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The splitting of separatrices for analytic diffeomorphisms

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Abstract. We study families of diffeomorphisms close to the identity, which tend to it when the parameter goes to zero, and having homoclinic points. We consider the analytical case and we find that the maximum separation between the invariant manifolds, in a given region, is exponentially small with respect to the parameter. The exponent is related to the complex singularities of a flow which is taken as an unperturbed problem. Finally several examples are given.

1. *Introduction*

In a previous paper [**5**] we considered families of differentiable diffeomorphisms with hyperbolic points which reduce to the identity for a certain value of the parameter. We studied the separation between the stable and unstable manifolds near homo-heteroclinic points associated with the hyperbolic ones. Now we shall study the analytic and conservative case. We refer the reader to [**5**] for the motivations of this problem. The results were inspired by the previous work by Lazutkin [**10**] for the standard map.

We study the complex invariant manifolds through the Birkhoff normal form of a diffeomorphism in a neighbourhood of a hyperbolic point. For that we need uniform behaviour of the normal form with respect to the parameter of the family. As in the differentiable case we use an auxiliary family of diffeomorphisms defined through the flow of an autonomous vector field with a homoclinic orbit. We compare the invariant manifolds of the two families, first locally and then globally. From the Birkhoff normal form we get a local first integral for the diffeomorphisms, which can be extended in a neighbourhood of the invariant manifolds as a multivalued function. With a suitable parametrization of the invariant manifolds, the evaluation of the first integral extended along one manifold on the other manifold gives a periodic function. The Fourier coefficients of that function can be bounded by using integration over complex paths. From that we get the bound of the separation between the invariant manifolds. Now we state the main result in a precise way.

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If F is a diffeomorphism defined in a neighbourhood of $p \in \mathbb{C}^n$ and p is a hyperbolic fixed point of F, we denote by $W^s_F(p)$ and $W^u_F(p)$ the stable and unstable invariant manifolds of p for F. When no confusion is possible we shall skip some indices.

Let $F_\varepsilon : U \to \mathbb{R}^2$ be a family of diffeomorphisms with $U \subset \mathbb{R}^2$ and $0 < \varepsilon < \varepsilon_0$ having the form

$$F_\varepsilon(x) = A(x) + \varepsilon^\alpha f(x) + \varepsilon^{\alpha+1} g(x, \varepsilon) \tag{1.1}$$

with $A(x) = (\lambda x_1, \lambda^{-1} x_2)$, $f(0) = g(0, \varepsilon) = 0$, $Df(0) = D_x g(0, \varepsilon) = 0$, $\alpha > 0$, and verifying

$H1$ F_ε preserves area,

$H2$ F_ε is real analytic in U and depends analytically with respect to ε,

$H3$ $\lambda = 1 + a\varepsilon^\alpha + O(\varepsilon^{\alpha+1})$, $a > 0$ and $\alpha \in \mathbb{N}$.

From (1.1) and $H3$ the origin is a hyperbolic fixed point of F_ε (if ε_0 is small enough). Let W^s and W^u be the corresponding invariant manifolds.

$H4$ For all $\varepsilon \in (0, \varepsilon_0)$ there exists a homoclinic point, q_ε, associated with the origin such that the pieces of W^u and W^s from the origin to q_ε are contained in a compact set contained in U. (We include the trivial case when two branches of the invariant manifolds coincide.)

We can prove

PROPOSITION. *Under hypotheses $H1$–$H4$ the vector field given by*

$$\dot{x} = x + \frac{1}{a} f_1(x, y),$$

$$\dot{y} = -y + \frac{1}{a} f_2(x, y),$$

where f_1 and f_2 denote the components of f, is conservative, has the origin as a hyperbolic point and has an analytic homoclinic orbit σ such that for ε small enough, the real invariant manifolds of F_ε are *ε-close to $\sigma(\mathbb{R})$.*

Let δ_0 be the distance from the real axis to the nearest singularity of σ to it. Furthermore we suppose that (1.1) verifies

$H5$ F_ε can be extended analytically to a neighbourhood U of $\{\sigma(t), |\mathrm{Im}\, t| < \delta\}$, with $0 < \delta < \delta_0$.

THEOREM A. *Let F_ε be as* (1.1) *verifying hypothesis* H1–H5. *Then given $p \in \sigma$ there exists a neighbourhood V of p not depending on ε such that for $\eta > 0$ there exist $\varepsilon_0 > 0$ and $N > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the distance between W^s and W^u in V is less than*

$$N \exp(-2\pi(\delta - \eta)/\ln \lambda).$$

We notice that the bound obtained in Theorem A is exponentially small with respect to ε and that such a bound cannot be obtained by a classical theory of perturbations.

We remark that we only consider pieces of W^s, W^u that have only one component in a small fixed neighbourhood of the origin. That is, we do not consider successive approaches of the invariant manifolds to the origin.

For families of diffeomorphisms obtained as Poincaré maps of Hamiltonian systems with two degrees of freedom or nonautonomous perturbations of conservative planar vector fields we can think of applying the Melnikov method [**14**, **6**]. Although we could compute the Melnikov function, the validity of the method is not proved for the situation described in this paper [**17**].

The form (1.1) may seem restrictive but in general families with hyperbolic and homo-heteroclinic points such that for some value of the parameter a hyperbolic point becomes parabolic can be put in form (1.1) through linear (ε depending) changes of variables.

In the examples studied we have found numerically that the angle between the invariant manifolds at homoclinic points (which is related to the maximum separation between them) behaves asymptotically as

$$A(\ln\lambda)^B \exp(-C/\ln\lambda),$$

with $A>0$, B real and $\operatorname{Re} C>0$. Furthermore we have found that $\operatorname{Re} C = 2\pi\delta_0$ so that the exponential part of the bound in Theorem A is optimal. Lazutkin has proved this kind of behaviour for the standard map [**10**].

The authors are aware of very interesting recent results in the same direction obtained by Holmes, Marsden and Scheurle announced in [**9**]. They consider the case of high frequency periodic time depending perturbations of 1 degree of freedom Hamiltonian systems.

In § 2 we recall the Birkhoff normal form for analytic diffeomorphisms and we give a parametrization of the invariant manifolds to be used later. In § 3 we prove the uniform behaviour of the normal form. In § 4 we study the proximity of the local invariant manifolds of diffeomorphisms of two families. Next, in § 5 we prove Theorem A and finally in § 6 we present several examples: the conservative (orientation preserving) Hénon map, the Duffing equation perturbed with a high frequency periodic function, a generalized standard map and the Hénon-Heiles problem.

We shall use the definitions and notations introduced in [**5**]. However we indicate that we shall represent the modulus of a complex number z by $|z|$. If

$$F:\mathbb{C}^n\to\mathbb{C}^n,\ |F(z)|=\left(\sum_{k=1}^{n}|F_k(z)|^2\right)^{1/2} \quad \text{where } F_k(z)\in\mathbb{C}.$$

Finally we shall write $|F|_U<K$ if $\sup_{z\in U}|F(z)|<K$.

2. *Birkhoff normal form and invariant manifolds*

First we recall the Birkhoff normal form of an analytic diffeomorphism in a neighbourhood of a hyperbolic fixed point.

THEOREM 2.1. [**18**]. *Let $U\subset\mathbb{C}^2$ an open set containing the origin and $F:U\to\mathbb{C}^2$ be an area preserving analytic diffeomorphism which takes the form*

$$F(x,y)=(\lambda x+P(x,y),\ \lambda^{-1}y+Q(x,y)), \tag{2.1}$$

with $P(0,0)=Q(0,0)=0$, $DP(0,0)=DQ(0,0)=0$ and $\lambda>1$. Then there exists an analytic change of variables C (non unique) defined in a neighbourhood of the origin *such that $N = C^{-1}FC$ takes the form*

$$N(\xi, \eta) = (\xi W(\xi\eta), \eta / W(\xi\eta)), \tag{2.2}$$

where $W(\xi\eta) = \lambda + \alpha_2\xi\eta + \alpha_4(\xi\eta)^2 + \cdots$ is analytic.

If F is real analytic so are N and C.

Furthermore N is unique.

N is called Birkhoff normal form of F at the origin.

Remarks

(1) N is area preserving but C need not to be. However there is always an area preserving C such that $N = C^{-1}FC$.

(2) If we write $C(\xi, \eta) = (\Phi(\xi, \eta), \Psi(\xi, \eta))$, C is unique provided that the Taylor series of $D_\xi\Phi - 1$ and $D_\eta\Psi - 1$ do not contain powers of $\xi\eta$ alone.

From (2.2) we have that $N(\xi, 0) = (\lambda\xi, 0)$ and $N(0, \eta) = (0, \lambda^{-1}\eta)$ and hence $\xi = 0$ and $\eta = 0$ represent $W^s_N(0,0)$ and $W^u_N(0,0)$ respectively. Then $C(0, \eta)$ and $C(\xi, 0)$ are parametrizations of $W^s_F(0,0)$ and $W^u_F(0,0)$. C is not unique but nevertheless we have

PROPOSITION 2.2. *Under the hypothesis of Theorem 2.1, if we suppose that $DC(0,0) = I$ then $C(\xi, 0)$ and $C(0, \eta)$ are unique.*

Proof. We prove that $C(\xi, 0)$ is unique. The proof for $C(0, \eta)$ is analogous. From $CN = FC$ we have

$$CN(\xi, 0) = C(\lambda\xi, 0) = FC(\xi, 0). \tag{2.3}$$

Let $C(\xi, 0) = (\sum_{n\geq 1} a_n\xi^n, \sum_{n\geq 1} b_n\xi^n)$ with $a_1 = 1$ and $b_1 = 0$ and $F(x, y) = (\sum_{k,l\geq 1} c_{kl}x^ky^l, \sum_{k,l\geq 1} d_{kl}x^ky^l)$ with $c_{10} = \lambda$, $d_{01} = \lambda^{-1}$ and $c_{01} = d_{10} = 0$.

The first order terms in (2.3) agree. The second order terms verify

$$a_2\lambda^2 = \lambda a_2 + c_{20},$$
$$b_2\lambda^2 = \lambda^{-1}b_2 + d_{20},$$

so that a_2 and b_2 are uniquely determined since λ is not a root of 1.

Assuming that a_k and b_k are uniquely determined until order $n-1$ we have

$$a_n\lambda^n = \lambda a_n + \text{known terms},$$
$$b_n\lambda^n = \lambda^{-1}b_n + \text{known terms},$$

which as before determine a_n and b_n uniquely. □

As a consequence of Theorem 2.1 we shall obtain new parametrizations of $W^s_F(0,0)$ and $W^u_F(0,0)$.

PROPOSITION 2.3. *Let $U \subset \mathbb{C}^2$ be an open set containing the origin and $F: U \to \mathbb{C}^2$ be an area preserving analytic diffeomorphism. Suppose that the origin is a hyperbolic fixed point of F. Then*

(1) *There exist analytic functions $u: V_1 \subset \mathbb{C} \to \mathbb{C}^2$ and $v: V_2 \subset \mathbb{C} \to \mathbb{C}^2$ such that $W^u_F(0,0) = \{u(t), t \in V_1\}$ and $W^s_F(0,0) = \{v(t), t \in V_2\}$. Furthermore, if $u(t)$, $v(t) \in U$, $F(u(t)) = u(t+h)$ and $F(v(t)) = v(t+h)$ where $h = \ln\lambda$ and λ is the eigenvalue of $DF(0,0)$ bigger than 1.*

(2) *There exists an analytic function E defined in a neighbourhood of the invariant manifolds such that $E(F(x,y))=E(x,y)$ and $E(u(t))=E(v(t))=0$.*

Proof. There exists a linear change C_1 such that $\bar{F}=C_1^{-1}FC_1$ is in form (2.1). Then by Theorem 2.1 there exists C_2 such that $N=C_2^{-1}\bar{F}C_2$ is in Birkhoff normal form. Let $C=C_2C_1$. Let $r>0$ be such that N and C are analytic in $B(r)$. We represent $W_F^u(0,0)$ by $u(t)=C(\exp(t-t_0),0)$ for $\operatorname{Re} t<\operatorname{Re} t_0+\ln r$ and $W_F^s(0,0)$ by $v(t)=C(0,\exp(t_0'-t))$ for $\operatorname{Re} t>\operatorname{Re} t_0'-\ln r$. We have that

$$\begin{aligned} F(u(t)) &= FC(\exp(t-t_0),0)=CN(\exp(t-t_0),0)\\ &= C(\exp(t-t_0+\ln\lambda),0)=u(t+h) \end{aligned}$$

and the same holds for $v(t)$. This relation lets us continue analytically u while $u(t)$ belongs to the domain of F.

Let $\bar{E}(\xi,\eta)=\xi\eta$. We define $E=\bar{E}\circ C^{-1}$. Since $\bar{E}\circ N=\bar{E}$ we have that

$$E\circ F=\bar{E}\circ C^{-1}\circ F=\bar{E}\circ N\circ C^{-1}=\bar{E}\circ C^{-1}=E.$$

Furthermore $E(u(t))=\bar{E}\circ C^{-1}\circ C(\exp(t-t_0),0)=0$. The expression $E\circ F=E$ allows us to continue analytically E to a neighbourhood of invariant manifolds. If there are homoclinic points E will be, in general, a multivalued function. □

Remark. From the proof it is clear that if F is an entire function so are u and v.

3. *Uniform behaviour of the Birkhoff normal form for families of diffeomorphisms*

In the last section we dealt with only one diffeomorphism. Now we will study the Birkhoff normal form for a near the identity family.

PROPOSITION 3.1. *Let $U\subset\mathbb{C}^2$ be an open set which contains the origin and $F_\varepsilon:U\to\mathbb{C}^2$ be a family of area preserving analytic diffeomorphisms depending analytically on ε around $\varepsilon=0$ of the form*

$$F_\varepsilon(x,y)=A(x,y)+\varepsilon^\alpha f(x,y)+\varepsilon^{\alpha+1}g(x,y,\varepsilon), \tag{3.1}$$

where $A(x,y)=(\lambda x,\lambda^{-1}y)$, $f(0,0)=g(0,0,\varepsilon)=0$, $Df(0,0)=Dg(0,0,\varepsilon)=0$ and $\lambda=1+a\varepsilon^\alpha+O(\varepsilon^{\alpha+1})$ with $a>0$, $\alpha>0$. Then there exist $\varepsilon_0>0$ and $r_0>0$ (independent of ε) such that if $0<\varepsilon<\varepsilon_0$ the change C_ε and the normal form N_ε given by Theorem 2.1 are analytic in $B(r_0)$.

The proof closely follows that of the convergence of the Birkhoff normal form in [**18**]. Before beginning the proof we introduce some notation. Given an analytic function S we write

$$S(\xi,\eta)=\sum_{k,l\ge 0}a_{k,l}\xi^k\eta^l=\sum_{n=-\infty}^{\infty}S_n(\xi,\eta),$$

where S_n is the sum of the terms $a_{kl}\xi^k\eta^l$ with $k-l=n$, $n\in Z$. Notice that if $u=u(\xi\eta)$ with $u(0)\neq 0$,

$$S_n(u\xi,u^{-1}\eta)=u^nS_n(\xi,\eta).$$

We shall represent the series $\sum_{k,l\ge 0}|a_{kl}|\xi^k\eta^l$ by $\overline{|S|}$, $\overline{|S|}(\xi,\eta)$ or $\overline{|S(\xi,\eta)|}$. If

$$S(\xi,\eta)=\sum_{k,l\ge 0}a_{kl}\xi^k\eta^l,\quad \hat{S}(\xi,\eta)=\sum_{k,l\ge 0}b_{kl}\xi^k\eta^l\quad\text{and}\quad |a_{kl}|\le b_{kl}\quad\text{for all } k,l\ge 0$$

we shall say that $\hat{S}$ is a majorant of S and we shall write $S\prec\hat{S}$.

We shall use an analogous notation for a series of one or more variables.

Proof. In order to simplify the notation we shall not write the dependence of C_ε and N_ε and the quantities related to them with respect to ε. We write

$$F(x, y) = A(x, y) + (P(x, y), Q(x, y)),$$
$$N(\xi, \eta) = (\xi u(\xi\eta), \eta v(\xi\eta)),$$

with

$$u(\xi\eta) = \lambda + \alpha_2\xi\eta + \alpha_4(\xi\eta)^2 + \cdots \quad \text{and } v(\xi\eta) = 1/u(\xi\eta),$$
$$C(\xi, \eta) = \left(\sum_{n\in Z} \Phi_n(\xi, \eta), \sum_{n\in Z} \Psi_n(\xi, \eta)\right).$$

It is formally proved in [18] that $N = C^{-1}FC$ and that C is unique if

$$D_\xi\Phi - 1, \quad D_\eta\Psi - 1 \tag{3.2}$$

do not contain powers of $\xi\eta$ alone.

We write

$$P(x, y, \varepsilon) = \varepsilon^\alpha \sum_{\substack{k+l\geq 2\\ h\geq 0}} b_{klh}x^k y^l \varepsilon^h.$$

There exist constants M_1, x_0, y_0, ε_1 such that $|b_{klh}| < M_1/x_0^k y_0^l \varepsilon_1^h$. Let ε_0 be such that $\varepsilon_0/\varepsilon_1 < 1$, and we define $\bar{b}_{kl}(\varepsilon) = \sum_{h\geq 0} b_{klh}\varepsilon^h$ for $|\varepsilon| < \varepsilon_0$. It is clear that

$$|\bar{b}_{kl}| \leq \sum_{h\geq 0} (M_1/x_0^k y_0^l \varepsilon_1^h)\varepsilon^h = M_2/x_0^h y_0^l \quad \text{with } M_2 = M_1/(1-\varepsilon_0/\varepsilon_1).$$

Let $c_{kl} = M_2/x_0^k y_0^l$. Then $\varepsilon^\alpha \sum_{k+l\geq 2} c_{kl}x^k y^l$ is a majorant of P which in turn has a majorant of the form

$$G(x, y) = \varepsilon^\alpha c_0(x+y)^2/(1-c_1(x+y)),$$

with c_0 and c_1 positive and independent of ε. We assume that c_0 and c_1 are chosen in such a way that G is also a majorant of Q. Writing $CN = FC$ explicitly

$$\Phi(u\xi, v\eta) = P(\Phi, \Psi) + \lambda\Phi,$$
$$\Psi(u\xi, v\eta) = Q(\Phi, \Psi) + \lambda^{-1}\Psi,$$

and taking into account that $\Phi_n(u\xi, u^{-1}\eta) = u^n\Phi_n(\xi, \eta)$ we have

$$\begin{aligned}(u^n - \lambda)\Phi_n &= (P(\Phi, \Psi))_n,\\ (u^n - \lambda^{-1})\Psi_n &= (Q(\Phi, \Psi))_n, \quad n \in Z.\end{aligned} \tag{3.3}$$

From (3.2) it is easy to see that $\Phi_1(\xi, \eta) = \xi$ and $\Psi_{-1}(\xi, \eta) = \eta$.

Now we shall prove that there exists $c_2 > 0$ (independent of ε) such that

$$(u^n - \lambda)^{-1} < c_2\varepsilon^{-\alpha}/(1 - c_2\varepsilon^{-\alpha}s), \quad n \neq 1 \tag{3.4}$$

$$(u^n - \lambda^{-1})^{-1} < c_2\varepsilon^{-\alpha}/(1 - c_2\varepsilon^{-\alpha}s), \quad n \neq -1 \tag{3.5}$$

where $s = \overline{|v - \lambda^{-1}|}$. Indeed, if $n < 0$,

$$(u^n - \lambda)^{-1} = -\lambda^{-1}(1 - v^{|n|}\lambda^{-1})^{-1} < \lambda^{-1} \sum_{k\geq 0} (\overline{|v|}^{|n|}\lambda^{-1})^k$$

$$< \frac{1}{1 - \overline{|v|}} < \frac{1}{1 - \lambda^{-1} - s} < \frac{c_2\varepsilon^{-\alpha}}{1 - c_2\varepsilon^{-\alpha}s},$$

where $c_2 \geq \varepsilon^\alpha/(1 - \lambda^{-1})$, the final bound being directly checked for $n = 0$.

If $n>1$

$$
\begin{aligned}
(u^n-\lambda)^{-1} &= u^{-n}(1-\lambda u^{-n})^{-1} < \overline{|v|}^n \sum_{k\ge 0} (\lambda \overline{|v|^n})^k \\
&< \sum_{k\ge 0} (\lambda \overline{|v|^n})^k < \sum_{k\ge 0} (\lambda^{1/n} \overline{|v|})^k \\
&= \frac{1}{1-\lambda^{1/n}\overline{|v|}} < \frac{1}{1-\lambda^{1/2}\overline{|v|}} < \frac{c_2\varepsilon^{-\alpha}}{1-c_2\varepsilon^{-\alpha}s},
\end{aligned}
$$

where $c_2 \ge \lambda\varepsilon^{\alpha}/(\lambda^{1/2}-1)$. (3.5) is proved analogously.

From (3.3) $[c_2\varepsilon^{-\alpha}/(1-c_2\varepsilon^{-\alpha}s)]G(\overline{|\Phi|}, \overline{|\Psi|})$ is a majorant of both $\overline{|\Phi|}-\xi$ and $\overline{|\Psi|}-\eta$ and $G(\overline{|\Phi|}, \overline{|\Psi|})$ can also be taken as a majorant of $\eta s = \eta|v-\lambda^{-1}|$.

We take $\xi=\eta$ and defining $W(\xi) = (\overline{|\Phi|}-\xi+\overline{|\Psi|}-\xi)/\xi + c_2\varepsilon^{-\alpha}s$ we look for a majorant of W. First

$$
\xi W(\xi) < \frac{2c_2\varepsilon^{-\alpha}}{1-c_2\varepsilon^{-\alpha}s} G(\overline{|\Phi|}, \overline{|\Psi|}) + c_2\varepsilon^{-\alpha}\xi s < \frac{3c_2\varepsilon^{-\alpha}}{1-c_2\varepsilon^{-\alpha}s} G(\overline{|\Phi|}, \overline{|\Psi|}).
$$

By the definition of W, $(\overline{|\Phi|}-\xi)/\xi < W$, $(\overline{|\Psi|}-\xi)/\xi < W$ and $c_2\varepsilon^{-\alpha}s < W$ so that $\overline{|\Phi|} < \xi(W+1)$. Then

$$
\begin{aligned}
\xi W(\xi) &< \frac{3c_2\varepsilon^{-\alpha}}{1-c_2\varepsilon^{-\alpha}s} G(\xi(W+1), \xi(W+1)) < \frac{3c_2\varepsilon^{-\alpha}}{1-W} \frac{\varepsilon^{\alpha}c_0 4\xi^2(W+1)^2}{1-2c_1\xi(W+1)} \\
&< \frac{12c_0c_2\xi^2(W+1)^2}{1-W-2c_1\xi(W+1)}
\end{aligned}
$$

so that

$$
W(\xi) < \frac{12c_0c_2\xi(W+1)^2}{1-W-2c_1\xi(W+1)}.
$$

We define V such that

$$
V(\xi) = \frac{12c_0c_2\xi(V+1)^2}{1-V-2c_1\xi(V+1)}.
$$

It is readily seen that $W(\xi) < V(\xi)$. V satisfies a second order equation whose coefficients are independent of ε. Hence it is analytic in a ball $B(r_0)$ with r_0 independent of ε and on this ball it is bounded by some constant independent of ε. From the previous majorants we get that N and C are analytic in $B(r_0)$ and that $N_\varepsilon(\xi,\eta) = (\lambda\xi, \lambda^{-1}\eta) + O(\varepsilon^{\alpha})$ and $C_\varepsilon(\xi,\eta) = (\xi,\eta) + O(\varepsilon^0)$. □

4. *Proximity of the local invariant manifolds*

In this section we shall compare the local invariant manifolds of two families of diffeomorphisms. First we prove

PROPOSITION 4.1. *Let F_ε and G_ε be two families of diffeomorphisms as in Theorem 3.1. Suppose that $|F_\varepsilon - G_\varepsilon|_U = O(\varepsilon^{\alpha+1})$ and that* Spec $DF_\varepsilon(0,0) =$ Spec $DG_\varepsilon(0,0)$. *Then, if $C_{1\varepsilon}$ and $C_{2\varepsilon}$ are changes that transform F_ε and G_ε to their normal forms, there* *exist* ε_0, r, $K>0$, *independents of* ε, *such that*

$$|C_{1\varepsilon}(\xi,0)-C_{2\varepsilon}(\xi,0)|<K\varepsilon,$$
$$|C_{1\varepsilon}(0,\eta)-C_{2\varepsilon}(0,\eta)|<K\varepsilon,$$

for $|\xi|,|\eta|<r$ and $0<\varepsilon<\varepsilon_0$.

Proof. As before we shall not write the dependence of ε. In this proof if T is analytic we shall write

$$T(\xi,\eta)=\sum_{k,l\geq 0} a_{kl}\xi^k\eta^l=\sum_{n\geq 0} T(\xi,\eta))_n \quad \text{where}$$
$$(T(\xi,\eta))_n=\sum_{k+l=n} a_{kl}\xi^k\eta^l.$$

We notice that

$$(T(\lambda\xi,0))_n=\lambda^n(T(\xi,0))_n. \tag{4.1}$$

We introduce the following notation

$$F(x,y)=(\lambda x,\lambda^{-1}y)+(P_1(x,y),Q_1(x,y)),$$
$$G(x,y)=(\lambda x,\lambda^{-1}y)+(P_2(x,y),Q_2(x,y)),$$
$$C_1(\xi,\eta)=(\Phi_1(\xi,\eta),\Psi_1(\xi,\eta)),$$
$$C_2(\xi,\eta)=(\Phi_2(\xi,\eta),\Psi_2(\xi,\eta)).$$

From $C_1N_1=FC_1$ and $C_2N_2=GC_2$ we have

$$(\Phi_i(\lambda\xi,0),\Psi_i(\lambda\xi,0))=(P_i(\Phi_i,\Psi_i)+\lambda\Phi_i,\,Q_i(\Phi_i,\Psi_i)+\lambda^{-1}\Psi_i),\quad i=1,2, \tag{4.2}$$

where here and from now on all the expressions are evaluated at $(\xi,0)$ except the ones given explicitly. From the nth order homogeneous parts of (4.2) and taking into account (4.1) we have

$$\begin{aligned}(\lambda^n-\lambda)&(\Phi_1(\xi,0)-\Phi_2(\xi,0))_n\\ &=(P_1(\Phi_1,\Psi_1))_n-(P_2(\Phi_2,\Psi_2))_n=(P_1(\Phi_1,\Psi_1)-P_1(\Phi_1,\Psi_2))_n\\ &\quad+(P_1(\Phi_1,\Psi_2)-P_1(\Phi_2,\Psi_2))_n+(P_1(\Phi_2,\Psi_2)-P_2(\Phi_2,\Psi_2))_n.\end{aligned}$$

We look for a majorant of $\Phi_1(\xi,0)-\Phi_2(\xi,0)$. First we observe that

$$\begin{aligned}(P_1(x_1,y_1)&-P_1(x_1,y_2))/(y_1-y_2)\\ &=(\textstyle\sum a_{kl}(x_1^ky_1^l-x_1^ky_2^l))/(y_1-y_2)\\ &=\textstyle\sum a_{kl}x_1^k(y_1^{l-1}+y_2^{l-2}+\cdots+y_2^{l-1})<\sum|a_{kl}|x_1^k(y_1+y_2)^{l-1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad<\overline{|D_yP_1(x_1,y_1+y_2)|}\end{aligned}$$

and analogously

$$(P_1(x_1,y_2)-P_1(x_2,y_2))/(x_1-x_2)<\overline{|D_xP_1(x_1+x_2,y_2)|}.$$

Then

$$\begin{aligned}(\Phi_1&(\xi,0)-\Phi_2(\xi,0))_n\\ &<\frac{1}{\lambda^n-\lambda}[(\overline{|D_yP_1|}(\overline{|\Phi_1|},\overline{|\Psi_1|}+\overline{|\Psi_2|})\overline{|\Psi_1-\Psi_2|})_n\\ &\quad+(\overline{|D_xP_1|}(\overline{|\Phi_1|}+\overline{|\Phi_2|},\overline{|\Psi_2|})\overline{|\Phi_1-\Phi_2|})_n+(\overline{|P_1-P_2|}(\overline{|\Phi_2|},\overline{|\Psi_2|}))_n].\end{aligned}$$

Since $\lambda = 1 + a\varepsilon^\alpha + \cdots$ there exists $0 < a' < a$ such that if ε is small enough $\lambda > 1 + a'\varepsilon^\alpha$ and hence $\lambda^n - \lambda > a'\varepsilon^\alpha$ for $n \geq 2$. From that we have

$$\begin{aligned}\overline{|\Phi_1(\xi,0) - \Phi_2(\xi,0)|} < \frac{1}{a'\varepsilon^\alpha}[&\overline{|D_y P_1|}(\overline{|\Phi_1|}, \overline{|\Psi_1|} + \overline{|\Psi_2|})\overline{|\Psi_1 - \Psi_2|} \\ &+ \overline{|D_x P_1|}(\overline{|\Phi_1|} + \overline{|\Phi_2|}, \overline{|\Psi_2|})\overline{|\Phi_1 - \Phi_2|} \\ &+ \overline{|P_1 - P_2|}(\overline{|\Phi_2|}, \overline{|\Psi_2|})]\end{aligned}$$

and an analogous expression for $\overline{|\Psi_1(\xi,0) - \Psi_2(\xi,0)|}$. P_1 and Q_1 are of the form

$$\varepsilon^\alpha \sum_{\substack{k+l\geq 2\\ h\geq 0}} b_{klh} x^k y^l \varepsilon^h.$$

By Proposition 3.1, Φ_1, Φ_2, Ψ_1 and Ψ_2 are analytic in a neighbourhood of the origin and have the form $(\xi, \eta) + O(\varepsilon^0)$. Then there exists r_0 such that

$$\overline{|D_y P_1|}(\overline{|\Phi_1|}, \overline{|\Psi_1|} + \overline{|\Psi_2|}), \quad \overline{|D_x P_1|}(\overline{|\Phi_1|} + \overline{|\Phi_2|}, \overline{|\Psi_2|}), \quad \overline{|D_y Q_1|}(\overline{|\Phi_1|}, \overline{|\Psi_1|} + \overline{|\Psi_2|}) \quad \text{and} \quad \overline{|D_x Q_1|}(\overline{|\Phi_1|} + \overline{|\Phi_2|}, \overline{|\Psi_2|})$$

are analytic for $|\xi| < r_0$. Let $\varepsilon^\alpha c_1\xi/(1 - c_1\xi)$ be a common majorant. $P_1 - P_2$ and $Q_1 - Q_2$ have a common majorant of the form $\varepsilon^{\alpha+1} c_2(x+y)^2/(1 - c_2(x+y))$. We define $W(\xi) = \overline{|\Phi_1 - \Phi_2|}(\xi, 0) + \overline{|\Psi_1 - \Psi_2|}(\xi, 0)$. It is clear from the last observations that

$$W(\xi) < [(\varepsilon^\alpha c_1\xi/(1 - c_1\xi))\, W(\xi) + 2\varepsilon^{\alpha+1} c_2(\Phi_2 + \Psi_2)^2/(1 - c_2(\Phi_2 + \Psi_2))]/(a'\varepsilon^\alpha).$$

Let $c_3\xi^2/(1 - c_3\xi)$ be a majorant of $c_2(\Phi_2 + \Psi_2)^2/(1 - c_2(\Phi_2 + \Psi_2))$. We define V such that

$$V(\xi) = [(c_1\xi/(1 - c_1\xi))\, V(\xi) + 2\varepsilon c_3\xi^2/(1 - c_3\xi)]/a'.$$

It is immediate that

$$V(\xi) = 2\varepsilon c_3\xi^2(1 - c_1\xi)/[(1 - c_3\xi)(a' - c_1\xi(1 + a'))],$$

where the constants c_i are independent of ε. Then there exist r, $K_1 > 0$ such that

$$|(\Phi_1 - \Phi_2)(\xi, 0)| < \overline{|\Phi_1 - \Phi_2|}(|\xi|, 0) < K_1\varepsilon,$$
$$|(\Psi_1 - \Psi_2)(\xi, 0)| < \overline{|\Psi_1 - \Psi_2|}(|\xi|, 0) < K_1\varepsilon,$$

for $|\xi| < r$. From that we get $|(C_1 - C_2)(\xi, 0)| < \sqrt{2}K_1\varepsilon$ for $|\xi| < r$ and analogously $|(C_1 - C_2)(0, \eta)| < \sqrt{2}K_2\varepsilon$. We take $K = \max(\sqrt{2}K_1, \sqrt{2}K_2)$. □

PROPOSITION 4.2. *Let F_ε be as in Proposition 3.1 and G_ε be the flow time $h = \ln \lambda$ of a conservative vector field X of the form*

$$\begin{aligned}\dot{x} &= x + g_1(x, y) \\ \dot{y} &= -y + g_2(x, y),\end{aligned} \tag{4.3}$$

with $Dg_1(0,0) = Dg_2(0,0) = 0$. Suppose that X has a real homoclinic orbit, σ, and suppose that $|F_\varepsilon - G_\varepsilon|_U = O(\varepsilon^{\alpha+1})$. Let u and v be the parametrizations of W_F^u and W_F^s given by Proposition 2.3. Then there exists r, $K > 0$ independent of ε such that

$$|u(t) - \sigma(t)| < K\varepsilon \quad \text{for Re } t < \ln r + t_0,$$
$$|v(t) - \sigma(t)| < K\varepsilon \quad \text{for Re } t > \ln r + t_0',$$

where t_0 and t_0' are suitable parameters (Note that $\sigma(t)$ is not defined for arbitrary Im *t. See also Proposition* 4.3).

Proof. First we express σ in a suitable way for the comparison with u and v. Since (4.3) is analytic and conservative there exists H analytic such that

$$\dot{x} = \frac{\partial H}{\partial y} = x + g_1(x, y),$$

$$\dot{y} = -\frac{\partial H}{\partial x} = -y + g_2(x, y).$$

In [15] it is proved that there exists an analytic canonical change of variables of the form

$$C(\xi, \eta) = \left(\xi + \sum_{k\geq 2} \Phi_k(\xi, \eta), \eta + \sum_{k \geq 2} \Psi_k(\xi, \eta)\right)$$

such that

$$\bar{H}(\xi, \eta) = H \circ C(\xi, \eta) = a_1 \xi\eta + a_2(\xi\eta)^2 + \cdots.$$

It is not difficult to prove that in our case $a_1 = 1$. We take $w = \xi\eta$. Then

$$\begin{aligned} \dot{\xi} &= \frac{\partial \bar{H}}{\partial \eta} = \xi \frac{\partial \bar{H}}{\partial w} \\ \dot{\eta} &= -\frac{\partial \bar{H}}{\partial \xi} = -\eta \frac{\partial \bar{H}}{\partial w} \end{aligned} \tag{4.4}$$

and the flow solution is

$$\bar{\varphi}(t, \xi, \eta) = (\xi \exp(H_w(w)t), \eta \exp(-H_w(w)t)). \tag{4.5}$$

In particular the homoclinic orbit is given locally, in these variables, by

$$\bar{\sigma}(t) = (\xi \exp t, 0) = (\pm\exp(t - t_0), 0) \quad \text{for Re}\,(-t) \text{ big enough},$$

$$\bar{\sigma}(t) = (0, \eta \exp(-t)) = (0, \pm\exp(t_0' - t)) \quad \text{for Re}\, t \text{ big enough}.$$

Supposing that $\xi, \eta > 0$ we have

$$\begin{aligned} \sigma(t) &= C(\exp(t - t_0), 0) \quad \text{for Re}\,(-t) \text{ big enough}, \\ \sigma(t) &= C(0, \exp(t_0' - t)) \quad \text{for Re}\, t \text{ big enough}. \end{aligned} \tag{4.6}$$

Now we reinterpret the change C. Let φ_t and $\bar{\varphi}_t$ be the flow solutions of (4.3) and (4.4). It is clear that $\varphi_t = C\bar{\varphi}_t C^{-1}$. Let $\bar{G}_\varepsilon$ be the flow time h of (4.4). Then $\bar{G}_\varepsilon = C^{-1} G_\varepsilon C$. From (4.5) we see that $\bar{G}_\varepsilon$ is in normal form and hence C is the change which transforms G_ε into its normal form. Notice that it is independent of ε. We take $C_2 = C$ and C_1 the change (depending on ε) which transforms F into its normal form. The proof finishes using Proposition 4.1, (4.6) and the definitions of u and v. □

As a consequence we have

Proposition 4.3. *Let σ be a homoclinic orbit of a vector field as* (4.3). *Then there exists $\delta > 0$ such that σ can be extended analytically to* $\{t \in \mathbb{C}, |\text{Im}\, t| < \delta\}$.

Proof. By the proof of Proposition 4.2 we know that there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $\sigma(t) = C(\exp(t-t_0), 0)$ if $\operatorname{Re} t < \tau_1$ and $\sigma(t) = C(0, \exp(t_0'-t))$ if $\operatorname{Re} t > \tau_2$ where C is analytic. Then σ is analytic on

$$\{t \in \mathbb{C}, \operatorname{Re} t < \tau_1\} \cup \{t \in \mathbb{C}, \operatorname{Re} t > \tau_2\}.$$

Furthermore σ is defined on $[\tau_1, \tau_2] \subset \mathbb{R}$. Since this interval is compact we can extend σ to $\{t \in \mathbb{C}, \tau_1 \leq \operatorname{Re} t \leq \tau_2, |\operatorname{Im} t| < \delta\}$. □

Remark. The maximal value of δ which verifies Proposition 4.3 is the distance from the real axis to the nearest singularity to it.

5. *The distance between splitted separatrices for analytic diffeomorphisms*

In the proof of Theorem A a very important role will be played by the vector field described in the next proposition.

PROPOSITION 5.1. *Under the hypothesis* H1–H4 *the vector field given by*

$$\begin{aligned} \dot{x} &= x + \frac{1}{a} f_1(x, y) \\ \dot{y} &= -y + \frac{1}{a} f_2(x, y), \end{aligned} \tag{5.1}$$

where f_1, f_2 denote the components of f, is conservative, the origin is hyperbolic and has a homoclinic orbit σ such that if ε is small enough the real invariant manifolds of F_ε are ε-close to $\sigma(\mathbb{R})$. More precisely: for each point p of $\sigma(\mathbb{R})$ there is a neighbourhood V of p in $\sigma(\mathbb{R})$ and parametrizations of V and a piece of the unstable invariant manifold of F_ε which are ε-close. The same is true for the stable manifold of F_ε.

Proof. From the condition $\det DF_\varepsilon \equiv 1$ we have

$$1 + \varepsilon^\alpha(a + D_x f_1 - a + D_y f_2) + O(\varepsilon^{\alpha+1}) \equiv 1$$

and since F_ε is analytic with respect to ε we get $D_x f_1 + D_y f_2 \equiv 0$. It is immediate that the origin is hyperbolic. The proximity between the invariant manifolds and σ is a consequence of Theorems A and A' of [5]. Indeed, we define G_ε as the flow time $a\varepsilon^\alpha$ of (5.1). It is clear that the origin is a hyperbolic fixed point of G_ε and that the corresponding invariant manifolds coincide with the ones of (5.1) so that they are independent of ε. The hypothesis of theorems A and A′ are an immediate consequence of Proposition 2.3 of [5] and the definitions. Then the invariant manifolds of F_ε are ε-close to those of G_ε. This implies that the invariant manifolds of G_ε coincide and hence we have σ. □

Proof of Theorem A. First we note that Lemmas 5.2, 5.3 and 5.4 will be used and proved along the proof of Theorem A.

Let $B(r_1)$ be a ball centered at $(0,0)$ such that the change C_ε and the normal form of F_ε described in Proposition 3.1 are analytic on it. We know that

$$|C_\varepsilon - I|_{B(r_1)} = O(\varepsilon^0) \quad \text{and} \quad |C_\varepsilon - I|_{B(r_1)} = O(|(x, y)|^2)$$

and then there exists $r_2 > 0$ such that $C_\varepsilon(B(r_1)) \subset B(r_2)$. We suppose that r_2 is such that the property of Proposition 4.2 is verified for $0 < \varepsilon < \varepsilon_0$ with a suitable ε_0.

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Furthermore

$$|C_\varepsilon^{-1}-I|_{B(r_2)}=O(\varepsilon^0) \quad \text{and} \quad |C_\varepsilon^{-1}-I|_{B(r_2)}=O(|(x,y)|^2).$$

Let C_i^{-1} be the ith component of C_ε^{-1}. Then

$$(C_1^{-1}C_2^{-1})(x,y)=xy+O(|x,y)|^3) \quad \text{and} \quad D_y(C_1^{-1}C_2^{-1})(x,y)=x+O(|(x,y)|^2).$$

From that one has that there exist $r_3\in(0,r_2]$ and $A>0$ such that if $(x,y)\in B(r_3)$ then $|D_y(C_1^{-1}C_2^{-1})(x,y)-x|<A|(x,y)|^2$. Therefore if

$$(x,y)\in\Sigma=B(r_3)\cap\{(x,y),\quad |x|\le|y|\}-B(r_3/2) \quad \text{then}$$
$$|D_y(C_1^{-1}C_2^{-1})(x,y)|>|x|-A|(x,y)|^2>r_3(1-4Ar_3)/4.$$

We assume that r_3 is so small that $1-4Ar_3>0$. Let $d=r_3(1-4Ar_3)/4$. Consider l_1 a connex real piece of σ contained in Σ and let $t_1<t_2$ be such that $\sigma(t_1)$ and $\sigma(t_2)$ are the extremes of l_1 and $\sigma(t)\in B(r_3)$ for $t<t_2$. Since for $\operatorname{Re} t<\tau_1$ one has

$$\sigma(t)=\left(\sum_{n\ge1}a_n\,\mathrm{e}^{nt},\ \sum_{n\ge2}b_n\,\mathrm{e}^{nt}\right)$$

for a suitable τ_1 then $|\sigma(t)|$, and therefore $|\operatorname{Im}\sigma(t)|$, tend to zero when $\operatorname{Re} t$ tends to $-\infty$. We can suppose, taking a smaller r_3 if necessary, that for any small positive η one has $\sigma(t)\in\Sigma$ if

$$t\in\Delta_0=\{t\in\mathbb{C},\ t_1\le\operatorname{Re} t\le t_2,\quad |\operatorname{Im} t|<\delta-\eta\}.$$

Let $T_1>0$ and $t_0\in(t_1+T_1,\ t_2+T_1)$ be such that $p=\sigma(t_0)$, where p is the point in σ introduced at the statement. Since $|\operatorname{Im}\sigma(t)|$ also tends to zero when $\operatorname{Re} t$ tends to ∞ there exists $T_2>0$ such that $\sigma(t)\in B(r_2)$ if

$$t\in\Delta_2=\{t\in\mathbb{C},\ t_1+T_1+T_2<\operatorname{Re} t<t_2+T_1+T_2,|\operatorname{Im} t|<\delta-\eta\}.$$

Let $\tilde\Omega_0=\{z=\sigma(t),\ t\in\Delta_0\}$ and let φ be the flow solution of (5.1). It is clear that φ is defined on $[0,\ T_1+T_2]\times\tilde\Omega_0$. By continuity there exists $r>0$ such that φ is defined on

$$[0,\ T_1+T_2]\times(\tilde\Omega_0+r),\ \varphi(t,z)\in B(r_2) \quad \text{if} \quad (t,z)\in[T_1+T_2,\ t_2-t_1+T_1+T_2]\times(\tilde\Omega_0+r)$$

and

$$\Omega=\{z=\varphi(t,w),\ w\in\tilde\Omega_0+r,\ 0\le t\le T_1+T_2\}$$

verifies $\Omega\subset\tilde U$ (see hypothesis H5).

To simplify the notation we shall write the equation (5.1) in the form $\dot z=\tilde f(z)$ with $z=(x,y)$. Then

$$F_\varepsilon(z)=z+a\varepsilon^\alpha\tilde f(z)+\varepsilon^{\alpha+1}\tilde g(z,\varepsilon) \quad \text{and}$$
$$F_\varepsilon^{-1}(z)=z-a\varepsilon^\alpha\tilde f(z)+\varepsilon^{\alpha+1}\tilde{\tilde g}(z,\varepsilon).$$

Let $M_2,M_3>0$ be such that $|f|_\Omega\le M_2$ and $|Df|_\Omega\le M_3$. Let $M_4=aM_3+1$ so that if ε_0 is small enough $|DF_\varepsilon|_\Omega<1+M_4\varepsilon^\alpha$ and $|DF_\varepsilon^{-1}|_\Omega<1+M_4\varepsilon^\alpha$. Let Ω^0, Ω_1^0 and $s>0$ be such that Ω_1^0 is open, $\tilde\Omega_0\subset\Omega_1^0\subset\Sigma$ and

$$\Omega^0=\{z=\varphi(t,w),\ 0\le t\le T_1+T_2,\ w\in\Omega_1^0\}\subset\Omega-s.$$

We define G_ε as the flow time h (see Proposition 2.3) of (5.1). We notice that this definition of G_ε is slightly different from the one used in Proposition 5.1.

LEMMA 5.2. *There exist M_1, $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

$$|F_\varepsilon - G_\varepsilon|_\Omega < M_1 \varepsilon^{\alpha+1},$$
$$|F_\varepsilon^{-1} - G_\varepsilon^{-1}|_{\Omega^0} < M_1 \varepsilon^{\alpha+1}.$$

Proof. The first bound is proved in an analogous way as in Proposition 2.3 of [**5**]. The second one is a consequence of Proposition 2.2 of [**5**]. □

Let $M_5 = \max((M_1/M_4)\exp(M_4TC), M_2/D)$ with $T = \max(T_1, T_2)$ and let C, D be constants such that $0 < D < \varepsilon^\alpha/h < C$ for $0 < \varepsilon < \varepsilon_0$.

Reducing ε_0 if necessary we consider Ω^j and Ω_1^j, $j = 1, 2, 3$, such that Ω_1^j are open sets, $Cl(\tilde{\Omega}_0) \subset \Omega_1^3$, $Cl(\Omega_1^{j+1}) \subset \Omega_1^j$, $j = 0, 1, 2$, and

$$\Omega^j = \{z = \varphi(t, w), 0 \le t \le T_1 + T_2, w \in \Omega_1^j\} \subset \Omega^0 - 2jM_5\varepsilon_0.$$

Let $\Omega_2^j = \{z = \varphi(T_1, w), w \in \Omega_1^j\}$ and $\Omega_3^j = \{z = \varphi(T_1 + T_2, w), w \in \Omega_1^j\}$. Let N_1 be the integer part of T_1/h and N_2 the integer part of T_2/h.

LEMMA 5.3. *For $0 < \varepsilon < \varepsilon_0$ we have*
(1) *If $z \in \Omega_1^1$, $F_\varepsilon^n(z) \in \Omega$ for $0 \le n \le N_1$,*
(2) *If $z \in \Omega_1^1$, $|F_\varepsilon^n(z) - G_\varepsilon^n(z)| < M_5\varepsilon$ for $0 \le n \le N_1$,*
(3) $F_\varepsilon^{N_1}(\Omega_1^1) \supset \Omega_2^2$,
(4) $F_\varepsilon^{-N_2}(\Omega_3^3) \subset \Omega_2^2$.

Proof. We shall prove (1) and (2) by induction. In fact we shall prove that

$$|F_\varepsilon^n(z) - G_\varepsilon^n(z)| \le M_1 \sum_{k=0}^{n-1} (1 + M_4\varepsilon^\alpha)^k \varepsilon^{\alpha+1}.$$

It is clear that this last expression is less than $M_1 e^{T_1 M_4 C}\varepsilon/M_4 \le M_5\varepsilon$. For $n = 1$ it is a consequence of Lemma 5.2. Assuming that it is true for $n-1$, $n \le N_1$ we can write

$$\begin{aligned}
|F_\varepsilon^n(z) - G_\varepsilon^n(z)| &\le |F_\varepsilon F_\varepsilon^{n-1}(z) - F_\varepsilon G_\varepsilon^{n-1}(z)| + |F_\varepsilon G_\varepsilon^{n-1}(z) - G_\varepsilon G_\varepsilon^{n-1}(z)| \\
&\le |DF_\varepsilon|_\Omega |F_\varepsilon^{n-1}(z) - G_\varepsilon^{n-1}(z)| + M_1\varepsilon^{\alpha+1} \\
&\le (1 + M_4\varepsilon^\alpha) M_1 \sum_{k=0}^{n-2} (1 + M_4\varepsilon^\alpha)^k \varepsilon^{\alpha+1} + M_1\varepsilon^{\alpha+1} \\
&= M_1 \sum_{k=0}^{n-1} (1 + M_4\varepsilon^\alpha)^k \varepsilon^{\alpha+1},
\end{aligned}$$

because as $|F_\varepsilon^{n-1}(z) - G_\varepsilon^{n-1}(z)| < M_5\varepsilon$, the segment which connects those points is contained in Ω and so we can apply the mean value theorem to F_ε. This proves (2) and also that $F_\varepsilon^n(z) \in B(M_5\varepsilon, G_\varepsilon^n(z)) \subset \Omega$.

To prove (3), first we check as before that

$$|F_\varepsilon^{-n}(z) - G_\varepsilon^{-n}(z)| \le M_1 \sum_{k=0}^{n-1} (1 + M_4\varepsilon^\alpha)^k \varepsilon^{\alpha+1}$$

and from that we see that if $z \in \Omega_2^2$,

$$F_\varepsilon^{-N_1}(z) \in B(M_5\varepsilon, G_\varepsilon^{-N_1}(z)) \subset \Omega^1.$$

Since

$$\begin{aligned}
|G_\varepsilon^{-N_1}(z) - \varphi(-T_1, z)| &= |\varphi(-N_1h, z) - \varphi(-T_1 + N_1h, \varphi(-N_1h, z))| \\
&= |z_0 - \varphi(-\theta h, z_0)|,
\end{aligned}$$

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with $z_0 = \varphi(-N_1 h, z)$ and $0 < \theta < 1$, and furthermore

$$|\varphi(-\theta h, z_0) - z_0| \le M_2 \theta h < M_5 D h < M_5 \varepsilon^{\alpha},$$

we have that $B(M_5\varepsilon, G_\varepsilon^{-N_1}(z)) \subset B(2M_5\varepsilon, \varphi(-T_1, z)) \subset \Omega_1^1$.

To prove (4), analogously as before we begin by showing that if $z \in \Omega_3^3$

$$|F_\varepsilon^{-n}(z) - G_\varepsilon^{-n}(z)| \le M_1 \sum_{k=0}^{n-1} (1 + M_4 \varepsilon)^k \varepsilon^{\alpha+1}$$

and from that $|F_\varepsilon^{-N_2}(z) - G_\varepsilon^{-N_2}(z)| < M_5 \varepsilon$. Finally, following the same ideas as in (3) we have

$$F_\varepsilon^{-N_2}(z) \in B(M_5\varepsilon, G_\varepsilon^{-N_2}(z)) \subset B(2M_5\varepsilon, \varphi(-T_2, z)) \subset \Omega_2^2. \qquad \square$$

Now let v be the parametrization of $W^{\mathrm{s}}_{F_\varepsilon}(0, 0)$ given by Proposition 2.3. It is clear that $F_\varepsilon^{-N_2}(v(t)) = v(t - N_2 h)$. By Proposition 4.2 if s and ε_0 are small enough one has $v(t) \in \Omega_3^3$ if $t \in \Delta_2$ and hence $v(t - N_2 h) \in \Omega_2^2$ if $t \in \Delta_2$. Let E be the local first integral given by Proposition 2.3. We extend E analytically to a neighbourhood of $W^{\mathrm{u}}_{F_\varepsilon}(0, 0)$ until Ω_2^2 by using $E(z) = E(F_\varepsilon^{-N_1}(z))$ and (3) of Lemma 5.3.

Now we define the analytic function Ψ by $\Psi(t) = E(v(t))$ on

$$\Omega_0^* = \{t \in \mathbb{C},\ t_1 + T_1 + T_2 - N_2 h < \operatorname{Re} t < t_2 + T_1 + T_2 - N_2 h, |\operatorname{Im} t| < \delta - \eta\}.$$

LEMMA 5.4. *For $0 < \varepsilon < \varepsilon_0$ we have*

(1) *Ψ is h-periodic,*

(2) *There exists $K_1 > 0$ such that $|\Psi(t)| \le K_1$ on Ω_0^*,*

(3) *There exists $K > 0$ such that $|\Psi(t)| < K \exp(-2\pi(\delta - \eta)/h)$ on $\Omega_0^* \cap \mathbb{R}$.*

Proof.

(1) Let us suppose that ε_0 is small and let t be such that t, $t + h \in \Omega_0^*$. Then

$$\Psi(t+h) = E(v(t+h)) = E(F(v(t))) = E(v(t)) = \Psi(t).$$

(2) If $t \in \Omega_0^*$, $\Psi(t) = E(v(t)) = E(F_\varepsilon^{-N_1}(v(t)))$ with $F_\varepsilon^{-N_1}(v(t)) \in \Omega_1^1$.
E is bounded in Ω^1 since it is analytic in a neighbourhood of Ω^1.

(3) By (1) we can write $\Psi(t) = \sum_{n=-\infty}^{\infty} c_n \exp(\mathrm{i}n2\pi t/h)$ with

$$c_n = h^{-1} \int_t^{t+h} \Psi(s) \exp(-\mathrm{i}n2\pi s/h)\, \mathrm{d}s.$$

To evaluate the last integral we consider, for $n > 0$, the complex path given in figure 1, where $\delta_1 = \delta - \eta$. By Cauchy's theorem the integral along the path is zero and by periodicity the integrals along the vertical sides cancel. Hence we have

$$c_n + h^{-1} \int_{t+h}^{t} \Psi(-\mathrm{i}\delta_1 + s) \exp(-\mathrm{i}n2\pi(-\mathrm{i}\delta_1 + s)/h)\, \mathrm{d}s = 0.$$

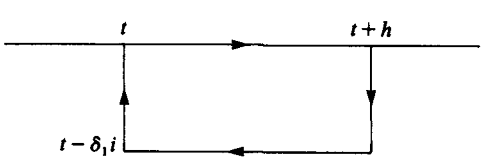

FIGURE 1

Therefore

$$|c_n| \le h^{-1} \exp(-n2\pi\delta_1/h) \int_t^{t+h} |\Psi(-i\delta_1+s)|\, ds$$
$$\le K_1 \exp(-n2\pi\delta_1/h).$$

For $n<0$ we consider the path given in figure 2 and we obtain $|c_n| \le K_1 \exp(-|n|2\pi\delta_1/h)$. Then

$$|\Psi(t)-c_0| \le 2K_1 \exp(-2\pi\delta_1/h)/(1-\exp(-2\pi\delta_1/h)) < 4K_1 \exp(-2\pi\delta_1/h),$$

if ε_0 is small enough. Since there are real homoclinic points, there are real values of t such that $\Psi(t)=0$. Then $|c_0|<4K_1 \exp(-2\pi\delta_1/h)$ and the lemma follows easily. □

Let $S \in W^s_{F_\varepsilon}(0,0) \cap \Omega_2^2 \cap \mathbb{R}^2$ and $R \in W^u_{F_\varepsilon}(0,0) \cap \Omega_2^2 \cap \mathbb{R}^2$ such that if

$$R'=(R_1', R_2') = F_\varepsilon^{-N_1}(R) \quad \text{and} \quad S'=(S_1', S_2') = F_\varepsilon^{-N_1}(S)$$

we have $R_1'=S_1'$. From the definition of E we have

$$D_y E = D_y(\bar{E} \circ C^{-1}) = D_y(C_1^{-1} C_2^{-1}).$$

On the other hand

$$E(S)-E(R) = E(S')-E(R') = D_y E(\theta_0)(S_2'-R_2') \quad \text{with } \theta_0 \in \Omega_1^1.$$

Then

$$|S_2'-R_2'| = |D_y E(\theta_0)|^{-1} |E(S)-E(R)| < d^{-1}|E(S)|$$
$$< (K/d) \exp(-2\pi(\delta-\eta)/h)$$

since $E(R)=0$. Finally, if $\theta_1 \in \Omega_1^1$

$$|DF_\varepsilon^{N_1}(\theta_1)| < (1+M_4\varepsilon^\alpha)^{N_1} < e^{M_4T_1}$$

and so

$$|S-R| = |F_\varepsilon^{N_1}(S') - F_\varepsilon^{N_1}(R')| \le |DF_\varepsilon^{N_1}(\theta_1)|\, |S_2'-R_2'|$$
$$< e^{M_4T_1}(K/d) \exp(-2\pi(\delta-\eta)/h).$$

Remark 1. If instead of hypothesis H5 of Theorem A, F_ε can only be extended to

$$\tilde{\tilde{U}} = \{(z_1, z_2) \in \mathbb{C}^2, (\operatorname{Re} z_1, \operatorname{Re} z_2) \in U, |\operatorname{Im} z_1|, |\operatorname{Im} z_2| < r\}$$

with r independent of ε, the conclusion is that there exist ε_0, N, $\bar{\delta}>0$ such that the separation between the invariant manifolds is less than $N \exp(-2\pi\bar{\delta}/\ln\lambda)$ for $0<\varepsilon<\varepsilon_0$. The value of $\bar{\delta}$ is related to the singularities of σ and to a value $\bar{\tau}$ such that $|\operatorname{Im} t| < \bar{\tau}$ implies $\sigma(t) \in \tilde{\tilde{U}}$.

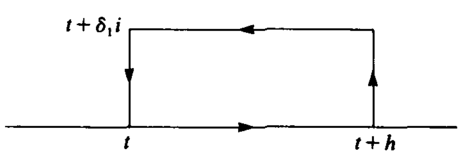

FIGURE 2

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Remark 2. In the examples, hypothesis H4 is not easy to verify. However we can substitute H4 by

H4′ (5.1) has a homoclinic orbit σ contained in a compact set contained in U.

Indeed, as in Proposition 5.1, the invariant manifolds of F_ε are ε-close to the ones of G_ε. Since two branches of the invariant manifolds of G_ε coincide with σ, those of F_ε are ε-close between them. Since F_ε preserves area they intersect or coincide [**13**].

Remark 3. A similar bound is true for the angle at a homoclinic point in V with a somewhat bigger η. A bound of the same type is also true for the area of the loops between $W^u_{F_\varepsilon}$ and $W^s_{F_\varepsilon}$. If we consider the stable manifold after returning just once to a neighbourhood of the hyperbolic point, the tongues entering V are at a distance from the unstable manifold of the same order, i.e., bounded by $\tilde{N}\exp(-2\pi(\delta-\eta)/\ln\lambda)$ for $0<\varepsilon<\varepsilon_0$.

Remark 4. With minor changes Theorem A can be proved for the heteroclinic case.

6. *Examples*

(1) The non-trivial quadratic diffeomorphisms of the plane which preserve area and orientation with a fixed point can be put after a linear change of variables into the one parameter family [**4**]

$$F_c(x,y)=(y,-x+2y^2+2cy)\quad c\ge 1. \tag{6.1}$$

In particular that holds for the Hénon map $F_{a,b}(x,y)=(1+y-ax^2,bx)$ when $b=-1$ and $a\ge -1$.

F_c has one fixed point, $(0,0)$, for $c=1$ which is parabolic and two fixed points, $(0,0)$ and $(1-c,1-c)$, for $c>1$. The first one is hyperbolic for all c and the second is elliptic for $1<c<3$, parabolic for $c=3$ and hyperbolic for $c>3$. We are interested in the invariant manifolds of the origin which we call W^u and W^s. The eigenvalues of $DF_\varepsilon(0,0)$ are

$$\lambda_\pm=c\pm(c^2-1)^{1/2}=1\pm\sqrt{2}\varepsilon+O(\varepsilon^2)\quad\text{where } \varepsilon=(c-1)^{1/2}.$$

Putting the linear part in diagonal form and scaling by $C_\varepsilon(x,y)=(\varepsilon^2x,\varepsilon^2y)$ we get

$$\begin{aligned}\bar{F}_\varepsilon(x,y)&=(\lambda x+\varepsilon(2+\varepsilon^2)^{-1/2}(\lambda x+\lambda^{-1}y)^2,\lambda^{-1}y-\varepsilon(2+\varepsilon^2)^{-1/2}(\lambda x+\lambda^{-1}y)^2)\\&=(\lambda x,\lambda^{-1}y)+2^{-1/2}\varepsilon((x+y)^2,-(x+y)^2)+O(\varepsilon^2).\end{aligned} \tag{6.2}$$

We consider the vector field given by

$$\dot{x}=x+(1/2)(x+y)^2,$$
$$\dot{y}=-y-(1/2)(x+y)^2,$$

which comes from the Hamiltonian

$$H(x,y)=xy+(1/6)(x+y)^3.$$

Its homoclinic orbit σ is given by

$$\sigma_1(t)=-3(\cosh(t/2)+\sinh(t/2))/4\cosh^3(t/2),$$
$$\sigma_2(t)=-3(\cosh(t/2)-\sinh(t/2))/4\cosh^3(t/2).$$

It is immediate that $\delta_0 = \pi$. Since $\bar{F}_\varepsilon$ is defined in $\mathbb{C}^2$ we can take $\delta = \delta_0$. By Theorem A and Remark 2 after it, given $\eta > 0$ there exist ε_0, $N > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the separation between the invariant manifolds of $\bar{F}_\varepsilon$ in a neighbourhood of a given point is less than

$$N \exp(2\pi(-\pi+\eta)/\ln\lambda) \simeq N \exp((-2\pi^2+\eta')/\sqrt{2}\varepsilon).$$

The bound of the separation between W^u and W^s is the same as before by a factor $O(\varepsilon^2)$ due to the changes to send F_c to $\bar{F}_\varepsilon$.

Now we compare the bound with the numerical results. Because of the symmetries there is a homoclinic point on the line $y = x$. We have computed the angle between W^u and W^s at that point. This gives a measure of the maximum separation between W^u and W^s in a neighbourhood of it. Some of the results, computed using arbitrary precision routines, are shown in table 1.

If we fit the data in the expression

$$A(\ln\lambda)^B \exp(-C/\ln\lambda)$$

and we determine the values of A, B and C from the data corresponding to $\lambda = 1.3$, $\lambda = 1.29$ and $\lambda = 1.28$ we get $A = 87\,692\,502$, $B = -8.268891734$ and $C = 19.78502818$. From that the angle predicted for $\lambda = 1.04$ is 0.3102×10^{-199} which is strikingly accurate. On the other hand the theoretical value of C is essentially $2\pi^2 = 19.739209$. The role of η in the theoretical bound is to bound the factor $(\ln\lambda)^B$ if $B < 0$. If we determine C from the data corresponding to $\lambda = 1.050$, $\lambda = 1.045$ and $\lambda = 1.040$ we get $C = 19.739507$. We notice that the relative error is only 1.51×10^{-5}. The related values of A and B are 1.08346×10^8 and -8.0082713.

(2) We consider the following perturbed Duffing equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \varepsilon \cos(t/\varepsilon). \end{aligned} \tag{6.3}$$

For ε small it has a hyperbolic periodic orbit of period $2\pi\varepsilon$ near the origin (Propositions 5.1 and 5.2 of [**5**]). We are interested in the existence of homoclinic orbits and in the separation between the invariant manifolds in a given neighbourhood. For that we consider the map time $2\pi\varepsilon$ of (6.3) in suitable variables so that

TABLE 1

λ	ε	c	Angle
1.3	0.18605210	1.03461538	0.9944869868 E−20
1.29	0.18054611	1.03259690	0.1293376504 E−20
1.28	0.175	1.03062500	0.1443569127 E−21
1.2	0.12909945	1.01666667	0.8470514533 E−33
1.1	0.06741999	1.00454545	0.1833626865 E−73
1.050	0.03450327	1.00119048	0.6795065162 E−157
1.045	0.03112715	1.00096890	0.1368763698 E−175
1.040	0.02773501	1.00076923	0.5260040656 E−199

we could apply Theorem A. In order to simplify the notation we write (6.3) in the more compact form

$$\dot{z}=f(z)+\varepsilon g(t/\varepsilon), \tag{6.4}$$

with $z=(x, y)$ not necessarily real. First we translate the periodic orbit $\gamma_\varepsilon(t)$ to the origin to have it fixed. The new equation is

$$\dot{z}=f(z+\gamma_\varepsilon(t))+\varepsilon g(t/\varepsilon)-\dot{\gamma}_\varepsilon(t). \tag{6.5}$$

Let $\varphi_1(t, \tau, z, \varepsilon)$ be the solution of (6.5) such that $\varphi_1(\tau, \tau, z, \varepsilon)=z$. Since (6.5) is $2\pi\varepsilon$-periodic we are interested only in $\tau\in[0, 2\pi\varepsilon)$. Then we write $\tau=2\pi\varepsilon\alpha$ with $\alpha\in[0,1)$. We also consider the equation

$$\dot{z}=f(z) \tag{6.6}$$

and its solution $\varphi_2(t, \tau, z, \varepsilon)$ such that $\varphi_2(\tau, \tau, z, \varepsilon)=z$. When no confusion is possible we shall write for short $\varphi_1(t)$ and $\varphi_2(t)$. For any $\alpha\in[0,1)$ we define

$$F_\varepsilon^\alpha(z)=\varphi_1(2\pi\varepsilon\alpha+2\pi\varepsilon, 2\pi\varepsilon\alpha, z, \varepsilon) \quad \text{and} \quad G_\varepsilon(z)=\varphi_2(2\pi\varepsilon\alpha+2\pi\varepsilon, 2\pi\varepsilon\alpha, z, \varepsilon)$$
$$=\varphi_2(2\pi\varepsilon, 0, z, \varepsilon)$$

on suitable domains. Since (6.5) and (6.6) are Hamiltonian then F_ε^α and G_ε preserve measure. To prove that F_ε^α is analytic and depends analytically on ε we begin by considering the equation

$$\dot{z}=\varepsilon f(z)+\varepsilon^2 g(t), \tag{6.7}$$

which is obtained by scaling time in (6.4). Let $\varphi_3(t, \tau, z, \varepsilon)$ and $\varphi_4(t, \tau, z, \varepsilon)$ be the solutions of (6.4) and (6.7), respectively. Since (6.7) is analytic, φ_4 is analytic and depends analytically on ε. We define

$$\bar{H}_\varepsilon^\alpha(z)=\varphi_4(2\pi\alpha+2\pi, 2\pi\alpha, z, \varepsilon) \quad \text{and} \quad H_\varepsilon^\alpha(z)=\varphi_3(2\pi\varepsilon\alpha+2\pi\varepsilon, 2\pi\varepsilon\alpha, z, \varepsilon).$$

The periodic orbit of (6.7) is $\bar{\gamma}_\varepsilon(t)=\gamma_\varepsilon(\varepsilon t)$. To prove that it depends analytically on ε we look for the initial condition $z(\varepsilon)$ of this orbit for $t=\tau$. It must satisfy that $\bar{H}_\varepsilon^\alpha(z(\varepsilon))=z(\varepsilon)$. For that we consider the function defined by $\psi(\varepsilon, z)=(\bar{H}_\varepsilon^\alpha(z)-z)/\varepsilon$ for $\varepsilon\neq0$ and $\psi(0, z)=2\pi f(z)$. ψ is analytic since $\bar{H}_\varepsilon^\alpha$ is analytic and $\bar{H}_\varepsilon^\alpha(z)=z+2\pi\varepsilon f(z)+O(\varepsilon^2)$. Also $D_z\psi(0, z)=2\pi Df(z)$ so that $D_z\psi(0,0)=2\pi Df(0)$ is invertible and, by the implicit function theorem, $z^\alpha(\varepsilon)$ is analytic with respect to ε. Then $\bar{\gamma}_\varepsilon(t)=\varphi_4(t, 2\pi\alpha, z^\alpha(\varepsilon), \varepsilon)$ is analytic. Furthermore

$$\varphi_3(t, \tau, z, \varepsilon)=\varphi_4(t/\varepsilon, \tau/\varepsilon, z, \varepsilon)$$

and

$$\varphi_1(t, \tau, z, \varepsilon)=\varphi_3(t, \tau, z+\gamma_\varepsilon(\tau), \varepsilon)-\gamma_3(t)$$

and hence

$$F_\varepsilon^\alpha(z)=\varphi_4(2\pi\alpha+2\pi, 2\pi\alpha, z+\bar{\gamma}_\varepsilon(2\pi\alpha), \varepsilon)-\bar{\gamma}_\varepsilon(2\pi\alpha)$$

is also analytic. To compute the eigenvalues of $DF_\varepsilon^\alpha(0)$ we compare $DF_\varepsilon^\alpha(0)$ with $DG_\varepsilon(0)$. We have that

$$D_tD_z\varphi_1(t, \tau, 0, \varepsilon)=Df(\varphi_1(t, \tau, 0, \varepsilon)+\gamma_\varepsilon(t))D_z\varphi_1(t, \tau, 0, \varepsilon)$$

and

$$D_t D_z \varphi_2(t, \tau, 0, \varepsilon) = Df(\varphi_2(t, \tau, 0, \varepsilon)) D_z \varphi_2(t, \tau, 0, \varepsilon)$$

with

$$D_z \varphi_1(\tau, \tau, 0, \varepsilon) = D_z \varphi_2(\tau, \tau, 0, \varepsilon) = I.$$

Since $\varphi_1(t, \tau, 0, \varepsilon) = \varphi_2(t, \tau, 0, \varepsilon) = 0$ we can write

$$D_z \varphi_2(t) - D_z \varphi_1(t) = \int_\tau^{\tau+t} [Df(0) D_z \varphi_2(s) - Df(\gamma_\varepsilon(s)) D_z \varphi_2(s)] \, ds$$

and hence

$$\begin{aligned} \|D_z \varphi_2(t) - D_z \varphi_1(t)\| \le & \int_\tau^{\tau+t} \|Df(0) - Df(\gamma_\varepsilon(s))\| \cdot \|D_z \varphi_2(s)\| \, ds \\ & + \int_\tau^{\tau+t} \|Df(\gamma_\varepsilon(s))\| \cdot \|D_z \varphi_2(s) - D_z \varphi_1(s)\| \, ds. \end{aligned}$$

From Theorem 5.1 of [**5**] we know that there exist $c > 0$ and $\varepsilon_0 > 0$ such that $\|\gamma_\varepsilon(s)\| < c\varepsilon^2$ for $0 < \varepsilon < \varepsilon_0$. Then taking a neighbourhood U_0 of 0 independent of ε and M_1 and M_2 such that $\|Df\|_{U_0} \le M_1$ and $\|D^2 f\|_{U_0} \le M_2$ and taking into account that $D_z \varphi_2(t) = \exp(Df(0)(t - \tau))$, by Gronwall's Lemma we get that

$$\|D_z \varphi_2(t) - D_z \varphi_1(t)\| \le 2\pi c M_2 \varepsilon^3 \exp(M_1 2\pi\varepsilon) \exp(M_1(t - \tau)) \quad \text{for } t \in [\tau, \tau + 2\pi\varepsilon]$$

and hence $\|DF_\varepsilon^\alpha(0) - DG_\varepsilon(0)\| \le M\varepsilon^3$ with $M > 2\pi c M_2 \exp(4\pi M_1 \varepsilon_0)$.

On the other hand it is not difficult to compute that

$$G_\varepsilon(x, y) = (x + 2\pi\varepsilon y + 2\pi^2\varepsilon^2(x - x^3), y + 2\pi\varepsilon(x - x^3) + 2\pi^2\varepsilon^2(y - 3x^2 y)) + O(\varepsilon^3)$$

so that

$$DF_\varepsilon(0) = \begin{pmatrix} 1 + 2\pi^2\varepsilon^2 & 2\pi\varepsilon \\ 2\pi\varepsilon & 1 + 2\pi^2\varepsilon^2 \end{pmatrix} + O(\varepsilon^3).$$

Then the eigenvalues of $DF_\varepsilon(0)$ are $1 \pm 2\pi\varepsilon + O(\varepsilon^2)$.

To apply Theorem A we should have F_ε^α in form (1.1). It is accomplished by a linear change of variables C_ε^α which put the linear part in diagonal form. It is easily seen that C_ε^α can be taken as

$$C_\varepsilon^\alpha(x, y) = (x + y, x - y) + O(\varepsilon)$$

and it is analytic with respect to ε. Clearly

$$(C_\varepsilon^\alpha)^{-1}(x, y) = ((x + y)/2, (x - y)/2) + O(\varepsilon).$$

Then we have that

$$\begin{aligned} \bar{F}_\varepsilon^\alpha(x, y) &= (C_\varepsilon^\alpha)^{-1} \circ F_\varepsilon^\alpha \circ C_\varepsilon^\alpha(x, y) \\ &= (x, y) + 2\pi\varepsilon(x - (x - y)^3/4, -y - (x - y)^3/4) + O(\varepsilon^2). \end{aligned}$$

We notice that $\bar{F}_\varepsilon^\alpha$ is the map time $2\pi\varepsilon$ of equation (6.5) after having performed the same change C_ε^α, that is,

$$\dot{z} = (C_\varepsilon^\alpha)^{-1}[f(C_\varepsilon^\alpha z + \gamma_\varepsilon(t)) + \varepsilon g(t/\varepsilon) - \dot{\gamma}_\varepsilon(t)].$$

Now we should consider the vector field given by

$$\dot{x} = x - (x-y)^3/4,$$
$$\dot{y} = -y - (x-y)^3/4 \tag{6.8}$$

and the map time h of it which we call $\bar{G}_\varepsilon$. Hypotheses H1, H2 and H3 are then satisfied. It remains to verify hypotheses H4 and H5. Equation (6.8) has two homoclinic orbits given by

$$\sigma_+(t) = (e^{-t}/\cosh^2 t, -e^t/\cosh^2 t),$$
$$\sigma_-(t) = -\sigma_+(t).$$

We focus the attention on $\sigma_+(t)$. Now hypothesis H4 follows from Corollary 4.1 of [5]. Clearly $\delta_0 = \pi/2$. Given any η, $0<\eta<\delta_0$, by the existence theorem for ordinary differential equations in the complex domain there exist a neighbourhood Ω_η of $\Sigma_\eta = \{\sigma_+(t), |\text{Im } t| < \delta_0 - \eta/2\}$ (for instance $\Sigma_\eta + 1$) and $\varepsilon_1 > 0$ such that for $0<\varepsilon<\varepsilon_1$, $\bar{G}_\varepsilon$ and $\bar{F}_\varepsilon^\alpha$ are well defined on Ω_η. We emphasize that both for $\bar{G}_\varepsilon$ and $\bar{F}_\varepsilon^\alpha$ the initial conditions are complex but the increment of time is kept real. Then, by Theorem A, there exist $N>0$ and $\varepsilon_2>0$ such that if $0<\varepsilon<\varepsilon_2$ the separation between the invariant manifolds of $\bar{F}_\varepsilon^\alpha$ in a fixed neighbourhood is less than

$$N \exp(-2\pi(\delta_0 - \eta/2 - \eta/2)/\ln \lambda) \approx N \exp(-\pi/2\varepsilon).$$

The same is true for the invariant manifolds of F_ε^α and also for the ones of H_ε^α. Finally the same is also true for the invariant manifolds of (6.4) since their intersection with the plane $t=\tau$ coincide with the ones of H_ε^α.

The Melnikov function for (6.3) is

$$M(t_0) = \sqrt{2}(\pi/\varepsilon)\sin(t_0/\varepsilon)/\cosh(\pi/2\varepsilon).$$

We notice that the exponential part of $M(t_0)$ is essentially the same as that of the upper bound we have found. From that we conjecture that the Melnikov function gives the right measure of the separation.

(3) We consider

$$F_\varepsilon(x, y) = (x + \varepsilon y + \varepsilon^2 V'(x), y + \varepsilon V'(x)) \tag{6.9}$$

as a generalization of the standard map, where V is an even, 2π-periodic, entire function which has a unique minimum in $(-\pi, \pi]$ located at $x=0$ with $V''(0)>0$. F_ε is a family of analytic area preserving diffeomorphisms. The only real hyperbolic points are $(2k\pi, 0)$, $k \in Z$. We are interested in their invariant manifolds, which we consider as identified with the ones of $(0,0)$ by the periodicity of F_ε. If we set $a = V''(0)$, the eigenvalues of $DF_\varepsilon(0,0)$ are

$$\lambda_\pm = 1 + \varepsilon^2 a/2 \pm (\varepsilon^2 a + \varepsilon^4 a^2/4)^{1/2} = 1 \pm \sqrt{a}\,\varepsilon + O(\varepsilon^2).$$

Putting the linear part of (6.9) in diagonal form we have

$$\bar{F}_\varepsilon(x, y) = (\lambda x + \varepsilon(V'(x+y) - a(x+y))/2\sqrt{a} + O(\varepsilon^2),$$
$$\lambda^{-1} y - \varepsilon(V'(x+y) - a(x+y))/2\sqrt{a}) + O(\varepsilon^2)$$

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which leads us to consider the vector field given by

$$\begin{aligned} \dot{x} &= x + (V'(x+y) - a(x+y))/(2a), \\ \dot{y} &= -y - (V'(x+y) - a(x+y))/(2a), \end{aligned} \tag{6.10}$$

which comes from the Hamiltonian

$$H(x, y) = xy + (V(x+y) - (a/2)(x+y)^2)/(2a).$$

It has a homoclinic orbit. Remark 2 after Theorem A shows that $\bar{F}_\varepsilon$ and so F_ε have homoclinic points (or the invariant manifolds coincide). In [3] it is proved that if F_ε has homoclinic points, there must be one on the line $x = \pi$. (We remark, however, that W^s is not forcibly obtained from W^u by symmetry with respect to the y-axis). The singularities of the homoclinic orbit of (6.10) are given by

$$\sqrt{a} \int_b^\infty (V(\sqrt{2}u) - V(0))^{-1/2}\, du, \tag{6.11}$$

where $2^{-1/2}(x(0) + y(0)) = b$. The integral (6.11) is evaluated on a complex path from b to ∞. In general (6.11) depends on the path because $V(\sqrt{2}\,u) - V(0)$ may have complex zeros. To get the nearest singularity to the real axis we must consider all the possible paths and compare the results. In that way we obtain δ_0. If $V(x) = -\sum_{k=1}^m (c_k/k) \cos kx$, (6.11) becomes

$$2\sqrt{a}\, \mathrm{i} \int_1^\infty u^{-1}\left(\left(\sum_{k=0}^{m-1} d_k u^{2k}\right)(u^2 - 1)\right)^{-1/2} du,$$

where the coefficients d_k are obtained from the coefficients c_k.

If $V(x) = -\cos x - (c/2)\cos 2x$ we can do the calculations and we get that given $\eta > 0$ there exist ε_0, $N > 0$ such that in a given neighbourhood and for $0 < \varepsilon < \varepsilon_0$ the separation between the invariant manifolds of $\bar{F}_\varepsilon$ is less than

$$N \exp(-\pi(\arccos(-4c - 1) - 2\eta)/\ln \lambda) \quad \text{for } -1/2 < c \leq 0$$

and less than

$$N \exp(-(\pi^2 - 2\eta)/\ln \lambda) \quad \text{for } c \geq 0.$$

These bounds agree with the analytical results of Lazutkin [**11**].

(4) As a final example we consider the Hénon-Heiles problem [**8**]. We take for granted that a suitable modification of the argument used in Example 2 allows us to use the extension of Theorem A to heteroclinic connections. This is a 2 degrees of freedom Hamiltonian system with

$$H(x, y) = (x_1^2 + y_1^2 + x_2^2 + y_2^2)/2 + x_1^3/3 - x_1 x_2^2. \tag{6.12}$$

It is known [**8, 7, 1, 12**] that it has hyperbolic periodic orbits near the origin for small values of the energy. Their invariant manifolds intersect so that (6.12) has heteroclinic orbits. We wish to study the maximum separation of the invariant manifolds near one of such orbits. To apply Theorem A we shall construct a family of diffeomorphisms (the parameter being related to the energy) through Poincaré maps on suitable sections.

For that we consider small positive values of the energy h_0 and we take $\varepsilon = h_0^{1/2}$. Scaling variables through $x_i = \varepsilon q_i$, $y_i = \varepsilon p_i$, $i = 1, 2$, we pass to the equivalent system given by

$$K(q, p) = (q_1^2 + q_2^2 + p_1^2 + p_2^2)/2 + \varepsilon(q_1^3/3 - q_1 q_2^2)$$

on the energy level $K(q, p) = 1$.

For $\varepsilon = 0$ the system reduces to an harmonic oscillator. We consider the Poincaré return map on the section $q_2 = 0$, $p_2 = 0$. Let us call it F_ε. It is clear that $F_0 = I$.

Using the Gustavson Normal Form [7] up to sixth order, the fourth being not sufficient [**12**], and using the same method of [**1**] (see also [**2**]) we have that F_ε is approximated by the time 2π map of the Hamiltonian flow given by

$$\begin{aligned} K_6(q_1, p_1) = (7\varepsilon^2/6)[q_1^2 - q_1^2(q_1^2 + p_1^2)/2] \\ + (7\varepsilon^4/72)[-13q_1^2 - 36p_1^2 + (25q_1^4 + 121q_1^2p_1^2 + 96p_1^4)/2 \\ + (-4q_1^6 - 24q_1^4p_1^2 - 36q_1^2p_1^4 - 16p_1^6)]. \end{aligned}$$

The critical points of K_6 are [**12**]

$$P_2 = (0, 2^{-1/2}), \quad P_3 = (0, -2^{-1/2}), \quad P_4 = (0, 0),$$
$$P_5 = (0, -(3/2)^{1/2}), \quad P_6 = (0, (3/2)^{1/2}), \quad P_7 = (1 + O(\varepsilon^2), 0),$$
$$P_8 = (-1 + O(\varepsilon^2), 0).$$

The boundary of the disk where F_ε is defined is also a periodic orbit of K that we call P_1. These critical points of K_6 are related to nearby fixed points of F_ε and, hence, to periodic orbits of the Hamiltonian K. It is easily seen that P_1, P_2 and P_3 are elliptic with eigenvalues

$$\exp(\pm 14\pi 3^{-1/2}\varepsilon^3 \mathrm{i}(1 + O(\varepsilon)));$$

P_4, P_5 and P_6 are hyperbolic with eigenvalues

$$\exp(\pm 14\pi 3^{-1/2}\varepsilon^3(1 + O(\varepsilon)))$$

and, finally, P_7 and P_8 are elliptic with eigenvalues

$$\exp(\pm(14\pi/3)\varepsilon^2 \mathrm{i}(1 + O(\varepsilon)).$$

The invariant manifolds of the hyperbolic points originate pairwise heteroclinic points (see figure 3) and hence homoclinic points.

We concentrate our attention on the connection between P_4 and P_6. As this connection is very close to the p_1 axis we scale variables as follows: $q_1 = \varepsilon x$ and $p_1 = y$.

We denote by $\bar{F}_\varepsilon$ the family of diffeomorphisms in the new variables. It is readily seen that

$$\bar{F}_\varepsilon(x, y) = (x, y) + (7\pi\varepsilon^3/3)(-x^2y - 6y + 16y^3 - 8y^5, -2x + xy^2) + O(\varepsilon^4).$$

Hence $\bar{F}_\varepsilon$ is close to the time $7\pi\varepsilon^3/3$ map of the Hamiltonian flow given by

$$\bar{K}(x, y) = x^2(1 - y^2/2) - 3y^2 + 4y^4 - (4/3)y^6.$$

Now we look for the heteroclinic orbit of $\bar{K}$. Since P_4 and P_6 are on the level $\bar{K}(x, y) = 0$ we get $x = (3y - 2y^3)(3 - 3y^2/2)^{-1/2}$ on the separatrix. Therefore $\dot{y} = -(y - (2/3)y^3)(1 - y^2/2)^{1/2}$, which is easily integrated to obtain $3y - 2y^3 = \sqrt{2}/\cosh t$

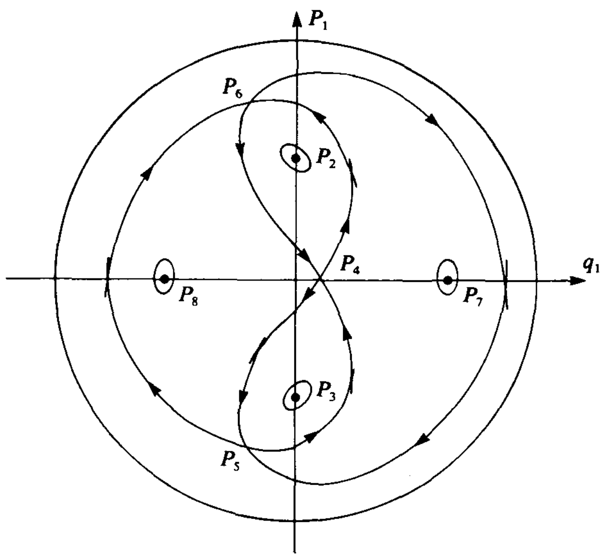

FIGURE 3

where the constant of integration has been selected to have $t=0$ when $y=2^{-1/2}$, and we have normalized the time in order to have the equations in the form (4.3) for the diagonal variables.

The singularities of the separatrix closer to the real axis are $\pm\pi i/2$ and hence $\delta_0=\pi/2$. By Theorem A (see Remark 4 after it), given $\eta>0$ the maximum separation between the invariant manifolds of $\bar{F}_\varepsilon$ will be bounded by

$$N \exp(-2\pi((\pi/2)-\eta)/\ln\lambda) \simeq N \exp(-\pi^2/(14\pi 3^{-1/2}\varepsilon^3)).$$

The separation for the invariant manifolds of F_ε and also the separation of the invariant manifolds of the periodic orbits of (6.9) are of the same kind.

In table 2 we reproduce part of the numerical results given in [12] for ε ranging from 0.35 to 0.23. Here α means the angle at a heteroclinic point. The value of λ given is obtained numerically because in that range the approximation $\ln\lambda \simeq 14\pi 3^{1/2}\varepsilon^3$ is too crude. Using the values $A=2^{-1/2}$ and $r=-5$ (a guess!) the formula

TABLE 2

ε	λ	Angle	ε	λ	Angle
0.35	3.6157	6.412 E−2	0.28	1.8993	8.49 E−5
0.34	3.2470	3.538 E−2	0.27	1.7706	1.54 E−5
0.33	2.9300	1.811 E−2	0.26	1.6596	2.06 E−6
0.32	2.6584	8.414 E−3	0.25	1.5637	1.87 E−7
0.31	2.4232	3.457 E−3	0.24	1.4808	1.07 E−8
0.30	2.2218	1.221 E−3	0.23	1.4089	3.5 E−10
0.29	0.0485	3.590 E−4			

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$\alpha = 2^{-1/2}\varepsilon^{-5}\exp(-\pi^2/\ln\lambda)$ gives results which agree with the computed ones with a relative error less of than 4%.

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