

DISTINGUISHED SUBMODULES

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Although there is no need for a 'distinguished' submodule to be given a formal definition in the present paper, we like to indicate the meaning attached to this concept here. Perhaps the shortest way of doing so is to say that a distinguished submodule is a (covariant idempotent) functor from the category of (left) R -modules into itself mapping each R -module into its R -submodule specified by a family of left ideals of R . If \mathcal{K} is a family of left ideals of R , then all elements of an R -module M of orders belonging to \mathcal{K} , do not, of course, in general form a submodule of M ; but, there are certain families $\overline{\mathcal{K}} \supseteq \mathcal{K}$ such that all the elements of orders from $\overline{\mathcal{K}}$ form a submodule in any R -module (distinguished submodules defined by \mathcal{K}). Consequently, no particular structural properties of the R -module are involved in the definition of such submodules. In this way we can define radicals (in the sense of Kuroš [4]) of a module. In particular, we feel that an application of this method is an appropriate way in defining the (maximal) torsion submodule of a module.

Thus, the present paper is, in fact, a study of subfamilies of the family \mathcal{L}_R of all left ideals of a ring R . We use this opportunity to deal with a certain duality in the set of subfamilies of \mathcal{L}_R ; this duality relates to problems connected with the problems of dependence over modules and, in particular, with the definition of the rank of a module which will be treated elsewhere (cf. [1]).

1. Basic definitions

Throughout the paper, R stands for an (associative) ring with unity ε . The family of all proper (that is, $\neq R$) left ideals of R is denoted by \mathcal{L}_R , or briefly by \mathcal{L} . For $L \in \mathcal{L}$ and $\rho \in R$, the symbol $L : \rho$ denotes the (right) ideal-quotient of L by ρ , that is, the left ideal of all $\chi \in R$ such that $\chi\rho \in L$.

We shall consider the following three properties of a subfamily $\mathcal{K} \subseteq \mathcal{L}$:

- (Q) $K \in \mathcal{K} \wedge \rho \in R \setminus K \rightarrow K : \rho \in \mathcal{K}$;
- (I) $K_1 \in \mathcal{K} \wedge K_2 \in \mathcal{K} \rightarrow K_1 \cap K_2 \in \mathcal{K}$;
- (E) $K_1 \in \mathcal{K} \wedge K_2 \in \mathcal{K} \wedge K_1 \subseteq K_2 \rightarrow K_2 \in \mathcal{K}$.

We can see immediately that

1.1. (i) If $\mathcal{X}_\omega, \omega \in \Omega$, satisfy (Q), (I), or (E), then $\bigcap_{\omega \in \Omega} \mathcal{X}_\omega$ satisfies (Q), (I), or (E), respectively.

(ii) If $\mathcal{X}_\omega, \omega \in \Omega$, satisfy (Q) or (E), then $\bigcup_{\omega \in \Omega} \mathcal{X}_\omega$ satisfies (Q) or (E), respectively.

In what follows we shall be interested in the Q -families of left ideals of R , i.e. in the subfamilies of \mathcal{L} satisfying (Q) and in the F -families (filters) of left ideals of R , i.e. in the subfamilies of \mathcal{L} satisfying (Q), (I) and (E). Denote the set of all subfamilies of \mathcal{L} by \mathbf{L} , the subset of \mathbf{L} of all Q -families by \mathbf{Q} and that of all F -families by \mathbf{F} . Hence, by 1.1,

1.2. The mappings c_Q and c_F defined on \mathbf{L} by

$$c_Q(\mathcal{X}) = \bigcap_{\mathcal{X} \subseteq \mathcal{X} \in \mathbf{Q}} \mathcal{X}$$

and

$$c_F(\mathcal{X}) = \bigcap_{\mathcal{X} \subseteq \mathcal{X} \in \mathbf{F}} \mathcal{X}$$

are (idempotent) closure operators in \mathcal{L} .

M will always denote a (unital left) R -module. The order of $m \in M$ is denoted by $O(m)$; hence, $O(m) \in \mathcal{L}$ if and only if $m \neq 0$. Moreover, evidently $O(\rho m) = O(m) : \rho$ for any non-zero $m \in M$ and $\rho \in R$. Since R will be fixed throughout the paper, we shall often speak briefly about submodules, homomorphisms etc. instead of R -submodules, R -homomorphisms etc. The R -module of all cosets of R modulo L will be denoted by $R \bmod L$. A submodule N of M is said to be essential in M if it meets any non-zero submodule of M non-trivially. Specializing to $M = R \bmod \{0\}$ we get the concept of an essential left ideal in R . If \mathcal{X} is a family of left ideals of R , then the subset of all elements of an R -module M whose orders belong to $\mathcal{X}^0 = \mathcal{X} \cup \{R\}$ will be denoted by $M_{\mathcal{X}} \subseteq M$.

2. An order and a Galois connection in \mathbf{Q}

Although our next consideration can easily be extended to \mathbf{L} , we shall, with regard to our further needs, restrict the definitions to \mathbf{Q} .

If $L \in \mathcal{L}$, then

$$c_Q(\{L\}) = c_Q(L) = \{L : \rho\}_{\rho \in R \setminus L};$$

let us call a Q -family of this type cyclic. Also, let us point out that, in view of 1.1, any family $\mathcal{X} \subseteq \mathcal{L}$ contains the greatest Q -subfamily: the (set-theoretical) union of all cyclic Q -families contained in \mathcal{X} . In particular, a Q -family is the union of its cyclic Q -subfamilies.

Now, define in \mathbf{Q} the following preorder \ll by

$$\mathcal{H}_1 \ll \mathcal{H}_2 \leftrightarrow \forall L (L \in \mathcal{L} \rightarrow c_Q(L) \not\subseteq \mathcal{H}_1 \setminus \mathcal{H}_2).$$

Clearly,

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \rightarrow \mathcal{H}_1 \ll \mathcal{H}_2;$$

moreover, we get immediately

$$\mathcal{H}_1 \ll \mathcal{H}_2 \rightarrow (\mathcal{H}_1 \ll \mathcal{H}_1 \cap \mathcal{H}_2 \wedge \mathcal{H}_1 \cup \mathcal{H}_2 \ll \mathcal{H}_2).$$

The preorder \ll yields an equivalence \approx in \mathbf{Q} , namely

$$\mathcal{H}_1 \approx \mathcal{H}_2 \leftrightarrow (\mathcal{H}_1 \ll \mathcal{H}_2 \wedge \mathcal{H}_2 \ll \mathcal{H}_1).$$

Again, we can easily prove that, for $\omega \in \Omega$,

$$\mathcal{H} \approx \mathcal{H}_\omega \rightarrow \mathcal{H} \approx \bigcup_{\omega \in \Omega} \mathcal{H}_\omega$$

and, provided Ω is finite, also

$$\mathcal{H} \approx \mathcal{H}_\omega \rightarrow \mathcal{H} \approx \bigcap_{\omega \in \Omega} \mathcal{H}_\omega.$$

Hence,

2.1. There is a greatest (with respect to order by inclusion) element in each equivalence class of \mathbf{Q} . If $\mathbf{K} \subseteq \mathbf{Q}$ is the equivalence class containing \mathcal{H} , then the greatest element $c(\mathcal{H})$ of \mathbf{K} is given by

$$c(\mathcal{H}) = \bigcup_{\mathcal{X} \approx \mathcal{H}} \mathcal{X} = \bigcup_{\mathcal{X} \in \mathbf{K}} \mathcal{X}.$$

Also, $\mathcal{H}_1 \cap \mathcal{H}_2 \in \mathbf{K}$ for every $\mathcal{H}_1 \in \mathbf{K}$, $\mathcal{H}_2 \in \mathbf{K}$ and, furthermore, any $\mathcal{X} \in \mathbf{Q}$ such that $\mathcal{H} \subseteq \mathcal{X} \subseteq c(\mathcal{H})$ belongs to \mathbf{K} , as well.

Denote the set of all the greatest elements in the equivalence classes of 2.1, i.e. the set of all $\mathcal{H} \in \mathbf{Q}$ such that $c(\mathcal{H}) = \mathcal{H}$, by \mathbf{C} and, furthermore, put

$$\mathbf{T} = \mathbf{C} \cap \mathbf{F}.$$

Evidently, \mathbf{C} can also be characterized as the set of all those families \mathcal{H} which satisfy, for any $\mathcal{X} \in \mathbf{L}$, the implication

$$\mathcal{X} \approx \mathcal{H} \rightarrow \mathcal{X} \subseteq \mathcal{H}$$

(or, even the stronger implication $\mathcal{X} \ll \mathcal{H} \rightarrow \mathcal{X} \subseteq \mathcal{H}$). Thus, we deduce that c is an (idempotent) closure operator; for, if $\mathcal{H}_\omega \in \mathbf{C} (\omega \in \Omega)$, then evidently

$$\bigcap_{\omega \in \Omega} \mathcal{H}_\omega \in \mathbf{C}.$$

Also, the preorder \ll induces an order in the set of all equivalence classes, and thus, it induces an order in \mathbf{C} . It turns out that this order in \mathbf{C} induced by \ll coincides with the order by inclusion.

Now, define the ‘duality’ map ∂ in \mathbf{Q} as follows: For $\mathcal{H} \in \mathbf{Q}$, let $\partial\mathcal{H}$ be given by

Clearly,
$$L \in \partial\mathcal{K} \rightarrow \mathfrak{c}_{\mathbf{Q}}(L) \cap \mathcal{K} = \emptyset.$$

Also

$$\partial\mathcal{K} \in \mathbf{Q} \text{ and } \mathcal{K} \cap \partial\mathcal{K} = \emptyset.$$

and

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \rightarrow \partial\mathcal{K}_1 \supseteq \partial\mathcal{K}_2$$

Furthermore,

$$\partial^2\mathcal{K} = \partial(\partial\mathcal{K}) \supseteq \mathcal{K}.$$

and

$$\partial\mathcal{K}_1 = \partial\mathcal{K}_2 \leftrightarrow \mathcal{K}_1 \approx \mathcal{K}_2$$

Hence,

$$\partial^2\mathcal{K} = \mathfrak{c}(\mathcal{K}).$$

2.2. The mapping ∂ defines for the elements of \mathbf{Q} a Galois connection (cf. Ore [5]). The operator \mathfrak{c} is the corresponding Galois closure operation and \mathbf{C} — the set of all closed elements.

In the introduction, the importance of the mapping ∂ in relation to dependence over modules was mentioned. It stems from the fact that, for any $\mathcal{K} \in \mathbf{Q}$, there exist maximal independent subsets consisting of elements whose orders belong to $\mathcal{K} \cup \partial\mathcal{K}$ in an arbitrary R -module (cf. [1]).

3. Distinguished submodules

The value of the concepts of an F -family and an T -family (i.e. a family belonging to $\mathbf{T} = \mathbf{C} \cap \mathbf{F}$) of left ideals will be apparent from the following theorems.

THEOREM 3.1. *Let $\mathcal{K} \in \mathbf{L}$. Then, in any R -module M , the subset $M_{\mathcal{K}} \subseteq M$ is an R -submodule of M if and only if \mathcal{K} is an F -family.*

PROOF. First, let M be an R -module, $\mathcal{K} \in \mathbf{F}$ and m_1, m_2 two non-zero elements of $M_{\mathcal{K}} \subseteq M$. Since $O(\rho m_1) = O(m_1) : \rho$, $\rho m_1 \in M_{\mathcal{K}}$ by (Q). Also, since $O(m_1+m_2) \supseteq O(m_1) \cap O(m_2)$, $m_1+m_2 \in M_{\mathcal{K}}$ in view of (I) and (E). Thus, if \mathcal{K} is an F -family, then $M_{\mathcal{K}}$ is a submodule of M .

On the other hand, let \mathcal{K} possess the property that, for any R -module M , $M_{\mathcal{K}}$ is always a submodule of M . Consider for a moment the R -module

$$M^* = R \text{ mod } L_1 \oplus R \text{ mod } L_2$$

with $L_i \in \mathcal{L}$ and denote by $\bar{\varepsilon}_i$ the class $\varepsilon + L_i$ of $R \text{ mod } L_i$ ($i = 1, 2$); thus, the general element of M^* can be written in the form $\rho_1 \bar{\varepsilon}_1 + \rho_2 \bar{\varepsilon}_2$. Clearly, since $O(\bar{\varepsilon}_i) = L_i$,

$$O(\rho_1 \bar{\varepsilon}_1) = L_1 : \rho_1$$

and

$$O(\bar{\varepsilon}_1 + \bar{\varepsilon}_1) = L_1 \cap L_2.$$

Also, $O(\bar{\varepsilon}_1 + \bar{\varepsilon}_2) = L_1$ provided that $L_1 \subseteq L_2$. Therefore, if $L_1 \in \mathcal{X}$, then necessarily $L_1 : \rho_1 \in \mathcal{X}^0 (= \mathcal{X} \cup \{R\})$ and if, moreover, $L_2 \in \mathcal{X}$, then also $L_1 \cap L_2 \in \mathcal{X}$; hence, \mathcal{X} satisfies (Q) and (I). Furthermore, if $L_1 \in \mathcal{X}$ and $L_1 \subseteq L_2$, then both $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_1 + \bar{\varepsilon}_2$ belong to $M_{\mathcal{X}}$ and thus

$$L_2 = O(\bar{\varepsilon}_2) = O((\bar{\varepsilon}_1 + \bar{\varepsilon}_2) - \bar{\varepsilon}_1) \in \mathcal{X}.$$

We conclude that \mathcal{X} satisfies (E) and consequently, it is an F -family.

The assumption in the following theorem is devised to fit the application in § 4.

THEOREM 3.2. *If $\mathcal{X} \in \mathbf{Q}$ satisfies (E) and, moreover, contains all proper essential left ideals of R , then $c(\mathcal{X})$ is an F -family, and thus a T -family. In particular, if \mathcal{X} is an F -family containing all proper essential left ideals of R , then $c(\mathcal{X})$ is a T -family.*

PROOF. First let us prove that (I) holds for $c(\mathcal{X})$. This follows immediately from the fact that, for any $\mathcal{X} \in \mathbf{Q}$, $\partial\mathcal{X}$ always satisfies (I):

Let $L = L_1 \cap L_2$ with $L_i \in \partial\mathcal{X}$ ($i = 1, 2$) and assume that $\rho \in R \setminus L$ exists such that $L : \rho \in \mathcal{X}$. Since

$$L : \rho = (L_1 : \rho) \cap (L_2 : \rho),$$

$(L_2 : \rho) \setminus (L_1 : \rho)$ is necessarily non-empty; but, for any element σ of this set

$$R \neq L_1 : \sigma \rho = (L_1 : \sigma \rho) \cap (L_2 : \sigma \rho) = L : \sigma \rho \in \mathcal{X},$$

a contradiction of $L_1 \in \partial\mathcal{X}$. Hence, $\partial\mathcal{X}$, as well as, $c(\mathcal{X}) = \partial(\partial\mathcal{X})$ satisfies (I).

Now, to complete the proof of the theorem only the validity of (E) for $c(\mathcal{X})$ need to be verified. Thus, let $K_1 \subseteq K_2$ with $K_1 \in c(\mathcal{X})$. In order to show that $K_2 \in c(\mathcal{X})$ it is sufficient to prove that

$$c_{\mathbf{Q}}(K_2) \ll c(\mathcal{X}).$$

Hence, take an arbitrary element $K_2 : \rho$ of $c_{\mathbf{Q}}(K_2)$; here

$$\rho \in R \setminus K_2 \subseteq R \setminus K_1.$$

Assume that $K_2 : \rho \notin \mathcal{X}$. Then, $K_2 : \rho$ is not essential and thus, there exists $\sigma_1 \in R$ such that

$$(K_2 : \rho) \cap R\sigma_1 = \{0\}.$$

Furthermore, since

$$\sigma_1 \in R \setminus (K_2 : \rho) \subseteq R \setminus (K_1 : \rho), \text{ i.e. } \sigma_1 \rho \in R \setminus K_1,$$

there is $\sigma_2 \in R$ such that $K_1 : \sigma_2 \sigma_1 \in \mathcal{X}$. Put $\sigma = \sigma_2 \sigma_1$. Hence,

$$(K_2 : \rho) : \sigma = K_2 : \sigma \rho \neq R,$$

and since $K_2 : \sigma \rho \supseteq K_1 : \sigma \rho$, it belongs to \mathcal{X} . The proof is completed.

THEOREM 3.3. *Let \mathcal{K} be a T -family. Then, for any R -module M , the quotient module $M/M_{\mathcal{X}}$ has no element of order from \mathcal{K} , i.e.*

$$(M/M_{\mathcal{X}})_{\mathcal{X}} = \{0\}.$$

Moreover, no other F -family equivalent to \mathcal{K} possesses this property.¹

PROOF. We present an indirect proof of the first part of the theorem. Suppose that $O(\bar{m}) \in \mathcal{K}$ for a certain $\bar{m} \in M/M_{\mathcal{X}}$ containing $m \in M$. We are going to show that, as a consequence,

$$c_{\mathcal{Q}}(O(m)) \ll \mathcal{K},$$

i.e. $O(m) \in \mathcal{K}$, contradicting the assumption of $\bar{m} \neq \bar{0}$. Hence, take $L \in c_{\mathcal{Q}}(O(m))$:

$$L = O(m) : \rho = O(\rho m) \text{ with } \rho m \neq 0,$$

and assume

$$O(\rho m) \not\in O(\rho \bar{m});$$

otherwise $O(\rho m) = O(\rho \bar{m}) \in \mathcal{K}$. Thus, for a $\sigma \in O(\rho \bar{m}) \setminus O(\rho m)$ we get

$$L : \sigma = O(\sigma \rho m) \in \mathcal{K},$$

i.e. $c_{\mathcal{Q}}(O(m)) \ll \mathcal{K}$, as required.

In order to prove the other part of the theorem, assume that there is an F -family \mathcal{K}_1 equivalent to \mathcal{K} such that

$$(M/M_{\mathcal{K}_1})_{\mathcal{K}_1} = \{0\}$$

for any R -module M . Then, there can be no left ideal $L \in \mathcal{K} \setminus \mathcal{K}_1$; for, otherwise the R -module

$$\bar{M} = R \text{ mod } L / (R \text{ mod } L)_{\mathcal{K}_1}$$

would be a non-zero R -module such that

$$O \neq \bar{m} \in \bar{M} \rightarrow O(\bar{m}) \in \mathcal{K} \setminus \mathcal{K}_1,$$

which is, because of $\mathcal{K} \approx \mathcal{K}_1$, impossible. Hence $\mathcal{K} = \mathcal{K}_1$.

The proof of the theorem is completed.

DEFINITION 3.4. Let $\mathcal{K} \in F$. An R -module M is said to be a \mathcal{K} -module if $M_{\mathcal{X}} = M$; M is said to be a \mathcal{K} -free-module if $M_{\mathcal{X}} = \{0\}$.

Notice that $\{0\}$ is the only R -module which is simultaneously a \mathcal{K} -module and a \mathcal{K} -free-module and that the orders of non-zero elements of a \mathcal{K} -free-module belong to $\partial \mathcal{K}$. We shall see that the concepts of a \mathcal{K} - and \mathcal{K} -free-modules will be particularly valuable in the case when $\mathcal{K} \in T$. Then, inspired by the radical theory of Kuroš [4] (cf. next Theorems 3.6

¹ The latter statement can be generalized (see [2]).

and 3.7) we can speak about \mathcal{K} -radical and \mathcal{K} -semisimple modules.²

The following three theorems will describe some fundamental properties of \mathcal{K} - and \mathcal{K} -free-modules.

THEOREM 3.5. (a) *A submodule of a \mathcal{K} -module, or a \mathcal{K} -free-module, is a \mathcal{K} -module, or a \mathcal{K} -free-module, respectively.*

(b) *An R -module generated by a family of its \mathcal{K} -submodules, or \mathcal{K} -free-submodules, is a \mathcal{K} -module, or a \mathcal{K} -free-module, respectively.*

(c) *The direct sum of \mathcal{K} -modules, or \mathcal{K} -free-modules, is a \mathcal{K} -module, or a \mathcal{K} -free-module, respectively.*

(d) *The direct product of \mathcal{K} -free-modules is a \mathcal{K} -free-module.*

(e) *An extension of a \mathcal{K} -free-module by a \mathcal{K} -free module is a \mathcal{K} -free-module.*

PROOF. The statements of (a), (b), (c) and (d) are obvious. In order to prove (e), consider an R -module M with a submodule $N \subseteq M$ such that both N and M/N are \mathcal{K} -free. Thus, $O(m) \notin \mathcal{K}$ for $m \in N$; also, if $m \in M \setminus N$, then $O(m) \subseteq O(\bar{m}) \neq R$, where $\bar{m} = m + N \in M/N$, and therefore, in view of $O(\bar{m}) \notin \mathcal{K}$, $O(m) \notin \mathcal{K}$, as well.

THEOREM 3.6. (a) *In any R -module M , $M_{\mathcal{K}}$ is the (unique) maximal \mathcal{K} -submodule of M (and, thus, contains any other \mathcal{K} -submodule of M).*

(b) *Let ϕ be a homomorphism of an R -module M into an R -module M' . Then, the restriction of ϕ to $M_{\mathcal{K}}$ is a homomorphism of $M_{\mathcal{K}}$ into $M'_{\mathcal{K}}$. In particular, every homomorphic image of a \mathcal{K} -module is again a \mathcal{K} -module.*

(c) *For any R -module M , every \mathcal{K} -free homomorphic image of M is a homomorphic image of $M/M_{\mathcal{K}}$.*

PROOF. (a) is evident. Taking into account the obvious fact that

$$O(m) \subseteq O(\phi(m)) \text{ for every } m \in M,$$

we deduce immediately that a homomorphic image of an element of $M_{\mathcal{K}}$ is either zero or an element of order from \mathcal{K} . Hence, both (b) and (c) follow.

THEOREM 3.7. *The following two equivalent statements hold if $\mathcal{K} \in \mathbf{T}$:*

(a) *An extension of a \mathcal{K} -module by a \mathcal{K} -module is a \mathcal{K} -module.*

(b) *For any R -module M , $M/M_{\mathcal{K}}$ is a \mathcal{K} -free-module.*

PROOF. First, let us prove the equivalence of the statements (a) and (b):

(a) \rightarrow (b). Assume that, for a certain R -module M , $M/M_{\mathcal{K}}$ is not \mathcal{K} -free, i.e.

² Another possibility would be to define, for any $\mathcal{K} \subseteq \mathcal{L}$ and any R -module M the \mathcal{K} -submodule and the \mathcal{K} -radical of M as the $c_{\mathcal{F}}(\mathcal{K})$ -submodule and the $c_{\mathcal{F}}(\mathcal{K})$ -submodule (in an obvious meaning) of M , respectively.

$$(M/M_{\mathcal{X}})_{\mathcal{X}} = M^*/M_{\mathcal{X}} \neq \{0\}.$$

Thus, by (a), since $M^*/M_{\mathcal{X}}$ is a \mathcal{X} -module, M^* is a \mathcal{X} -module, as well. But, on the other hand, clearly

$$(M^*)_{\mathcal{X}} = M_{\mathcal{X}} \neq M^*,$$

— a contradiction.

(b) \rightarrow (a). Let M be an R -module with a \mathcal{X} -submodule N such that M/N is a \mathcal{X} -module. Here, $N \subseteq M_{\mathcal{X}}$ and thus,

$$M/M_{\mathcal{X}} \cong (M/N)/(M_{\mathcal{X}}/N);$$

therefore, in view of Theorem 3.6(b), $M/M_{\mathcal{X}}$ is — as a homomorphic image of a \mathcal{X} -module — a \mathcal{X} -module. Also, according to (b), $M/M_{\mathcal{X}}$ is a \mathcal{X} -free-module. Hence, $M = M_{\mathcal{X}}$, as required.

Finally, an application of Theorem 3.3 completes the proof.

Here, before presenting an application of the obtained results, let us point out that a more detailed study of the Q -families, their equivalence classes and properties related to the ‘radical’ properties of modules can be found in [2].

4. Torsion and torsion-free modules

In the preceding section, a general method of defining a ‘torsion’ element of a module and thus, torsion and torsion-free modules was described. In what follows, we suggest a particular choice of such a definition which seems to be the most appropriate at present: a torsion element is an element of so-called maxi order. To justify the latter statement let us express our belief that the elements of essential orders should be classified as torsion elements; then, it turns out that the family \mathcal{T} of all maxi ideals belongs to \mathbf{T} and is the least family containing all proper essential ideals and satisfying the ‘radical’ requirements for $M_{\mathcal{F}}$. Moreover, the corresponding concepts of torsion and torsion-free modules enable us to extend some other features of the abelian group theory to modules (cf. [1]). We shall also show that $M_{\mathcal{F}}$ is, in fact, $Z_2(M)$ of Goldie introduced in [3] under some restricting conditions for R and M .

Let us start with the definition of a maxi ideal.

DEFINITION 4.1. An ideal $L \in \mathcal{L}$ is said to be maxi if, for each $\rho \in R \setminus L$, there is $\sigma \in R$ such that $L : \sigma\rho \neq R$ is essential. Also, $L \in \mathcal{L}$ is said to be a mini ideal if no quotient ideal $L : \rho \neq R$ is essential.

Denoting by \mathcal{E} , \mathcal{T} and \mathcal{F} the families of all proper essential ideals, all maxi ideals and all mini ideals, respectively, we can readily see that

$$\partial\mathcal{E} = \mathcal{F} \text{ and } \partial^2\mathcal{E} = \partial\mathcal{F} = \mathcal{F}.$$

Thus, since evidently $\mathcal{E} \in \mathbf{F}$, we get by virtue of Theorem 3.2 the following

THEOREM 4.2. *The family of all maxi ideals \mathcal{T} belongs to \mathbf{T} and thus, for any R -module M , $M_{\mathcal{T}} \subseteq M$ possesses the properties described in Theorems 3.5, 3.6 and 3.7.*

It is easy to see that $L \in \mathcal{L}$ is a maxi ideal if and only if the submodule $(R \bmod L)_{\mathcal{E}}$ is essential in $R \bmod L$. From here and the fact that an ideal $E \supseteq L$ is essential in R if and only if the submodule $E \bmod L$ is essential in $R \bmod L$, the two statements mentioned in the introduction to this section follow immediately ((a) follows also from Theorem 3.3):

THEOREM 4.3. (a) *Every F -family \mathcal{K} containing \mathcal{E} such that $M_{\mathcal{K}} \subseteq M$ satisfies the ‘radical’ properties of Theorem 3.7 contains \mathcal{T} (and is a T -family).*

(b) *An ideal $L \in \mathcal{L}$ is a maxi ideal if and only if there is an essential ideal E in R such that $L : \rho$ is essential for every $\rho \in E$.*

At the end, we like to include the following two remarks:

Consider briefly the case when R is the ring of all integers, i.e. the case of abelian groups G . Of course, $G_{\mathcal{T}}$ is the maximal torsion subgroup of G . Applying the method of § 3, we can show readily that any \mathcal{K} -submodule satisfying ‘radical’ properties of Theorem 3.7 is derived from $\mathcal{K} \in \mathbf{T}$, and that $G_{\mathcal{T}}$ is the ‘greatest radical’: Denote by Π^* a subset of the set Π of all primes and by \mathcal{P}^* the family of all (principal) ideals of the form

$$\langle p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} \rangle \text{ with } k_i > 0 \text{ and } p_i \in \Pi^* \quad (1 \leq i \leq n).$$

Clearly, $\mathcal{P}^* \in \mathbf{F}$ and, in fact, $\mathcal{P}^* \in \mathbf{T}$. Thus, for any selection Π^* of primes there is a \mathcal{P}^* -radical $G_{\mathcal{P}^*}$ of G :

If $G_{\mathcal{T}} = \sum_{p \in \Pi} G_p$ is the decomposition into the p -primary components, then $G_{\mathcal{P}^*} = \sum_{p \in \Pi^*} G_p$.

It is easy to see that every ‘radical’ must be of this form. In particular, $G_{\mathcal{P}^*} = \{0\}$ for $\Pi^* = \emptyset$ and $G_{\mathcal{P}^*} = G_{\mathcal{T}}$ for $\Pi^* = \Pi$.

The other remark concerns the fact that for some rings R , the families \mathcal{T} and \mathcal{E} of all maxi and all essential left ideals of R , respectively, coincide. As a matter of fact, we can give a simple necessary and sufficient condition for the equality $\mathcal{T} = \mathcal{E}$:

THEOREM 4.4. *The equality $\mathcal{T} = \mathcal{E}$ holds if and only if the singular ideal S of R equals to $\{0\}$; here, S is the ideal of all elements of R whose left annihilators are essential in R .*

Indeed, the sufficiency (cf. [3]) follows immediately from the fact that, for any non-essential $L \in \mathcal{T}$, there is a non-zero $\rho \in R$ such that

$$L : \rho = \{0\} : \rho$$

Hence, there exists $\sigma \in R$ such that $L : \sigma\rho \neq R$ is essential in R ; thus, $O \neq \sigma\rho \in S$, i.e. $S \neq \{0\}$.

In order to prove the necessity, assume that $S \neq \{0\}$. Denote by L a maximal left ideal of R such that $L \cap S = \{0\}$. Clearly, the left ideal E generated by L and S is essential in R . Also, for any $\rho \in E$, $\rho = \lambda + \sigma$ with $\lambda \in L$ and $\sigma \in S$,

$$L : \rho \supseteq \{0\} : \sigma$$

is essential in R . Hence, in view of Theorem 4.3(b), $L \in \mathcal{T}$. Since $L \notin \mathcal{E}$, we get $\mathcal{T} \neq \mathcal{E}$.

In conclusion, let us note that the condition $S = \{0\}$ is satisfied in a ring R if 0 is the only nilpotent element of R and the ascending chain condition holds for the annihilator left ideals of single elements of R . For, in any ring R , the assumption that $\rho \in R$ is not nilpotent and $\{0\} : \rho$ is essential in R implies readily that

$$\{0\} : \rho^{n+1} \neq \{0\} : \rho^n \text{ for every natural } n.$$

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