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ON THE HOMOTOPY TYPE OF THE p-SUBGROUP COMPLEX FOR FINITE SOLVABLE GROUPS

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Abstract

We provide a wedge decomposition of the homotopy type of the p-subgroup complex in the case of a finite solvable group G. In particular, this includes a new proof of the result of Quillen which says that this complex is contractible if and only if there is a non-trivial normal p-subgroup in G. We also provide reduction formulas for the G-module structure of the homology groups. Our results are obtained with diagram-methods by gluing the p-subgroup complex of G along the p-subgroup complex of $\bar{G} = G/N$ for a normal p'-subgroup of G.

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1. Introduction

For a finite group G and a prime p, we denote by $S_p(G) = \{P \leq G \mid |P| = p^i \neq 1\}$ the partially ordered set (poset for short) of all non-trivial p-subgroups of G, ordered by inclusion. For a poset P the *order complex* $\Delta(P)$ is the set of all chains of P, that is, the set of linearly ordered subsets. In particular, $\Delta(P)$ is a simplicial complex. In his influential paper [15] Quillen conjectured that $\Delta(S_p(G))$ is contractible if and only if there is a non-trivial (that is, $\neq 1$) normal p-subgroup in G. Quillen [15] himself verified this conjecture for several classes of groups, among them the class *of solvable groups. We provide in Theorem 1.1 an extension of Quillen's result to a* partial analysis of the homotopy type of the complex. The methods of the proof can also be used to provide information about the representation of G on the homology groups of $\Delta(S_p(G))$. Recall that a result of Quillen [15, Proposition 2.1] shows that $\Delta(S_p(G))$ and the complex $\Delta(A_p(G))$ are homotopic for all finite groups G. The

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basic idea of our approach, already employed by Quillen, is to study a covering of the order complex of the poset $A_p(G)$ of all non-trivial elementary abelian p-subgroups by order complexes of posets of type $A_p(NA)$, where N is a normal solvable p'-subgroup N and $A \in A_p(G)$. We transform this covering by homotopy co-limit methods (see Bousfield and Kan [6], Vogt [23] and Ziegler and Živaljević [26]) until we reach structurally nice formulas for homotopy type and Euler-characteristic (Theorem 1.1 and Theorem 1.2). In case $G = NA$, a semidirect product of a solvable p'-group N and an elementary abelian p-group A, Quillen [15, Theorem 11.2] showed that $S_p(G)$ is homotopically Cohen-Macaulay. We use the methods applied in the proof of Theorem 1.1 in order to retrieve Quillen's result in this case and additionally provide in Theorem 1.2 numerical information on the Euler-characteristic of the complex. In particular, we deduce in Theorem 1.2 that $\Delta(A_p(NA))$ is not contractible if A acts faithfully on N.

Further motivation for the study of $S_p(G)$ is provided in the papers of Brown [7, 8], which initiated the investigation $S_p(G)$, and more recently in connection with modular representation theory, in the work of Knörr and Robinson [11] and Thévenaz and Webb (see [20, 21, 25]). The, so far, most general result on Quillen's conjecture can be found in the work of Aschbacher and Smith [2].

Before we state our main result we introduce some basic notation. In general, for a poset P and an element $r \in P$ we write $P_{<r}$ for the poset $\{q \in P \mid q < r\}$. Analogously defined are the posets $P_{\leq r}$, $P_{>r}$ and $P_{\geq r}$. If $f : P \to Q$ is a map between posets then we will always implicitly assume that f is a map in the category of posets, which means $x \leq y \Rightarrow f(x) \leq f(y)$. We write $\Delta_1 * \Delta_2$ for the *join* of the simplicial complexes Δ_1 and Δ_2. In contrast to the usual definition [13], we define the join of a simplicial complex Δ_1 and the empty space to be the complex Δ_1. By $\Delta_1 \vee \Delta_2$ we denote the *wedge* of simplicial complexes Δ_1 and Δ_2. Note that for spaces which are not path-connected the wedge is only well defined if additionally wedge points c_i are specified. This will become crucial in our situation. In our wedge decompositions of the homotopy type of $\Delta(S_p(G))$ in Theorem 1.1 the wedge of spaces is not formed by using a single wedge point. Instead, we have to specify for each space in the wedge to where it is wedged to. This is just for technical reasons, since we also have to deal with the case of disconnected spaces. For Theorem 1.2 this problem does not arise. We write $\widetilde{\chi}(\Delta)$ to denote the reduced *Euler-characteristic* of the simplicial complex Δ. Via the functor $\Delta(\cdot)$ we are able to speak about homological and topological properties of posets. We write $\widetilde{H}_i(P)$ to denote the ith reduced homology group of $\Delta(P)$ and $\widetilde{\chi}(P)$ to denote the reduced Euler-characteristic of $\Delta(P)$. For a subgroup H of a group G we use $\mathrm{N}_G(H)$ for the normalizer of H in G and $\mathrm{C}_G(H)$ for the centralizer. Now we are in position to state our main results.

The first result provides a partial analysis of the homotopy type of $\Delta(S_p(G))$ for general finite solvable groups G. Note that by Quillen's Theorem 11.2 [15] the spaces $\Delta(A_p(NA))$ occurring in the description of the homotopy type are known to be homotopic to a wedge of spheres.

THEOREM 1.1. *Let G be a finite group with a solvable normal p'-subgroup N. For $A \leq G$ set $\bar{A} = AN/N$. Then $A_p(G)$ is homotopically equivalent to the wedge*

$$\Delta(A_p(\bar{G})) \vee \bigvee_{\bar{A} \in A_p(\bar{G})} \Delta(A_p(NA)) * \Delta(A_p(\bar{G})_{>\bar{A}}),$$

where for each $\bar{A} \in A_p(\bar{G})$ an arbitrary point $c_{\bar{A}} \in \Delta(A_p(NA))$ is identified with $\bar{A} \in \Delta(A_p(\bar{G}))$.

Note that by the way the wedge is formed in Theorem 1.1 the space is disconnected if $\Delta(A_p(\bar{G}))$ is disconnected.

Our second theorem replaces Quillen's Theorem 11.2 [15] and adds numerical information on $\Delta(S_p(G))$ for finite solvable groups G that are split extensions of p'-groups by elementary abelian p-groups. We mention that a proof of Quillen's conjecture for p-solvable groups with abelian p-Sylow subgroups, which relies purely on combinatorial information about the numerical value of $\widetilde{\chi}(S_p(G))$, can be found in the work of Hawkes and Isaacs [10]. Our formula has the advantage of providing more structural insight, since only positive terms occur and all terms in our formula can—up to homotopy—be interpreted as certain parts of the space. On the other hand, for computational purposes the alternating sum given by Hawkes and Isaacs [10] may be much more efficient.

THEOREM 1.2. *Let $G = KA$ be a semidirect product of a normal solvable p'-group K and an elementary abelian p-group A. Then $A_p(KA)$ is homotopically Cohen-Macaulay. If N is a G-invariant subgroup of K, then the number of spheres in the wedge-decomposition of $A_p(KA)$ equals the absolute value of its reduced Euler-characteristic which equals*

$$\left|\widetilde{\chi}\left(A_p(G)\right)\right| = \sum_{\bar{B} \in A_p(\bar{G}) \cup \{\hat{0}\}} \left|\widetilde{\chi}\left(A_p(NB)\right) \cdot \widetilde{\chi}\left(A_p(C_{\bar{G}}(\bar{B})/\bar{B})\right)\right|.$$

Here we set $\bar{B} = BN/N$ for $B \leq A$. Moreover, this number is 0 if and only if $\Delta(A_p(G))$ is contractible, which happens if and only if A does not act faithfully on K.

In Proposition 4.2 we provide results similar to Theorem 1.1 and Theorem 1.2 for the G-module structure of the homology groups of $\Delta(A_p(G))$, These formulas are not direct consequences of Theorem 1.1 and Theorem 1.2, but are obtained by similar methods.

We mention that in recent work of Segev and Webb [17] a wedge decomposition of the complex $\Delta(A_p(G))$ is given in a situation 'dual' to the situations treated in Theorem 1.1 and Theorem 1.2. Their reduction formulas work best for normal subgroups 'close' to $O^{p'}(G)$ (that is, the minimal normal subgroup N of G such that $\bar{G}$ is a p'-group); whereas our formulas work only for normal subgroups that are p'-groups. Our Theorem 1.1 reduces the study of the Quillen complex of G to the study of upper intervals in the poset $A_p(\bar{G})$. Thévenaz has raised the question whether $\Delta(A_p(G))$ is always homotopic to a wedge of spheres (of possibly different dimensions). We already know by Quillen's result [15] that $\Delta(A_p(NA))$—in the situation of Theorem 1.2—is homotopic to a wedge of spheres. It is also well known that the join of a wedge of spheres with a wedge of spheres is homotopic to a wedge of spheres. Thus if one would know by induction that the order complex of an upper intervals in $A_p(\bar{G})$ is homotopic to a wedge of spheres, the question of Thévenaz would have a positive answer for the group G. Finally we give two examples which show that even in cases where it is possible to verify that $\Delta(A_p(G))$ is homotopic to a wedge of spheres, it is not so easy to guess which dimensions these spheres will have. In particular, there are solvable groups G such that $\Delta(A_p(G))$ is homotopic to a wedge of spheres, but not all their dimensions reflect the ranks of maximal elementary abelian p-subgroups.

2. Topological tools

Our basic methods are some lemmata from [6, 16, 22, 26] in the theory of diagrams of spaces and homotopy co-limits. Since these methods are not standard in the combinatorial theory of order complexes, we recall the basic constructions here (see [6] and [23] for more details). A *diagram of spaces* $\mathscr{D}$ is a poset P together with spaces D_r for $r \in P$ and maps $d_{rq} : D_r \to D_q$ for $q \leq r$ such that $d_{r'r''} \circ d_{rr'} = d_{rr''}$ and $d_{rr} = id_{D_r}$ for $r'' \leq r' \leq r$ in P. The setting of Bousfield and Kan [6] and Vogt [23] is more general, they consider instead of a poset an arbitrary small category, but for our applications the restriction to posets is just the right setting. Some of the lemmata listed below are formulated in [6] or [23] for small categories, therefore we always add the reference to the work of Ziegler and Živaljević [26], where the results are formulated in the poset situation.

Our basic example for a diagram of spaces will be the following. Let $\mathscr{U} : X = \bigcup_{i \in I} X_i$ be a covering of a space X by a finite number of subspaces. The *intersection poset* of the covering $\mathscr{U}$ is the partially ordered set $P^{\mathscr{U}}$ on the set of intersections $\bigcap_{i \in J} X_i$ for $J \subseteq I$ with the reversed inclusion as the order relation. We write X_J to denote the intersection $\bigcap_{i \in J} X_i$. In particular the space X_I is the unique maximal element of $P^{\mathscr{U}}$. Note that for different subsets $J, J' \subseteq I$ we may have $X_J = X_{J'}$; in this case X_J and $X_{J'}$ represent that same element of $P^{\mathscr{U}}$. If $X_I = \emptyset$, then $\emptyset$ is the unique maximal element of $P^{\mathscr{U}}$. There is a natural diagram $D^{\mathscr{U}}$ on the poset $P^{\mathscr{U}}$.

For $X \in P^{\mathcal{U}}$ we set $D_r = X$ and for $r > r'$ let $d_{rr'}$ be the the natural inclusion.

The *homotopy co-limit* $\underrightarrow{\mathrm{hocolim}}\, \mathcal{D}$ of an arbitrary diagram $\mathcal{D}$ over a poset P is constructed from the space $\coprod_{r \in P} D_r \times \Delta(P_{\leq r})$ by identifying points according to the equivalence relation '$\equiv$'. The equivalence relation '$\equiv$' is generated by $\alpha(u, v) \equiv \beta(u, v)$, where

$$\alpha : D_r \times \Delta(P_{\leq q}) \hookrightarrow D_r \times \Delta(P_{\leq r}),$$

for $q \leq r$, is induced by the identity in the first and by the inclusion in the second component, and

$$\beta : D_r \times \Delta(P_{\leq q}) \to D_q \times \Delta(P_{\leq q}),$$

for $q \leq r$, is induced by d_{rq} in the first and by the identity in the second component. Now if a covering $\mathcal{U} : X = \bigcup_{i \in I} X_i$ is sufficiently nice the space X and the homotopy co-limit of the diagram $\mathcal{D}^{\mathcal{U}}$ have the same homotopy type.

PROPOSITION 2.1 (Projection Lemma, see [6, 16, 26]). *Let $\mathcal{U} : X = \bigcup_{i \in I} X_i$ be a covering of the CW-complex X by a finite number of closed subcomplexes X_i, then* $\underrightarrow{\mathrm{hocolim}}\, \mathcal{D}^{\mathcal{U}} \simeq X$.

In a next step we would like to modify the maps d_{rq} and the spaces D_r of a given diagram in a way that preserves the homotopy type of the homotopy co-limit.

PROPOSITION 2.2 (Homotopy Lemma, see [6, 22, 26]). *Let $\mathcal{D}$ and $\mathcal{D}'$ be diagrams of spaces on the poset P. Assume that there are maps $f_r : D_r \to D'_r$ such that $d'_{rq} \circ f_r = f_q \circ d_{rq}$ for $q \leq r$ in P and f_r induces a homotopy equivalence between D_r and D'_r. Then* $\underrightarrow{\mathrm{hocolim}}\, \mathcal{D} \simeq \underrightarrow{\mathrm{hocolim}}\, \mathcal{D}'$.

Note that we must have $d'_{rq} \circ f_r = f_q \circ d_{rq}$ in the formulation of the Homotopy Lemma but no commutativity condition is imposed on the homotopies induced by the maps f_r.

Finally the following lemma allows the computation of the homotopy type if some strong assumptions are satisfied.

PROPOSITION 2.3 (Wedge Lemma, see [26]). *Let $\mathcal{D}$ be a diagram of spaces over the poset P with maximal element $\hat{1}$ such that for each $q \in P$, $q \neq \hat{1}$, there is a point $c_q \in D_q$ such that $d_{rq}(x) = c_q$ for all $r > q$ and $x \in D_r$. Then* $\underrightarrow{\mathrm{hocolim}}\, \mathcal{D}$ *is homotopy equivalent to*

$$\bigvee_{r \in P} D_r * \Delta(P_{<r}),$$

where the wedge is formed by identifying for every $r < \hat{1}$ the point $c_r \in D_r * \Delta(P_{<r})$ with $r \in D_{\hat{1}} * \Delta(P_{<\hat{1}})$.

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Combining the preceding three propositions we obtain the following corollary.

COROLLARY 2.4. *Let $f : P \to Q$ be a map of (finite) posets. Assume:*

(i) *Q is a meet-semilattice, that is all infima exist in Q. In particular, Q contains a least element $\widehat{0}$.*

(ii) *All elements $q \in Q$ with the possible exception of $\widehat{0}$ are in the image of f.*

(iii) *For $q \in Q_{>\widehat{0}}$ there exists an element $c_q \in f^{-1}(Q_{\leq q})$ such that the inclusion map $\iota_q : \Delta(f^{-1}(Q_{<q})) \hookrightarrow \Delta(f^{-1}(Q_{\leq q}))$ is homotopic to the constant map sending $\Delta(f^{-1}(Q_{<q}))$ to c_q.*

Then $\Delta(P)$ is homotopy equivalent to the wedge

$$\bigvee_{q \in Q} \Delta(f^{-1}(Q_{\leq q})) * \Delta(Q_{>q}),$$

where for $q \in Q_{>\widehat{0}}$ the point $c_q \in \Delta(f^{-1}(Q_{\leq q})) \subseteq \Delta(f^{-1}(Q_{\leq q})) * \Delta(Q_{>q})$ is identified with $q \in \Delta(f^{-1}\{\widehat{0}\}) * \Delta(Q_{>\widehat{0}})$.

REMARK. Condition (ii) avoids that arbitrary points can be adjoined to Q. It can be replaced by the somewhat weaker condition:

(ii') If $\Delta(f^{-1}(Q_{\leq q}))$ is not contractible, then $q \in f(P)$.

PROOF. $\mathscr{U} := (\Delta(f^{-1}(Q_{\leq q})))_{q \in Q}$ is a finite covering of $\Delta(P)$ by closed subcomplexes, hence $\underrightarrow{\mathrm{hocolim}}\, \mathscr{D}^{\mathscr{U}} \simeq \Delta(P)$ by the Projection Lemma, Proposition 2.1. By (i) and (ii) the intersection poset $P^{\mathscr{U}}$ of $\mathscr{U}$ can be identified with the dual poset Q^* of Q, this means Q^* is the poset on the ground set Q ordered by '$\leq^*$', which is defined by $q \leq^* q'$ if and only if $q' \leq q$. This can be seen as follows. By definition for each $q \in Q$ there is an element in $P^{\mathscr{U}}$. Since Q is a meet-semilattice we have

$$f^{-1}(Q_{\leq q}) \cap f^{-1}(Q_{\leq q'}) = f^{-1}(Q_{\leq q \wedge q'}).$$

We define a second diagram $\mathscr{D}'$ of spaces over Q^* by setting $D'_q := \Delta(f^{-1}(Q_{\leq q}))$ for $q \in Q^*$ and $d'_{qq'} := c_{q'}$ for $q' <^* q$. By (iii) there is homotopy $f_{q'} : D_{q'} \to D'_{q'}$ such that $f_{q'}(d_{qq'}(D_q)) = c_{q'}$ for all $q' <^* q$. This defines a map of diagrams from $\mathscr{D}$ to $\mathscr{D}'$ satisfying the conditions of the Homotopy Lemma, Proposition 2.2. We conclude $\underrightarrow{\mathrm{hocolim}}\, \mathscr{D}' \simeq \underrightarrow{\mathrm{hocolim}}\, \mathscr{D}^{\mathscr{U}}$ ($\simeq \Delta(P)$). Finally, the application of the Wedge Lemma, Proposition 2.3, to diagram $\mathscr{D}'$ shows the claimed homotopy equivalence. $\square$

It would be interesting to generalize Corollary 2.4 to G-homotopy types, but the authors see no way how to do this. Therefore, we can state and prove the main result of Section 3 only for homotopy, not for G-homotopy equivalence.

In the situation of Corollary 2.4 let us replace (iii) by the stronger condition that $\Delta(f^{-1}(Q_{\leq q}))$ is contractible for each $q > \widehat{0}$. Then the corollary yields $\Delta(P) \simeq$

$\Delta(f^{-1}\{\widehat{0}\}) * \Delta(Q_{>\widehat{0}})$ (note that the join of two spaces is contractible if at least one of the spaces is contractible). If additionally $\Delta(f^{-1}\{\widehat{0}\}) = \emptyset$, or equivalently if f is a map into $Q_{>\widehat{0}}$, then $\Delta(P) \simeq \Delta(Q_{>\widehat{0}})$. But this is well-known by the Fiber Lemma of Quillen which we state below in the equivariant version due to Thévenaz and Webb [21], since we will need some facts about G-homotopy equivalence for our results on Steinberg-Modules in Section 4.

By a *G-poset* P we understand a poset P on which the group G acts order-preserving, which means $x \leq y$ if and only if $x^g \leq y^g$. A map $f : P \to Q$ between the G-sets P and Q is called *equivariant* if $f(x^g) = f(x)^g$ for all $g \in G$ and $x \in P$.

PROPOSITION 2.5 (Fiber Lemma [15, Proposition 1.6] and [21, Theorem 1]). *Let $f : P \to Q$ be an equivariant map of G-posets. If for all $q \in Q$ the order complex $\Delta(f^{-1}(Q_{\leq q}))$ is* $\mathrm{Stab}_G(q)$*-contractible then f induces a G-homotopy equivalence between $\Delta(P)$ and $\Delta(Q)$.*

In the lemma $\mathrm{Stab}_G(q)$ denotes the stabilizer of q in G. Recall that a G-homotopy equivalence is a homotopy equivalence between G-spaces which is equivariant.

3. Applications of the topological tools on the poset $A_p(G)$

In this section let G always denote a finite group and N a normal p'-subgroup of G. Recall that we use the bar-notation, that is, we write $\bar{H}$ for the image HN/N of a subgroup H of G under the homomorphism $\pi : G \to \bar{G} = G/N$. Further if a subgroup $\bar{H}$ of $\bar{G}$ is given, we write HN for its preimage $\pi^{-1}(\bar{H})$.

The image $\bar{A}$ under π of an elementary abelian p-subgroup A of G is isomorphic to A. Therefore π induces a map $f : A_p(G) \to A_p(\bar{G})$, $A \mapsto \bar{A}$. We want to apply Corollary 2.4 in this situation, but before doing this we first observe the following simple, but crucial facts:

LEMMA 3.1. *Let G be a finite group with a normal p'-subgroup N. Then*:

(a) *$A_p(G) \cup \{\widehat{0}\}$, where $\widehat{0}$ serves as a minimal element, is a meet-semilattice.*

(b) *$A_p(G)_{>A} = A_p(\mathrm{C}_G(A))_{>A}$ for elementary abelian p-subgroups A of G.*

(c) *If the p-Sylow subgroups of G are elementary abelian, then the poset $A_p(G)_{>A}$ is $\mathrm{N}_G(A)$-isomorphic to $A_p(\mathrm{C}_G(A)/A)$ for each elementary abelian p-subgroup of G (via the map $B \mapsto B/A$).*

(d) *If A is not the intersection of maximal elementary abelian p-subgroups, then $A_p(G)_{>A}$ is $\mathrm{N}_G(A)$-contractible.*

(e) *$f : A_p(G) \to A_p(\bar{G})$, $A \mapsto \bar{A}$ is an surjective map.*

(f) *If we write $AN := \pi^{-1}(\bar{A})$ for an elementary abelian $\bar{A} \leq \bar{G}$, then*

$$f^{-1}(A_p(\bar{G})_{\leq \bar{A}}) = f^{-1}(A_p(\bar{A})) = A_p(AN).$$

PROOF. Each statement except (d) is immediately clear by the definitions (respectively, the surjectivity of f is clear by Sylow's Theorems).

For part (d) let $g : A_p(G)_{>A} \to A_p(G)_{>A}$ be the map that assigns to each subgroup C the intersection of all maximal elementary abelian p-subgroups that contain C. Since A is not the intersection of all maximal elements of $A_p(G)_{>A}$, the image $g(A_p(G)_{>A})$ is a poset with a unique minimal element $\hat{0}_A$ which corresponds to the intersection of all maximal elements of $A_p(G)_{>A}$. Thus $\Delta(g(A_p(G)_{>A}))$ is a cone with apex $\hat{0}_A$ and therefore $\mathrm{N}_G(A)$-contractible. The map g is a poset map and $\mathrm{N}_G(A)$-equivariant. Let $C \in A_p(G)_{>A}$. If C is the intersection of maximal elementary abelian p-subgroups, then $g^{-1}(A_p(G)_{\geq C}) = A_p(G)_{\geq C}$. Thus $\Delta(g^{-1}(A_p(G)_{\geq C}))$ is a cone over C and hence $\mathrm{N}_G(C)$-contractible. If C is not the intersection of maximal elementary abelian p-subgroups, then $g^{-1}(A_p(G)_{\geq C})$ is contractible by induction. Thus the Fiber Lemma, Proposition 2.5, applies and shows that $\Delta(A_p(G)_{>A})$ and $\Delta(g(A_p(G)_{>A}))$ are $\mathrm{N}_G(A)$-homotopy equivalent. □

By (a) and (e) of the preceding lemma, we know that for an application of Corollary 2.4 to the map $f : A_p(G) \to A_p(\bar{G}) \cup \{\hat{0}\}$ it remains to verify condition (iii) of Corollary 2.4. We claim that this is done, once we know that $\Delta(A_p(NA))$ is homotopic to a wedge of spheres of dimension rank $A - 1$, if $N \rtimes A$ is a semidirect product of a normal p'-group N and an elementary abelian p-group A. By $f^{-1}(A_p(\bar{G})_{<\bar{A}}) = \bigcup_{\bar{B}<\bar{A}} f^{-1}(A_p(\bar{G})_{\leq \bar{B}}) = \bigcup_{\bar{B}<\bar{A}} A_p(BN)$ it follows that $\Delta(f^{-1}(A_p(\bar{G})_{<\bar{A}}))$ is a (rank $A-2$)-dimensional space. Now a wedge of (rank $A-1$)-spheres is (rank $A-2$)-connected. This implies by standard topology the following lemma.

LEMMA 3.2. *If* $\Delta(A_p(NA))$ *is homotopic to a wedge of* (rank A − 1)*-spheres, then the embedding* $\Delta(f^{-1}(A_p(\bar{G})_{<\bar{A}})) \hookrightarrow \Delta(f^{-1}(A_p(\bar{G})_{\leq\bar{A}})) = \Delta(A_p(NA))$ *is homotopic to a constant map.*

The fact that $\Delta(A_p(NA))$ is homotopic to a wedge of (rank A − 1)-spheres in the case of a solvable group N is due to Quillen [15, Theorem 11.2]. Indeed he proves the stronger conclusion that $\Delta(A_p(NA))$ is homotopically Cohen-Macaulay (see Question 3.6 below). Whether the same is true for non-solvable N is a problem [15, Problem 12.3] raised by Quillen. But—as far the authors know—for rank $A \geq 3$ even the weaker conjecture [1] of Aschbacher that $A_p(NA)$ is simply-connected remains unproved until now.

Later we will show how to obtain the Cohen-Macaulayness in the solvable case by our methods, but first we prove the following weak version of the result.

LEMMA 3.3. *For semidirect products* $G = K \rtimes A$ *where* K *is a solvable* p'*-group and* A *is an elementary abelian* p*-group,* $\Delta(A_p(KA))$ *is homotopic to a wedge of* (rank A − 1)*-spheres.*

PROOF. The proof is by induction on the order of KA.

If A does not act faithfully on K, then the kernel $B \leq A$ of this action is in the center of KA. The maximal elementary abelian p-subgroups of G are the Sylow p-subgroups which intersect in $O_p(G) \geq B$ (indeed '=' B). Hence by Lemma 3.1 (d) $A_p(KA)$ is contractible and therefore the wedge of 0 spheres (of arbitrary dimension).

So assume that A acts faithfully on K. If K contains a proper A-invariant normal subgroup N then we can apply Corollary 2.4 to the map $f : A_p(KA) \to A_p(KA/N) \cup \{\widehat{0}\}$, $B \mapsto KB/N$. The assumptions (i) and (ii) of Corollary 2.4 are true by Lemma 3.1 (a) and (e), and (iii) is proved by Lemma 3.2 and the induction hypothesis. As usual we will denote for a subgroup B of G by $\bar{B}$ the image NB/N of B under the projection $G \to \bar{G} = G/N$. Hence Corollary 2.4 yields

$$\begin{aligned}
\Delta(A_p(KA)) &\simeq \bigvee_{\bar{B} \in A_p(\overline{KA}) \cup \{\widehat{0}\}} \Delta(f^{-1}(A_p(\overline{KA})_{\leq \bar{B}})) * \Delta(A_p(\overline{KA})_{>\bar{B}}) \\
&\overset{3.1\,(c)}{\simeq} \bigvee_{\bar{B} \in A_p(\overline{KA}) \cup \{\widehat{0}\}} \Delta(f^{-1}(A_p(\bar{B}))) * \Delta(A_p(C_{\overline{KA}}(\bar{B})/\bar{B})) \\
&= \bigvee_{\bar{B} \in A_p(\overline{KA}) \cup \{\widehat{0}\}} \Delta(A_p(NB)) * \Delta(A_p(C_{\bar{K}}(\bar{B})\bar{A}/\bar{B})).
\end{aligned}$$

By the induction hypothesis the simplicial complexes $\Delta(A_p(NB))$ and $\Delta(A_p(C_{\bar{K}}(\bar{B}) \rtimes \bar{A}/\bar{B}))$ are homotopic to a wedge of (rank $B-1$)-spheres and (rank $A -$ rank $B - 1$)-spheres. As the join between a wedge of m spheres of dimensions n with a wedge of m' spheres of dimension n' is homotopic to a wedge of mm' spheres of dimension $(n + n' + 1)$, we conclude the claim in this case.

Finally assume that K does not possess a proper $K \rtimes A$-invariant subgroup. Then K is characteristically simple, hence (as a solvable group) an elementary abelian q-group where q is a prime $\neq p$, or equivalently K is a $\mathbb{F}_q$-vector space on which A acts faithfully and irreducibly. Hence A is cyclic by [9, Theorem 3.2.2] and $A_p(KA)$ is an antichain, which as a topological space is the wedge of $(|\operatorname{Syl}_p(KA)| - 1)$ 0-spheres. $\square$

Now we are in the position to prove our main result:

THEOREM 1.1. *Let G be a finite group with a solvable normal p'-subgroup N. Then $A_p(G)$ is homotopically equivalent to the wedge*

$$\Delta(A_p(\bar{G})) \vee \bigvee_{\bar{A} \in A_p(\bar{G})} \Delta(A_p(NA)) * \Delta(A_p(\bar{G})_{>\bar{A}}),$$

where for each $\bar{A} \in A_p(\bar{G})$ an arbitrary chosen point $c_{\bar{A}} \in \Delta(A_p(NA))$ is identified with $\bar{A} \in \Delta(A_p(\bar{G}))$.

PROOF. We apply Corollary 2.4 to the mapping $f : A_p(G) \to A_p(\bar{G}) \cup \{\widehat{0}\}$. By the Lemmata 3.1 (a) and (e), 3.2 and 3.3, the assumptions of Corollary 2.4 are fulfilled, hence

$$A_p(G) \simeq \bigvee_{\bar{A} \in A_p(\bar{G}) \cup \{\widehat{0}\}} \Delta(f^{-1}(A_p(\bar{G})_{\le \bar{A}}) * \Delta(A_p(\bar{G})_{>\bar{A}}).$$

Then Lemma 3.1 (f) and our convention $X * \emptyset = X$ show the claimed homotopy formula. □

REMARK 3.4. (a) In the homotopy formula of Theorem 1.1 many summands in the wedge may be identical and some may be contractible. Therefore, it is sometimes possible to write the wedge with a much smaller number of summands.

The poset $A_p(NA)$ is contractible if (and only if) A does not act faithfully on N by the first part of the proof of Lemma 3.3 (respectively, Corollary 3.6 below). Further $A_p(\bar{G})_{>\bar{A}}$ is contractible if A is not the intersection of maximal elementary abelian p-subgroups by Lemma 3.1 (d). As the join of any space with a contractible space is contractible, we have only to sum up over $\bar{A}$ which are such intersections of maximal elementary abelian p-subgroups of $\bar{G}$ such that A acts faithfully on N.

There is a second way to reduce the number of summands. It is given by the fact that for $\bar{A}$ and $\bar{A}'$ which are conjugated by a $\bar{g} \in \bar{G}$ the spaces $A_p(NA)$ and $A_p(NA')$ (respectively, $A_p(\bar{G})_{>\bar{A}}$ and $A_p(\bar{G})_{>\bar{A}'}$) are homeomorphic via the conjugation by g. Writing short $n \cdot X$ instead of $\bigvee_{i=1}^{n} X$ we obtain the formula

$$A_p(G) \simeq A_p(\bar{G}) \vee \bigvee_{\bar{A} \in \mathscr{R}} n_{\bar{A}} \cdot \left(A_p(NA) * A_p(\bar{G})_{>\bar{A}}\right),$$

where $n_{\bar{A}}$ is the number of conjugates of $\bar{A}$ in $\bar{G}$ and $\mathscr{R}$ denotes a set of representatives of the orbits of the action of $\bar{G}$ on the set of all $\bar{A} \in A_p(\bar{G})$ which are intersections of maximal elementary abelian p-subgroups and which act faithfully on N.

(b) Our formula suggests that the homotopy-type of $A_p(G)$ can be calculated recursively for solvable groups if one knows the structure of upper intervals in $A_p(G)$. It is possible to generalize the formula for upper intervals (by Lemma 3.1 (c) upper intervals in $A_p(KA)$ are again homotopic to a wedge of spheres in the proper dimension), but for the sake of simplicity we state only the global version.

Note however that our decomposition is only non-trivial if N is non-trivial. In particular, we will not be able to say anything about upper intervals $A_p(G)_{>A}$ for groups where $C_G(A)$ has no non-trivial normal p'-subgroup.

Now we want to show which additional informations can be derived by our methods for the special case of semidirect products. For doing this we need some definitions. A poset P is called *ranked* if all maximal chains in P are of like length, its rank

$\mathrm{rank}(P)$ is the cardinality of a maximal chain in P. A ranked poset P is called *homotopically Cohen-Macaulay*, if for all $r \leq q \in P$ the order complexes of the posets $P_{<r}$, $P_{>r} \cap P_{<q}$, $P_{>q}$ and P are homotopic to a wedge of spheres of dimensions $\mathrm{rank}(P_{<r}) - 1$, $\mathrm{rank}(P_{<q}) - \mathrm{rank}(P_{<r}) - 1$, $\mathrm{rank}(P) - \mathrm{rank}(P_{<q}) - 1$ and $\mathrm{rank}(P) - 1$. The far reaching implications of being homotopically Cohen-Macaulay will not be needed here. We refer the reader to [4] for an excellent survey. The non-enumerative part of the following Theorem 1.2 is due to Quillen [15, Theorem 11.2].

THEOREM 1.2. *Let $G = KA$ be a semidirect product of a normal solvable p'-group K and an elementary abelian p-group A. Then $A_p(KA)$ is homotopically Cohen-Macaulay. If N is a G-invariant subgroup of K, the number of spheres in the wedge-decomposition of $A_p(KA)$ equals the absolute value of its reduced Euler-characteristic which equals*

$$\left|\widetilde{\chi}\left(A_p(G)\right)\right| = \sum_{\bar{B} \in A_p(\bar{G}) \cup \{\widehat{0}\}} \left|\widetilde{\chi}\left(A_p(NB)\right) \cdot \widetilde{\chi}\left(A_p(\mathrm{C}_{\bar{G}}(\bar{B})/\bar{B})\right)\right|.$$

Moreover, this number is 0 if and only if $\Delta(A_p(G))$ is contractible, which happens if and only if A does not act faithfully on K.

PROOF. Let $B \leq C$ be two elementary abelian p-subgroups of G. Then $A_p(G)_{<B}$ and $A_p(G)_{>B} \cap A_p(G)_{<C}$ are of rank $\mathrm{rank}(B) - 1$ and $\mathrm{rank}(C) - \mathrm{rank}(B) - 1$ (here we denote for $B \in A_p(G)$ by $\mathrm{rank}(B)$ the rank of $A_p(G)_{<B}$) and proper parts of a geometric modular lattice. It is well known that in this case the order complexes are homotopic to a wedge of spheres of $\mathrm{rank}(B) - 1$ and $\mathrm{rank}(C) - \mathrm{rank}(B) - 1$. By Lemmata 3.1 (c) and 3.3 the complex $\Delta(A_p(G)_{>C})$ (respectively, $\Delta(A_p(G))$) is homotopic to a wedge of spheres of rank $\mathrm{rank}(A_p(G)) - \mathrm{rank}(C) - 1$ (respectively, $\mathrm{rank}(A_p(G)) - 1$). Thus $A_p(G)$ is homotopically Cohen-Macaulay (see also [15, Proposition 10.1]).

The formula for the reduced Euler-characteristic is immediately clear by Theorem 1.1 and the fact that all occurring spaces in the wedge are spheres of the same dimension (Lemma 3.3).

For the last part of the claim we have already seen that if A acts not faithfully on K then $\mathrm{O}_p(G) \neq 1$, which implies that $A_p(G)$ is contractible. If A acts faithfully on K, let N be a minimal G-invariant subgroup of K. If $N = K$, then the A-module K (K is an elementary abelian p'-group, since K is solvable and characteristically simple) is irreducible. Therefore, by [9, Theorem 3.3.2] A is cyclic and the result follows. If $N < K$, define $B := \mathrm{C}_A(\bar{K})$. Since A acts faithfully on K, and K and A have coprime order, by [9, Theorem 5.3.2] the group B acts faithfully on N. By construction, $B = \mathrm{O}_p(\mathrm{C}_{\bar{K}}(B)A)$. Hence $\mathrm{C}_{\bar{K}}(B)A/B$ is the semidirect product of the p'-group $\mathrm{C}_{\bar{K}}(B)$ and an elementary abelian p-group C which acts faithfully on $\mathrm{C}_{\bar{K}}(B)$. The

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induction hypothesis yields, $\widetilde{\chi}(A_p(NB)) \neq 0$ and $\widetilde{\chi}(A_p(C_{\bar{K}}(B)A/B)) \neq 0$. Thus, the term corresponding to B does not vanish. In particular, $\widetilde{\chi}(A_p(NA)) \neq 0$. □

Actually a strong form of the first part of the preceding corollary is true. If $G = KA$ with $K = O_{p'}(G)$ solvable and A an elementary abelian p-Sylow-subgroup, then $A_p(G)$ is indeed shellable (see the paper of Björner [3] for an introduction of this concept). For abelian normal subgroups K this was first discovered by Özaydin and Kutin in [12] and later a different reasoning led to the general statement [14].

Concerning the Euler-characteristic formula we mention that it is possible to combine—as in Remark 3.4 (a) for the homotopy formula—many of the summands. But contrary to the general case, here the formula can be used recursively to calculate the Euler-characteristic completely since the Sylow p-subgroups are elementary abelian (and so Lemma 3.1 (c) applies).

Our formula has the advantage that all the summands are positive, but for computational reason the formula [10, Lemma 1.4] might be easier. Another way of proving our formula is to apply Corollary 3.2 in [24] together with Lemma 3.3.

Finally we want to draw a conclusion of Theorem 1.1 for solvable groups.

COROLLARY 3.5. *Let G be a finite solvable group with no non-trivial normal p-subgroup. Then $A_p(G)$ can be decomposed as a wedge of spaces, where for each maximal elementary abelian p-subgroup A of $A_p(G)$ there occurs (at least) one sphere of dimension* rank $A - 1$. *In particular, Quillen's conjecture is true for solvable groups.*

PROOF. As $O_p(G) = 1$ the Fitting subgroup $N := F(G)$ of G has order prime to p. As G is solvable, N has the property $C_G(N) \subseteq N$ ([9, Theorem 6.1.3]). Therefore, any p-subgroup acts faithfully on N. Let A be a maximal (with respect to inclusion) element of $A_p(G)$, that is, $\bar{A}$ is maximal in $A_p(\bar{G})$. Then the summand corresponding to $\bar{A}$ in the wedge-decomposition of Theorem 1.1 is $A_p(NA) * A_p(\bar{G})_{>\bar{A}} = A_p(NA) * \emptyset = A_p(NA)$. By the last corollary this summand is a non-trivial wedge of spheres of dimension rank $A - 1$. □

The preceding corollary is not true for general groups. For example for $G = \mathrm{Gl}_n(\mathbb{F}_q)$, $q = p^i$, the space $A_p(G)$ is homotopic the building of G. In particular, it is homotopic to a wedge of spheres of dimension $n - 1$. On the other hand, in all examples the authors know of, the space $A_p(G)$ is homotopic to a wedge of spheres (of possibly different dimension). So we want to mention a question raised by Thévenaz.

QUESTION 3.6. Is $A_p(G)$ always homotopic to a wedge of spheres (of possibly different dimension)?

In the case of a solvable group G, Theorem 1.1 reduces a positive answer to the question to the structure of upper intervals $A_p(C_G(A))_{>A}$, where $C_G(A)$ has no normal

p'-subgroup (compare Remark 3.4 (b)). But even for p-groups it is not clear, how those spaces look like.

In the second example of Section 5.2 we disprove the stronger conjecture that for solvable groups G only spheres of dimension rank $A-1$ occur where A runs over the maximal (with respect to inclusion) elements of $A_p(G)$. This shows that for solvable groups G the set $A_p(G)$ cannot be non-pure shellable in the sense of Björner and Wachs (see [5]) in general.

4. Formulas for the Steinberg module

We see no way to strengthen our results of the last to G-homotopy types. But via the standard spectral sequence for homotopy co-limits [6, 16] we obtain the following result on the homology modules. Here $\widetilde{H}(\cdot)$ denotes reduced simplicial homology with coefficients in the ring of integers $\mathbb{Z}$. An application of Segal's arguments in the given situation yields the following proposition, which is a reformulation of [18, Proposition 2.3] for our purposes. The result can also be obtained by a Mayer-Vietoris spectral sequence.

PROPOSITION 4.1. *Let $\mathcal{U} : X = \bigcup_{i=1}^{t} X_i$ be a covering of a CW-complex X by a finite number of closed subcomplexes X_i. Assume that the finite group G acts on X as a group of homeomorphisms, such that $X_i^g = X_{i(g)}$ for all $1 \le i \le t$, $g \in G$ and some $1 \le i(g) \le t$. Let $J_1 \subseteq J_2$ be arbitrary subsets of J. Assume*:

(i) *If $\bigcap_{j\in J_1} X_j \neq \bigcap_{j \in J_2} X_j$ then the inclusion map $\bigcap_{j\in J_2} X_j \hookrightarrow \bigcap_{j\in J_1} X_j$ is homologically trivial, that is it induces the constant map in homology.*

(ii) *All homology groups of $\bigcap_{j\in J_1} X_j$ are free.*

Let $\mathcal{D}^{\mathcal{U}}$ be the diagram associated to $\mathcal{U}$. Then there is an isomorphism of G-modules

$$\widetilde{H}_i(X) \cong_G \bigoplus_{r\in P^{\mathcal{U}}/G} \bigoplus_{k+l=i-1} \operatorname{ind}_{\mathrm{Stab}_G(r)}^{G} \left(\widetilde{H}_l(D_r) \otimes \widetilde{H}_k(P^{\mathcal{U}}_{<r})\right),$$

where $P^{\mathcal{U}}/G$ is a set of representatives of the G-orbits on $P^{\mathcal{U}}$ and $\mathrm{Stab}_G(r)$ *is the stabilizer of r in G.*

From the preceding proposition and facts from Theorem 1.1, Lemma 3.1, Lemma 3.3 we infer the following result. From Section 3 we adopt the notation $\bar{A}$ for the image NA/N of the subgroup A of the group G under the map $f : G \to \bar{G} = G/N$, $f(A) = \bar{A}$, for a normal subgroup N of G.

PROPOSITION 4.2. (A) *Let G be a finite group and let N be a solvable normal p'-subgroup. Then $\widetilde{H}_i(A_p(G))$ is as a G-module isomorphic to*

$$\widetilde{H}_i(A_p(\bar{G})) \oplus \bigoplus_{\bar{B}\in A_p(\bar{G})/G} \operatorname{ind}_{\mathrm{N}_G(NB)}^{G} \left(\widetilde{H}_{\mathrm{rank}(B)-1}(A_p(NB)) \otimes \widetilde{H}_{i-\mathrm{rank}(B)}(A_p(\bar{G})_{>\bar{B}})\right),$$

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where $A_p(\bar{G})/G$ denotes a set of representatives of G-orbits on $A_p(\bar{G})$.

(B) *Let $G = KA$ be the semidirect product of a solvable p'-group K and an elementary abelian p-group A. Let N be a G-invariant subgroup of K. Then $\widetilde{H}_i(A_p(G))$ is trivial for $i \neq \operatorname{rank}(A) - 1$. For $i = \operatorname{rank}(A) - 1$ the G-module $\widetilde{H}_i(A_p(G))$ is isomorphic to*

$$\bigoplus_{B \leq A} \operatorname{ind}^G_{N_G(NB)} \left(\widetilde{H}_{\operatorname{rank}(B)-1}(A_p(NB)) \otimes \widetilde{H}_{i-\operatorname{rank}(B)}(A_p(C_{\bar{K}}(B)A/B))\right).$$

PROOF. In either case we consider the covering $\mathscr{U}$ of $A_p(G)$ given by the preimages of $A_p(\bar{A})$ for the maximal abelian subgroups $\bar{A}$ of $\bar{G}$. Lemma 3.2 shows that conditions (i) and (ii) of Proposition 4.1 are fulfilled (note that homology groups of wedges of spheres are free). By the fact that $f^{-1}(A_p(\bigcap \bar{A}_i)) = \bigcap f^{-1}(A_p(\bar{A}_i))$ and by Lemma 3.1 (d) and (f) we can replace $P^{\mathscr{U}}$ by the dual poset of $A_p(\bar{G}) \cup \{\hat{0}\}$. The assertion follows. For part (B) we additionally use Lemma 3.1 (c).

Note that we can omit the sum over $k + l = i - 1$ from Proposition 4.1, since by Lemma 3.3 the reduced homology of $A_p(NB)$ is concentrated in dimension $\operatorname{rank}(B) - 1$. □

Again we can reduce the number of summands by the observations stated in Remark 3.4. A second proof of part (A) of the proposition can be given by Lemma 3.3 together with Corollary 3.4 in [19].

5. Examples

In this section we provide some examples where we apply our formulas. The first example shows that there are solvable groups for which $A_p(G)$ is homotopic to a wedge of spheres of more than one dimension—for homology this was already known from the work of Segev and Webb [17]; we use the same example, but our methods go beyond their results. The construction of this example goes back to Alperin who used it as a counterexample to a conjecture of Webb [25].

EXAMPLE 5.1. Let $G = (Z_q \rtimes Z_p) \operatorname{wr} Z_p$ be a wreath product of a Frobenius group $F = Z_q \rtimes Z_p$ with a cyclic group Z_p of prime order p dividing $q - 1$.

Then $A_p(G)$ is homotopic to the wedge of $(q-1)^p$ spheres of dimension $p - 1$ with $p^{p-1}(q^p - q^{p-1})$ spheres of dimension 1.

PROOF. Let N be Z_q^p the direct product of the p copies of the cyclic groups of order q coming from $Z_q \rtimes Z_p$. Then $\bar{G} \cong Z_p \operatorname{wr} Z_p$ has two conjugacy classes of maximal elementary abelian subgroups. The first conjugacy class consists of one group of

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order p^p—the base group of $\bar{G}$—and the second of p^{p-2} groups of order p^2—the centralizers of the elements of order p not lying in the base group.

Every two distinct maximal elementary abelian subgroups intersect in the center $Z(\bar{G}) \cong Z_p$ of $\bar{G}$ which is the diagonal subgroup of the base group.

Hence by Theorem 1.1 (respectively, Remark 3.4) we conclude

$$A_p(G) \cong A_p(N \rtimes Z_p^p) \vee p^{p-2} \cdot A_p(N \rtimes Z_p^2) \vee A_p(\bar{G})_{>Z(\bar{G})} * A_p(N \rtimes Z_p).$$

For the first summand we have (see also Quillen [15, Proposition 2.6])

$$\begin{aligned} A_p(Z_q^p \rtimes Z_p^p) &= A_p((Z_q \rtimes Z_p)^p) \simeq A_p(Z_p \rtimes Z_p)^{*p} \\ &\simeq ((q-1) \cdot S^0)^{*p} \simeq (q-1)^p \cdot S^p. \end{aligned}$$

The homotopy type of the second summand $A_p(N \rtimes Z_p^2)$, where Z_p^2 acts as diagonal multiplication (respectively, by permuting the vectors of the standard basis) can be calculated as $(p \cdot (q^p - q^{p-1}) - q^p + 1) \cdot S^1$. For this one observes that N is the direct sum of the one-dimensional subspaces which are the centralizers of the p subgroups of Z_p^2 of order p which are not generated by the diagonal multiplication. The diagonal multiplication centralizes only the $0 \in N \cong Z_q^p$, but stabilizes every one-dimensional subspace. Hence each of the direct summands is normal in $N \rtimes Z_p^2$ and by an induction using Theorem 1.1 one obtains the result (the induction is best done by taking normal subgroups of codimension 1 in $O_{p'}$).

In the third summand the second factor $A_p(N \rtimes Z_p) = \mathrm{Syl}_p(Z_q^p \rtimes Z_p)$ is a set of q^p points, which is as a topological space a wedge of $q^p - 1$ spheres S^0. This fact is implied by diagonal multiplication acting fixpointfree on $N \setminus \{0\}$ and $C_N(Z_p) = N_N(Z_p)$. As the intersection of two different maximal elementary abelian subgroups is $Z(\bar{G})$, every $A \in A_p(\bar{G})_{>Z(\bar{G})}$ is contained in a unique maximal elementary abelian subgroup. Thus by the Fiber Lemma, Proposition 2.5, $A_p(\bar{G})_{>Z(\bar{G})}$ is homotopic to a set of $p^{p-2} + 1$ points, that is, homotopic to a wedge of p^{p-2} spheres S^0. Hence the third summand is $((q^p - 1) \cdot S^0) * (p^{p-2} \cdot S^0) \simeq (q^p - 1) \cdot p^{p-2} \cdot S^1$ and the claimed formula is proved. □

Using Proposition 4.2 one also recovers the formula for the Steinberg module for the preceding example given in [17]. For $p = 2$ one additionally obtains that the short exact sequence used by Segev and Webb [17] splits.

The second example serves as a counterexample to the assumption that if $A_p(G)$ is homotopic to a wedge of spheres of dimensions $i_1, \dots, i_l$ then for each $1 \leq j \leq l$ there is a maximal elementary abelian p-subgroup of order p^{i_j+1}.

EXAMPLE 5.2. Let P be the group of all upper triangular 3×3 matrices over a finite field $\mathbb{F}_{p^n}$ of characteristic $p \neq 2$, that is P is a Sylow p-subgroup of $GL_3(\mathbb{F}_{p^n})$. Suppose that P acts faithfully (as a group of automorphisms) on a solvable p'-group N.

Then $A_p(G)$ is homotopic to the wedge of spheres of dimensions n and $2n-1$ and, for $n > 1$, in both dimensions there is at least one sphere.

PROOF. Since $p > 2$ all elements of $\bar{G} \cong P$ have order p. The center Z of P is the set of all matrices (a_{ij}) with $a_{12} = 0 = a_{21}$. It is contained in every maximal elementary abelian subgroup. P/Z is isomorphic to $\mathbb{F}_{p^n}^2$ and the maximal elementary abelian subgroups of P correspond to the one-dimensional $\mathbb{F}_{p^n}$-subspaces of P/Z; each two of them intersect in Z. Hence Theorem 1.1 and Remark 3.4 yield

$$A_p(G) \cong A_p(\bar{G})_{>Z(\bar{G})} * A_p(N \rtimes Z) \vee \bigvee_{A \in \mathscr{P}} A_p(NA),$$

where $\mathscr{P}$ denotes the set of 1-dimensional $\mathbb{F}_{p^n}$-subspaces of P/Z. By Lemma 3.3 the second summand has the homotopy type of a wedge of $(2n-1)$-spheres. Again by the Fiber Lemma, Proposition 2.5, $A_p(\bar{G})_{>Z(\bar{G})}$ is homotopic to the wedge of p^n spheres S^0. As Lemma 3.3 implies that $A_p(N \rtimes Z)$ is homotopic to a wedge of $(n-1)$-spheres, the claim is showed. $\square$

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