQ generated by the arrows of Q . An ideal $I \subset kQ$ is admissible if $P^r \subset I \subset P^2$, for some $r \geq 2$.

Proposition 8.1 ([1, Section III.1, Theorem 1.9]) *Let A be a finite-dimensional, split, basic k -algebra with quiver Q . Then $A \cong kQ/I$, where I is an admissible ideal of kQ .*

Let I be an admissible ideal of kQ . An element of I is a relation from x to y if it is a k -linear combination of paths in Q beginning at a vertex x and ending at a vertex y . Note that any element $\rho \in I$ can be written as a linear combination of relations since $x\rho y$ is a relation for any pair of vertices $x, y \in Q$. The following result combines Corollary 1.1 and Proposition 1.2 of [9].

Proposition 8.2 *Let Q be a quiver with no oriented cycles and let I be an admissible ideal. Suppose that R is a minimal set of relations generating I as a two-sided ideal of kQ . Then the number of relations from x to y in R is the dimension of the k -vector space $\text{Ext}_{kQ/I}^2(S_x, S_y)$.*

8.2 The Quiver of the Face Semigroup Algebra

Since every simple $k\mathcal{F}$ -module is of dimension one, $k\mathcal{F}$ is a split basic algebra. This section computes the quiver \mathcal{Q} of $k\mathcal{F}$, and the next section describes an ideal I such that $k\mathcal{Q}/I \cong k\mathcal{F}$.

Lemma 8.3 *For $X, Y \in \mathcal{L}$ and $p \geq 0$,*

$$\text{Ext}_{k\mathcal{F}}^p(k_X, k_Y) \cong \begin{cases} k & \text{if } Y \leq X \text{ and } \dim(X) - \dim(Y) = p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let $\text{codim}_X(W)$ denote the codimension of W in X and let C_p denote $\bigoplus_{\text{codim}_X(W)=p} k\mathcal{F}_W$. Applying the functor $\text{Hom}(-, k_Y)$ to the projective resolution of k_X in Proposition 7.4, gives the cocomplex

$$\cdots \xrightarrow{\partial_p^*} \text{Hom}_{k\mathcal{F}}(C_p, k_Y) \xrightarrow{\partial_{p+1}^*} \text{Hom}_{k\mathcal{F}}(C_{p+1}, k_Y) \xrightarrow{\partial_{p+2}^*} \cdots$$

Now $\text{Hom}_{k\mathcal{F}}(C_p, k_Y) \cong \bigoplus_{\text{codim}_X(W)=p} \text{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y)$ and

$$\text{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y) \cong \text{Hom}_{k\mathcal{F}}(k\mathcal{F}e_W, k_Y) \cong e_W \cdot k_Y = \begin{cases} k & \text{if } W = Y, \\ 0 & \text{otherwise,} \end{cases}$$

where we used the fact that $\chi_Y(e_W) = 0$ if $W \neq Y$ and 1 otherwise. (If $W \neq Y$, then $\chi_Y(e_Y) = 1$ implies $\chi_Y(e_W) = \chi_Y(e_W)\chi_Y(e_Y) = \chi_Y(e_W e_Y) = 0$.)

Since $\text{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y)$ vanishes unless $W = Y$, the entries in the above cocomplex vanish in all degrees except for that in which $k\mathcal{F}_Y$ appears. This degree is precisely $\text{codim}_X(Y) = \dim(X) - \dim(Y)$, in which case $\text{Hom}_{k\mathcal{F}}(k\mathcal{F}_Y, k_Y) \cong k$. ■

Corollary 8.4 *The quiver \mathcal{Q} of $k\mathcal{F}$ is given by the Hasse diagram of the intersection lattice \mathcal{L} . The cover relations are oriented by $X \rightarrow Y \iff X \succ Y$.*

Proof The vertices of \mathcal{Q} are in one-to-one correspondence with the isomorphism classes of simple $k\mathcal{F}$ -modules. These are indexed by the elements of \mathcal{L} . The number of arrows $X \rightarrow Y$ is

$$\dim \text{Ext}_{k\mathcal{F}}^1(k_X, k_Y) = \begin{cases} 1 & \text{if } X \succ Y, \\ 0 & \text{otherwise.} \end{cases} \quad \blacksquare$$

8.3 Quiver Relations

This section defines a k -algebra surjection $\varphi: k\mathcal{Q} \rightarrow k\mathcal{F}$ and identifies a minimal generating set of the kernel. The kernel is an admissible ideal of the path algebra $k\mathcal{Q}$, so this generating set gives the quiver relations.

8.3.1 First Version

Let $\partial: k\mathcal{F} \rightarrow k\mathcal{F}$ be the map

$$\partial(y) = \sum_{\substack{x \in k\mathcal{F} \\ x \succ y}} [y : x]x,$$

where $[y : x]$ is the incidence number defined in equation (7.1). Define a k -algebra morphism $\varphi: k\mathcal{Q} \rightarrow k\mathcal{F}$ by

$$\begin{aligned} \varphi(X) &= e_X \text{ for } X \in \mathcal{Q}_0, & \varphi(X \rightarrow Y) &= e_Y \partial(y) e_X, \\ \varphi(X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_r) &= \varphi(X_{r-1} \rightarrow X_r) \cdots \varphi(X_0 \rightarrow X_1), \end{aligned}$$

where y was chosen in the construction of e_Y . (Actually, y can be any element of support Y . This follows from the identity $xx' = x$ if and only if $\text{supp}(x) \geq \text{supp}(x')$.) Using Lemma 5.1 and that $e_Y = y - \sum_{Z \supset Y} ye_Z$, it follows that $e_Y \partial(y) e_X = ([y:x_1]x_1 + [y:x_2]x_2)e_X$ where x_1 and x_2 are the two faces of support X with common codimension one face y . In particular, this is nonzero.

Proposition 8.5 *Let $\varphi: k\mathcal{Q} \rightarrow k\mathcal{F}$ be the map defined above. For each interval $[Z, X]$ of length two in \mathcal{L} , the sum of all paths of length two from X to Z*

$$\sum_{Y \in (Z, X)} (X \rightarrow Y \rightarrow Z)$$

is an element of the kernel of φ . These elements form a minimal generating set of relations for the kernel of φ .

Proof If R is a minimal set of relations generating $\ker \varphi$, then Proposition 8.2 gives that the number of elements of $Z.R.X$ (the number of relations in R starting at X and ending at Z) is $\dim \text{Ext}_{k\mathcal{G}}^2(k_X, k_Z)$. This is 1 if $[Z, X]$ is an interval of length two and 0 otherwise. Therefore, we need only one relation for each interval of length two in \mathcal{L} .

Let z be the element of support Z chosen in the construction of e_Z . Then $\sum_{Y \in (Z, X)} \varphi(X \rightarrow Y \rightarrow Z)$ is a linear combination of elements of the form $\tilde{x}e_X$ with \tilde{x} of support X having z as a face. If \tilde{x} has z as a face, then z is of codimension two in \tilde{x} . Lemma 7.1 gives that \tilde{x} has exactly two codimension one faces \tilde{y} and \tilde{w} . Since

$$\begin{aligned} \varphi(\text{supp}(\tilde{y}) \rightarrow Z)\varphi(X \rightarrow \text{supp}(\tilde{y})) = \\ ([z:\tilde{y}]\tilde{y} + [z:y']y')([y:x_1]x_1 + [y:x_2]x_2)e_X \end{aligned}$$

and one of $\tilde{y}x_1$ or $\tilde{y}x_2$ must be \tilde{x} — suppose $\tilde{y}x_1 = \tilde{x}$ — we see that $\tilde{x}e_X$ appears in $\varphi(X \rightarrow \text{supp}(\tilde{y}) \rightarrow Z)$ with coefficient $[z:\tilde{y}][y:x_1]$. The identity (7.1) gives that this coefficient is $[z:\tilde{y}][\tilde{y}:\tilde{x}]$. Similarly, $\tilde{x}e_X$ appears in $\varphi(X \rightarrow \text{supp}(\tilde{w}) \rightarrow Z)$ with coefficient $[z:\tilde{w}][\tilde{w}:\tilde{x}]$. Lemma 7.1 shows that these two coefficients sum to zero. Therefore, $\sum_{Y \in (Z, X)} \varphi(X \rightarrow Y \rightarrow Z) = 0$. ■

Corollary 8.6 *The face semigroup algebra $k\mathcal{F}$ of a hyperplane arrangement depends only on the intersection lattice \mathcal{L} .*

Note that this implies that arrangements with the same intersection lattice but nonisomorphic face posets have isomorphic face semigroup algebras.

8.3.2 Second Version

In this section we note that the idempotents e_X used in the previous section to define φ can be changed slightly without affecting the kernel of φ . This will be used in a companion paper to construct idempotents for the descent algebra of a finite Coxeter group [23].

For each $X \in \mathcal{L}$ let L_X denote a nonempty set of elements of support X and let $\lambda_X = |L_X|$. In what follows we will need the fact that the characteristic of k does

not divide λ_X for all $X \in \mathcal{L}$. Let \tilde{X} denote the sum of the elements in L_X divided by λ_X . Then \tilde{X} is an idempotent and the elements $e_X = \tilde{X} - \sum_{Y>X} \tilde{X}e_Y$ form a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ (see Remark 5.3). Define $\varphi: k\mathcal{Q} \rightarrow k\mathcal{F}$ using these idempotents: the image of vertex X is the idempotent e_X ; the image of an arrow $X \rightarrow Y$ is $e_Y\partial(y)e_X$, where y is any element of support Y .

To see that the kernel of φ is described by Proposition 8.5, let $(X \rightarrow Y \rightarrow Z)$ be a path in \mathcal{Q} and note that $\varphi(X \rightarrow Y \rightarrow Z)$ can be written as

$$\frac{1}{\lambda_Z} \sum_{z \in L_Z} ([z:y_1^z]y_1^z + [z:y_2^z]y_2^z) ([y:x_1^y]x_1^y + [y:x_2^y]x_2^y) e_X,$$

where y_1^z and y_2^z are the two faces of support Y with z as a face and x_1^y and x_2^y are the two faces of support X with y as a face. (Use Lemma 5.1, the fact that $e_X = \tilde{X} - \sum_{Y>X} \tilde{X}e_Y$ for all $X \in \mathcal{L}$, and Proposition 4.2.)

Next we will show that the coefficient of $y_i^z x_j^y$ in the above is $\frac{1}{\lambda_Z} [z:y_i^z][y:x_j^y]$. This amounts to showing that if $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^{y'}$, then $z = z', i = i',$ and $j = j'$. Both z and z' are faces of $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^{y'}$, but no face can have two distinct faces of the same support. So $z = z'$. Also, y_i^z and $y_{i'}^{z'}$ are faces of $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^{y'}$ of the same support, so in fact $i = i'$. Since $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^{y'}$, it follows that x_j^y and $x_{j'}^{y'}$ are on the same side of Y . But, by definition, they are on different sides of Y . So $j = j'$.

Let $x \in \mathcal{F}$ have support X and suppose xe_X is a summand of $\varphi(X \rightarrow Y \rightarrow Z)$. Then $x = y_i^z x_j^y$ for some $i, j \in \{1, 2\}, z \in L_Z$. Since there are exactly two faces w_1 and w_2 in the open interval $\{w \in \mathcal{F} : z < w < x\}$, it follows that y_i^z is either w_1 or w_2 . In the former case the coefficient of xe_X is

$$\frac{1}{\lambda_Z} [z:w_1][y:x_j^y] = \frac{1}{\lambda_Z} [z:w_1][w_1 y' : w_1 x_j^y] = \frac{1}{\lambda_Z} [z:w_1][w_1 : x],$$

using equation (7.1). Similarly, if $y = w_2$, then the coefficient is $\frac{1}{\lambda_Z} [z:w_2][w_2 : x]$. Therefore, the coefficient of xe_X in $\sum_{X<Y<Z} \varphi(X \rightarrow Y \rightarrow Z)$ is, by Lemma 7.1,

$$\frac{1}{\lambda_Z} [z:w_1][w_1 : x] + \frac{1}{\lambda_Z} [z:w_2][w_2 : x] = 0.$$

So $\sum_{X<Y<Z} \varphi(X \rightarrow Y \rightarrow Z) = 0$ since $\{xe_X : \text{supp}(x) = X\}$ is a basis of $k\mathcal{F}e_X$.

9 The Ext-Algebra of the Face Semigroup Algebra

9.1 Koszul Algebras

Our treatment of Koszul algebras closely follows [3]. Let k be a field. A k -algebra A is a *graded k -algebra* if there exists a k -vector space decomposition $A \cong \bigoplus_{i \geq 0} A_i$ satisfying $A_i A_j \subset A_{i+j}$. Here $A_i A_j$ is the set of elements $\{\sum_l a_l a'_l \mid a_l \in A_i, a'_l \in A_j\}$. The subspace A_0 is considered to be an A -module by identifying it with the A -module $A / \bigoplus_{i > 0} A_i$.

If $A = \bigoplus_{i \geq 0} A_i$ is a graded k -algebra, then a *graded A -module* M is an A -module with a vector space decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ satisfying $A_i M_j \subset M_{i+j}$ for all

$i, j \in \mathbb{Z}$. A graded A -module M is *generated in degree i* if $M_j = 0$ for $j < i$ and $M_j = A_{j-i}M_i$ for all $j \geq i$. If M and N are graded A -modules, then an A -module morphism $f : M \rightarrow N$ has *degree p* if $f(M_i) \subset N_{i+p}$ for all i .

A graded A -module M has a *linear resolution* if M admits a projective resolution

$$\dots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

with P_i a graded A -module generated in degree i and d_i a degree 0 morphism for all $i \geq 0$. Observe that if M admits a linear resolution, then M is generated in degree 0.

Definition 9.1 A graded k -algebra $A = \bigoplus_{i \geq 0} A_i$ is a *Koszul algebra* if A_0 is a semisimple k -algebra and A_0 , considered as a graded A -module concentrated in degree 0, admits a linear resolution.

A *quadratic k -algebra* is a graded k -algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is semisimple and A is generated by A_1 over A_0 with relations of degree 2. Explicitly, $A = \bigoplus_{i \geq 0} A_i$ is quadratic if A_0 is semisimple and A is a quotient of the free tensor algebra $T_{A_0}A_1 = \bigoplus_{i \geq 0} (A_1)^{\otimes i}$ of the A_0 -bimodule A_1 by an ideal generated by elements of degree 2: $A \cong T_{A_0}A_1 / \langle R \rangle$ with $R \subset A_1 \otimes_{A_0} A_1$. Here $(A_1)^{\otimes i}$ denotes the i -fold tensor product of A_1 over A_0 .

Proposition 9.2 ([3, Corollary 2.3.3]) *Koszul algebras are quadratic.*

Not all quadratic algebras are Koszul algebras. Furthermore, it is not known for which algebras the notions of quadratic and Koszul coincide.

Let $A = T_{A_0}A_1 / \langle R \rangle$ be a quadratic algebra. If V is an A_0 -bimodule, let $V^* = \text{Hom}_{A_0}(V, A_0)$. For any subset $W \subset V$, let $W^\perp = \{f \in V^* \mid f(W) = 0\}$. The algebra $A^\dagger = T_{A_0}A_1^* / \langle R^\perp \rangle$ is the *quadratic dual* of A or the *Koszul dual* of A in the case when A is a Koszul algebra. (An important technicality is that in defining the quadratic dual the identification $(V_1^* \otimes \dots \otimes V_n^*) \cong (V_n \otimes \dots \otimes V_1)^*$ has been made, where $(f_1 \otimes \dots \otimes f_n)(v_n \otimes \dots \otimes v_1) = f_n(v_n f_{n-1}(v_{n-1} \dots f_1(v_1) \dots))$ for all $f_i \in V_i^*$ and $v_i \in V_i$.)

If A is a graded k -algebra, then the *Ext-algebra* of A is the graded k -algebra $\text{Ext}(A) = \bigoplus_n \text{Ext}^n(A_0, A_0)$ with multiplication given by Yoneda composition.

Theorem 9.3 ([3, Theorems 2.10.1 and 2.10.2]) *Suppose A is a Koszul algebra. Then the Koszul dual A^\dagger is a Koszul algebra isomorphic to the opposite of the Ext-algebra $\text{Ext}(A)$ of A and $\text{Ext}(\text{Ext}(A)) \cong A$.*

Before proceeding, we record how the quadratic dual of a quadratic algebra arising as the quotient of the path algebra of a quiver is constructed from the quiver and relations. Note that the path algebra kQ of a quiver Q is the free tensor algebra of the k -vector space kQ_1 spanned by the arrows of Q viewed as a bimodule over the k -vector space kQ_0 spanned by the vertices of Q . It follows that $A = kQ / \langle R \rangle \cong T_{kQ_0}kQ_1 / \langle R \rangle$, where R is a set of relations of paths of length two. Then the quadratic dual algebra $A^\dagger \cong T_{kQ_0}(kQ_1)^* / \langle R^\perp \rangle \cong kQ^{\text{opp}} / \langle R^\perp \rangle$ is a quotient of the path algebra kQ^{opp} of the opposite quiver Q^{opp} of Q and $R^\perp = \{s \in kQ_2^{\text{opp}} \mid s^*(r) = 0 \text{ for all } r \in R\}$. Here

$(pq)^* : kQ_2 \rightarrow k$ for a path pq of length two in Q^{opp} is the function that takes the value 1 on $qp \in Q$ and 0 otherwise. That is, the quiver of A^1 is Q^{opp} and the relations are the relations *orthogonal* to R . (This can be derived from the definitions. See also [17].)

9.2 The Face Semigroup Algebra is a Koszul Algebra

This section establishes that the face semigroup algebra of a hyperplane arrangement admits a grading making it a Koszul algebra. This is done by constructing a linear resolution for the degree 0 component with respect to the grading inherited from the path length grading on the path algebra of the quiver.

Proposition 9.4 *$k\mathcal{F}$ admits a grading making it a Koszul algebra.*

Proof The k -vector spaces

$$(k\mathcal{F})_i = \bigoplus_{\text{codim}_Y(X)=i} e_X k\mathcal{F}e_Y.$$

define a grading on $k\mathcal{F}$. (This is the grading inherited from the path length grading on the path algebra kQ of the quiver Q of $k\mathcal{F}$.) So $k\mathcal{F}$ is a graded k -algebra. The degree 0 component is

$$(k\mathcal{F})_0 = \bigoplus_{\text{codim}_Y(X)=0} e_X k\mathcal{F}e_Y = \bigoplus_{X \in \mathcal{L}} e_X k\mathcal{F}e_X \cong k^{|\mathcal{L}|},$$

hence is semisimple. It remains to show that $k^{|\mathcal{L}|}$ has a linear resolution. It suffices to show that each simple $k\mathcal{F}$ -module k_X has a linear resolution since $k^{|\mathcal{L}|} \cong \bigoplus_{X \in \mathcal{L}} k_X$.

Fix $X \in \mathcal{L}$ and consider the projective resolution of the simple $k\mathcal{F}$ -module k_X given by Proposition 7.4,

$$\dots \longrightarrow \left(\bigoplus_{\text{codim}_X(Y)=p} k\mathcal{F}e_Y \right) \xrightarrow{\partial} \dots \xrightarrow{\partial} k\mathcal{F}e_X \longrightarrow k_X \longrightarrow 0.$$

For each $k\mathcal{F}e_Y$ define k -subspaces

$$(k\mathcal{F}e_Y)_i = \bigoplus_{\text{codim}(W)=i} e_W k\mathcal{F}e_Y.$$

By Lemma 5.1, if $i < \text{codim}(Y)$, then the degree i component of $k\mathcal{F}e_Y$ is 0. For $i = \text{codim}(Y)$, $(k\mathcal{F})_i = e_Y k\mathcal{F}e_Y = \text{span}_k e_Y$ (Lemma 5.1 again). Since e_Y generates $k\mathcal{F}e_Y$ as a $k\mathcal{F}$ -module, $k\mathcal{F}e_Y$ is generated in degree $\text{codim}(Y)$. The boundary operator ∂ is a degree 0 morphism: if $e_W w \in e_W k\mathcal{F}e_Y$, then $\text{deg}(e_W w) = \text{codim}(W)$ and the degree of its image $\partial(e_W w) = e_W \partial(w) \in e_W \partial(k\mathcal{F}e_Y) \subset \bigoplus_{\text{codim}_X(Y')=p} e_W k\mathcal{F}e_{Y'}$ is $\text{codim}(W)$. ■

Remark 9.5. Notice that by creating the surjection $\varphi: k\mathcal{Q} \rightarrow k\mathcal{F}$ many choices were taken (in constructing the complete system of primitive orthogonal idempotents and in putting orientations on the subspaces in \mathcal{L}). These choices affect the grading inherited by $k\mathcal{F}$ from $k\mathcal{Q}$, but the corresponding graded algebras are isomorphic: two gradings on a k -algebra that both give rise to a Koszul algebra give isomorphic graded k -algebras. (See Corollary [3, Corollary 2.5.2].)

9.3 The Ext-algebra of the Face Semigroup Algebra

In this section we show that the Ext-algebra of $k\mathcal{F}$ is the incidence algebra of the opposite lattice \mathcal{L}^* of the intersection lattice \mathcal{L} .

The *incidence algebra* $I(P)$ of a finite poset P is the set of functions on the subset of $P \times P$ of comparable elements $\{(y, x) \in P \times P \mid y \leq x\}$ with multiplication $(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The identity element is the Krönercker δ -function. The incidence algebra $I(P)$ is a split basic algebra and the quiver Q of $I(P)$ has P as its set of vertices and exactly one arrow $x \rightarrow y$ if $y < x$. If I denotes the ideal of kQ generated by differences of parallel paths, then $I(P) \cong kQ/I$. This isomorphism is given by mapping a vertex x of Q to the function $y \mapsto \delta(x, y)$, and an arrow $x \rightarrow y$ of Q to the function $(u, v) \mapsto \delta(x, u)\delta(y, v)$.

Proposition 9.6 *The Ext-algebra of $k\mathcal{F}$ is the incidence algebra $I(\mathcal{L}^*)$ of the opposite lattice of the intersection lattice \mathcal{L} . Equivalently, it is the opposite algebra $I(\mathcal{L})^{\text{opp}}$ of the incidence algebra $I(\mathcal{L})$ of \mathcal{L} .*

Proof Since $k\mathcal{F}$ is a Koszul algebra (Proposition 9.4), its Ext-algebra is its Koszul dual algebra (Theorem 9.3), so we compute the Koszul dual of $k\mathcal{F}$.

Let \mathcal{Q} denote the quiver of $k\mathcal{F}$. From Proposition 8.5, $k\mathcal{F} \cong k\mathcal{Q}/\langle R \rangle$ is the quotient of the path algebra $k\mathcal{Q}$ by the ideal generated by the sums of all parallel paths of length two,

$$R = \left\{ \sum_{Z \in (Y, X)} (X \rightarrow Z \rightarrow Y) : X, Y \in \mathcal{L} \right\}.$$

Then $(k\mathcal{F})^\perp \cong k\mathcal{Q}^{\text{opp}}/\langle R^\perp \rangle$, where R^\perp is spanned by differences of parallel paths of length two in \mathcal{Q}^{opp} ,

$$R^\perp = \{ (X \rightarrow Z \rightarrow Y) - (X \rightarrow Z' \rightarrow Y) : X < Z, Z' < Y \in \mathcal{L} \}.$$

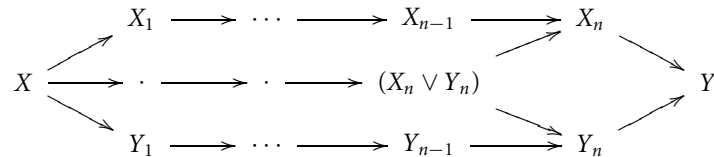
(See the discussion at the end of Subsection 9.1.)

Let $I(\mathcal{L}^*)$ denote the incidence algebra of \mathcal{L}^* . Then $I(\mathcal{L}^*) \cong k\mathcal{Q}^{\text{opp}}/I$, where I is the ideal generated by differences of parallel paths (not necessarily of length two). Therefore, the proof is complete once it is shown that R^\perp generates I .

If $p: X \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow Y$ and $q: X \rightarrow Y_1 \rightarrow \dots \rightarrow Y_n \rightarrow Y$ are parallel paths in \mathcal{Q} such that there exists an i with $X_j = Y_j$ for all $j \neq i$, then $p - q \in I$. If there exists a sequence of paths $p = p_0, p_1, \dots, p_j = q$ with p_{i-1} and p_i differing in exactly one place for $1 \leq i \leq j$, then $p - q = (p_0 - p_1) + \dots + (p_{j-1} - p_j) \in I$. Therefore, $I = \langle R^\perp \rangle$ if any path in \mathcal{Q}^{opp} can be obtained from any other path that is parallel to it by swapping one vertex at a time (without breaking the path). This

follows from the semimodularity of \mathcal{L}^* and by induction on the length of paths in \mathcal{Q}^{opp} . Recall that a finite lattice L is (*upper*) *semimodular* if for every x and y in L , if x and y cover $x \wedge y$, then $x \vee y$ covers x and y .

Let $X \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow Y$ and $X \rightarrow Y_1 \rightarrow \dots \rightarrow Y_n \rightarrow Y$ be parallel paths in \mathcal{Q}^{opp} . Since X_n and Y_n cover $X_n \wedge Y_n = Y$, semimodularity of \mathcal{L}^* gives that $X_n \vee Y_n$ covers both X_n and Y_n . Since $X \leq X_n$ and $X \leq Y_n$, it follows that $X \leq (X_n \vee Y_n)$. So there exists a path from X to $X_n \vee Y_n$. We are now in the following situation.



Induction on the length of paths gives that

$$(Y \rightarrow \dots \rightarrow Y_{n-1} \rightarrow Y_n \rightarrow Y) - (X \rightarrow \dots \rightarrow (X_n \vee Y_n) \rightarrow Y_n \rightarrow Y),$$

$$(X \rightarrow \dots \rightarrow X_{n-1} \rightarrow X_n \rightarrow Y) - (X \rightarrow \dots \rightarrow (X_n \vee Y_n) \rightarrow X_n \rightarrow Y)$$

are in $\langle R^\perp \rangle$. Clearly,

$$(X \rightarrow \dots \rightarrow (X_n \vee Y_n) \rightarrow X_n \rightarrow Y) - (X \rightarrow \dots \rightarrow (X_n \vee Y_n) \rightarrow Y_n \rightarrow Y) \in \langle R^\perp \rangle.$$

Therefore,

$$(Y \rightarrow \dots \rightarrow Y_{n-1} \rightarrow Y_n \rightarrow Y) - (X \rightarrow \dots \rightarrow X_{n-1} \rightarrow X_n \rightarrow Y)$$

is in $\langle R^\perp \rangle$. Therefore, $I = \langle R^\perp \rangle$ and $(k\mathcal{F})^! \cong k\mathcal{Q}^{\text{opp}} / \langle R^\perp \rangle = k\mathcal{Q}^{\text{opp}} / I \cong I(\mathcal{L}^*)$. ■

Corollary 9.7 *The Ext-algebra of $I(\mathcal{L}^*)$ is isomorphic to the face semigroup algebra $k\mathcal{F}$.*

9.4 The Hochschild (Co)Homology of the Face Semigroup Algebra

Let A be a k -algebra and M an A -bimodule. There is a complex of A -bimodules

$$\dots \xrightarrow{d_{i+1}} M \otimes_k A^{\otimes i} \xrightarrow{d_i} \dots \xrightarrow{d_1} M \otimes_k A \xrightarrow{d_0} M$$

with maps $d_i : M \otimes_k A^{\otimes i} \rightarrow M \otimes_k A^{\otimes i-1}$ defined by $d_0(m \otimes a) = am - ma$ for $m \in M$, $a \in A$ and for $i \geq 1$

$$d_i(m \otimes a_1 \otimes \dots \otimes a_i) = (ma_1 \otimes a_2 \otimes \dots \otimes a_i)$$

$$+ \sum_{j=1}^{i-1} (-1)^j (m \otimes a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_i) + (-1)^i (a_i m \otimes a_1 \otimes \dots \otimes a_{i-1}),$$

where $m \in M$ and $a_1, \dots, a_i \in A$. The Hochschild homology of A with coefficients in M is $\text{HH}_i(A, M) = \ker(d_i) / \text{im}(d_{i+1})$ for $i \geq 0$. Let $\text{HH}_i(A) = \text{HH}_i(A, A)$.

Similarly, there exists a cocomplex of A -bimodules

$$M \xrightarrow{d^0} \text{Hom}_k(A, M) \xrightarrow{d^1} \text{Hom}_k(A \otimes_k A, M) \xrightarrow{d^2} \dots$$

where $d^0 : M \rightarrow \text{Hom}_k(A, M)$ is the map $d^0(m)(a) = am - ma$ and d^i is the map $d^i : \text{Hom}_k(A^{\otimes i}, M) \rightarrow \text{Hom}_k(A^{\otimes i+1}, M)$ given by

$$\begin{aligned} (d^i f)(a_1 \otimes \dots \otimes a_{i+1}) &= a_1 f(a_2 \otimes \dots \otimes a_{i+1}) \\ &+ \sum_{j=1}^i (-1)^j f(a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes \dots \otimes a_i) a_{i+1}, \end{aligned}$$

where $f \in \text{Hom}_k(A^{\otimes i}, M)$ and $a_1, \dots, a_{i+1} \in A$. The Hochschild cohomology of A with coefficients in M is $\text{HH}^i(A, M) = \ker(d^i) / \text{im}(d^{i-1})$ for $i \geq 0$. Denote the Hochschild cohomology of A with coefficients in A by $\text{HH}^i(A) = \text{HH}^i(A, A)$.

Proposition 9.8 *The Hochschild homology $\text{HH}_i(k\mathcal{F})$ and cohomology $\text{HH}^i(k\mathcal{F})$ of $k\mathcal{F}$ vanish in positive degrees. In degree zero the homology is $\text{HH}_0(k\mathcal{F}) \cong k^{\#\mathcal{L}}$ and the cohomology is $\text{HH}^0(k\mathcal{F}) \cong k$.*

Proof Let \mathcal{Q} denote the quiver of $k\mathcal{F}$. The Hochschild homology of algebras whose quivers have no oriented cycles is known to be zero in positive degrees and k^q in degree 0, where q is the number of vertices in the quiver [12]. This establishes the Hochschild homology of $k\mathcal{F}$ since \mathcal{Q} has no oriented cycles.

Buchweitz [19, Section 3.5] proved that the Hochschild cohomology algebra of a Koszul algebra is the Hochschild cohomology algebra of its Koszul dual. Since $k\mathcal{F}$ is a Koszul algebra with Koszul dual the incidence algebra $I(\mathcal{L}^*)$ of the lattice \mathcal{L}^* , there is an isomorphism

$$\text{HH}^*(k\mathcal{F}) \cong \text{HH}^*(I(\mathcal{L}^*)) \cong \bigoplus_{i \geq 0} \text{HH}^i(I(\mathcal{L}^*)).$$

Gerstenhaber and Schack ([16]; see also [13, Corollary 1.4]) proved that the Hochschild cohomology $\text{HH}^i(I(\mathcal{L}^*))$ of $I(\mathcal{L}^*)$ is the simplicial cohomology of the simplicial complex $\Delta(\mathcal{L}^*)$ whose i -simplices are the chains of length i in the poset \mathcal{L}^* . Therefore,

$$\text{HH}^i(I(\mathcal{L}^*)) \cong H^i(\Delta(\mathcal{L}^*), k).$$

The latter is zero in positive degrees since $\Delta(\mathcal{L}^*)$ is a double cone (\mathcal{L}^* contains both a top and bottom element) and is k in degree zero since $\Delta(\mathcal{L}^*)$ is connected. It is easy to check directly that $\text{HH}^0(k\mathcal{F}) \cong k$, completing the proof. ■

10 Connections with Poset Cohomology

10.1 The Cohomology of a Poset

Let P denote a finite poset. The *order complex* $\Delta(P)$ of P is the simplicial complex with i -simplices the chains of length i in P . Suppose P has both a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$ and let k denote a field. The *order cohomology of P with coefficients in k* is the reduced simplicial cohomology with coefficients in k of the order complex $\Delta(P - \{\widehat{0}, \widehat{1}\})$ of $P - \{\widehat{0}, \widehat{1}\}$. The order cohomology of P has the following characterization in terms of the chains of P .

Suppose P contains at least two distinct elements. For $i \geq 0$, let $C_i(P)$ denote the k -vector space spanned by the i -chains of $P - \{\widehat{0}, \widehat{1}\}$,

$$C_i(P) = \text{span}_k \{ (x_0 < \cdots < x_i) \mid x_j \in P - \{\widehat{0}, \widehat{1}\} \}.$$

For $i = -1$, let $C_{-1}(P) = k$, the vector space spanned by the empty chain. If P consists of one element, then define $C_{-2}(P) = k$ and $C_i(P) = 0$ otherwise.

Define coboundary morphisms $\delta_i: C_i(P) \rightarrow C_{i+1}(P)$ by

$$\delta_i(x_0 < \cdots < x_i) = \sum_{j=0}^{i+1} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_i),$$

where $x_{-1} = \widehat{0}$ and $x_{i+1} = \widehat{1}$. It is straightforward to check that $\delta^2 = 0$. The order cohomology of P is $H^i(P) = H^i(P; k) = \ker(\delta_i) / \text{im}(\delta_{i-1})$.

Notice that if P consists of exactly one element, then $H^{-2}(P) = k$ and $H^i(P) = 0$ for $i \neq -2$. If $P = \{\widehat{0}, \widehat{1}\}$, then $H^{-1}(P) = k$ and $H^i(P) = 0$ for $i \neq -1$.

10.2 A Vector Space Decomposition of the Face Semigroup Algebra

Suppose the length of the longest chain in the poset P is $d+2$. Then $\ker(\delta_d)$ is spanned by the chains of length d in $P - \{\widehat{0}, \widehat{1}\}$ and $\text{im}(\delta_{d-1})$ is spanned by the elements,

$$\sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_{d-1}),$$

one for each chain $x_0 < \cdots < x_{j-1} < x_j < \cdots < x_{d-1}$ of length $d - 1$.

Put $P = \mathcal{L}$ in the above and identify the cover relations with the arrows in \mathcal{Q} . Then the top cohomology of \mathcal{L} corresponds to the quotient of the span of the maximal paths in \mathcal{Q} by the quiver relations. This gives a vector space isomorphism $e_{\widehat{0}} k\mathcal{F} e_{\widehat{1}} \cong H^{d-2}(\mathcal{L})$, where the length of the longest chain in \mathcal{L} is d . Folkman [15] showed that the cohomology of a geometric lattice is non-vanishing only in the top degree. Since \mathcal{L}^* is a geometric lattice and $\Delta(\mathcal{L}^*) = \Delta(\mathcal{L})$, the cohomology of \mathcal{L} is non-vanishing only in the top degree. Therefore, $e_{\widehat{0}} k\mathcal{F} e_{\widehat{1}} \cong H^*(\mathcal{L})$. Since every interval of a geometric lattice is also a geometric lattice, the result holds for every interval of \mathcal{L} . That is, $e_X k\mathcal{F} e_Y \cong H^*([X, Y])$.

Proposition 10.1 *$k\mathcal{F}$ has a k -vector space decomposition in terms of the order cohomology of the intervals of \mathcal{L} ,*

$$k\mathcal{F} \cong \bigoplus_{X, Y \in \mathcal{L}} H^*([X, Y]).$$

10.3 Another Cohomology Construction on Posets

In light of the above decomposition, the direct sum $\bigoplus_{X,Y \in \mathcal{L}} H^*([X, Y])$ inherits a k -algebra structure from $k\mathcal{F}$. This section shows that the algebraic structure can be obtained via the cup product of a cohomology algebra on the intersection lattice. This cohomology construction appears to be new.

Let P be a finite poset and let k denote a field. Let $D_i(P)$ denote the k -vector space of i -chains in P ,

$$D_i(P) = \{ (x_0 < \dots < x_i) \mid x_j \in P \}.$$

Define coboundary morphisms $d_i: D_i(P) \rightarrow D_{i+1}(P)$ by

$$d_i(x_0 < \dots < x_i) = \sum_{j=1}^i (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \dots < x_{j-1} < x < x_j < \dots < x_i).$$

Then $d^2 = 0$. The cohomology groups of the cocomplex (D_\bullet, d) will be denoted by $\mathcal{H}^i(P) = \mathcal{H}^i(P; k) = \ker(d_i) / \text{im}(d_{i-1})$.

The differences between $\mathcal{H}^i(P)$ and $H^i(P)$ are small but important. The former is defined for any poset P , not just a poset with $\hat{0}$ and $\hat{1}$. The vector space $D_i(P)$ is spanned by all the chains in P , not just those avoiding $\hat{0}$ and $\hat{1}$. The summation in the coboundary morphism $d_i: D_i(P) \rightarrow D_{i+1}(P)$ runs from $j = 1$ to $j = i$, whereas the summation runs from $j = 0$ to $j = i + 1$ in the coboundary morphism $\delta_i: C_i(P) \rightarrow C_{i+1}(P)$. However, there is a strong relationship between $\mathcal{H}(P)$ and $H(P)$.

Proposition 10.2 *Let P be a finite poset. Then for all $i \geq 0$,*

$$\mathcal{H}^i(P) \cong \bigoplus_{x,y \in P} H^{i-2}([x, y]).$$

Proof $D_i(P)$ decomposes into subspaces spanned by the i -chains of P beginning at x and terminating at y : $(x < x_1 < \dots < x_{i-1} < y)$. The differential d_i respects this decomposition and the subspaces are isomorphic to $C_{i-2}([x, y])$ (drop the x and y of each chain). This isomorphism commutes with the coboundary operators, establishing the proposition. ■

The benefit of working with $\mathcal{H}^*(P)$ is that the *simplicial cup product* (see [20, Section 49]) on the simplices of the order complex $\Delta(P)$ of P descends to a product on the cohomology.

Define a product $\smile: D_p(P) \times D_q(P) \rightarrow D_{p+q}(P)$ by

$$(x_0 < \dots < x_p) \smile (y_0 < \dots < y_q) = \begin{cases} (x_0 < \dots < x_p = y_0 < \dots < y_q) & x_p = y_0, \\ 0 & x_p \neq y_0. \end{cases}$$

Lemma 10.3 *For $c \in D_p(P)$ and $d \in D_q(P)$,*

$$\delta_{p+q}(c \smile d) = \delta_p(c) \smile d + (-1)^p c \smile \delta_q(d).$$

Proof Let $c = (x_0 < \dots < x_p)$ and $d = (x_p < \dots < x_{p+q})$. Then

$$\begin{aligned} & \delta_p(c) \smile d + (-1)^p c \smile \delta_q(d) \\ &= \sum_{j=1}^p (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \dots < x_{j-1} < x < x_j < \dots < x_p) \smile d \\ & \quad + (-1)^p c \smile \sum_{j=p+1}^{p+q} \sum_{x_{j-1} < x < x_j} (x_p < \dots < x_{j-1} < x < x_j < \dots < x_{p+q}) \\ &= \sum_{j=1}^p (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \dots < x_{j-1} < x < x_j < \dots < x_{p+q}) \\ & \quad + \sum_{j=p+1}^{p+q} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \dots < x_{j-1} < x < x_j < \dots < x_{p+q}) \\ &= \sum_{j=1}^{p+q} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \dots < x_{j-1} < x < x_j < \dots < x_{p+q}) \\ &= \delta_{p+q}(x_0 < \dots < x_{p+q}) \\ &= \delta_{p+q}(c \smile d). \quad \blacksquare \end{aligned}$$

Corollary 10.4 *The product $D_p(P) \times D_q(P) \xrightarrow{\sim} D_{p+q}(P)$ induces a well-defined product $\mathcal{H}^p(P) \times \mathcal{H}^q(P) \xrightarrow{\sim} \mathcal{H}^{p+q}(P)$ giving $\mathcal{H}^*(P) = \bigoplus_i \mathcal{H}^i(P)$ a k -algebra structure.*

10.4 The Face Semigroup Algebra as a Cohomology Algebra

Combining Propositions 10.1 and 10.2 gives the vector space isomorphism

$$\phi: \mathcal{H}^*(\mathcal{L}) \xrightarrow{\cong} \bigoplus_{X,Y \in \mathcal{L}} H^*([X, Y]) \xrightarrow{\cong} k\mathcal{Q}/I \xrightarrow{\varphi} k\mathcal{F}.$$

Recall that Proposition 10.1 identifies $\bigoplus_{X,Y} H^*([X, Y])$ with $k\mathcal{F}$ via the quiver \mathcal{Q} with relations of $k\mathcal{F}$. The isomorphism identifies an unrefinable chain in \mathcal{L} with the corresponding path in \mathcal{Q}

$$(X_0 \triangleleft X_1 \triangleleft \dots \triangleleft X_{j-1} \triangleleft X_j) \longmapsto (X_j \rightarrow X_{j-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0)$$

and maps the relations in $\mathcal{H}^*(\mathcal{L})$ to the quiver relations. Under this isomorphism the multiplication in $\mathcal{H}^*(\mathcal{L})$ maps to the multiplication in $k\mathcal{Q}/I$ (composition of chains in \mathcal{L} maps to composition of paths in \mathcal{Q}). Therefore, ϕ is a k -algebra isomorphism.

Proposition 10.5 *Let $k\mathcal{F}$ be the face semigroup algebra of a hyperplane arrangement with intersection lattice \mathcal{L} . Then $k\mathcal{F} \cong \mathcal{H}^*(\mathcal{L})$.*

10.5 Connection with the Whitney Cohomology of the Intersection Lattice

We finish this section by identifying the Whitney cohomology of \mathcal{L} in $k\mathcal{F}$. (See [2] and more recently [28].) The *Whitney cohomology* of a poset P with $\widehat{0}$ is the direct sum $WH^*(P) = \bigoplus_{X \in P} H^*([\widehat{0}, X])$. Since the Whitney homology of \mathcal{L}^* is isomorphic to the Orlik–Solomon algebra of \mathcal{L}^* [7, Section 7.10], the following result also explains how the dual of the Orlik–Solomon algebra embeds in the face semigroup algebra.

Corollary 10.6 *The Whitney cohomology of \mathcal{L}^* is isomorphic to the ideal of chambers in $k\mathcal{F}$. It is a projective indecomposable $k\mathcal{F}$ -module.*

Proof Since $H^*([X, Y]) \cong e_X k\mathcal{F} e_Y$ for all $X, Y \in \mathcal{L}$ (see the discussion preceding Proposition 10.1), the Whitney cohomology of \mathcal{L}^* is

$$WH^*(\mathcal{L}^*) \cong \bigoplus_{X \in \mathcal{L}} H^*([X, \widehat{1}]) \cong \bigoplus_{X \in \mathcal{L}} e_X k\mathcal{F} e_{\widehat{1}} \cong k\mathcal{F} e_{\widehat{1}} \cong k\mathcal{F}_{\widehat{1}}. \quad \blacksquare$$

11 Future Directions

These results can be extended to the semigroup algebra of the semigroup of covectors of an oriented matroid (see [8, Section 4.1] for the definition of this semigroup). This is a consequence of two observations. The first observation is that the exact sequence used to construct the projective resolutions of the simple modules (Section 7) can be extended to the semigroup algebra of an oriented matroid [11, Section 6]. The second observation is that the construction of the complete set of primitive orthogonal idempotents in Section 5.1 holds for a larger class of semigroups called *left regular bands* (see [24]).

By restricting attention to the reflection arrangement of a finite Coxeter group, the theory developed here yields results about the descent algebra of the Coxeter group. In a companion paper [23], we study the quiver and module structure of the descent algebra using this approach.

The cohomology construction introduced in Section 10.3 appears to be new. This construction is interesting, especially because the resulting cohomology algebra appears naturally as the face semigroup algebra of a hyperplane arrangement and deserves to be studied further. A natural starting point would be to mimic the theory of the order cohomology of a poset. We mention one possibility: if G is a group acting on a poset P , then the G -action on P induces a G -module structure on $\mathcal{H}^*(P)$, and the resulting G -module structure can be studied. This has been extensively studied for order homology and cohomology and is quite interesting [28].

For certain classes of posets $\mathcal{H}^*(P)$ has nice algebraic structure. For example, if P is a Cohen–Macaulay poset, then its incidence algebra $I(P)$ is a Koszul algebra [22, 29]. Hence, $\mathcal{H}^*(P)$ is the Koszul dual algebra of $I(P)$. This describes the Koszul dual algebra of $I(P)$ in terms of the order cohomology of P .

The construction also provides an extension of a result describing a part of the Lie algebra (co)homology of a certain subalgebra $N(P)$ of the incidence algebra of P in terms of the order (co)homology of P [18]. Hozo showed that if P contains $\widehat{0}$ and $\widehat{1}$, then the Lie algebra (co)homology of $N(P)$ contains the order (co)homology of P . His proof extends to show that for any poset P (not necessarily containing $\widehat{0}$ and $\widehat{1}$),

the Lie algebra (co)homology of $N(P)$ contains the (co)homology $\mathcal{H}^*(P)$. This is a further step towards describing the complete Lie algebra (co)homology of $N(P)$ in terms of the combinatorics of the poset P .

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