A COMMUTATOR FORMULA FOR A PAIR OF SUBGROUPS AND A THEOREM OF BLACKBURN

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Let $G = K_1(G) \supseteq K_2(G) \supseteq \dots \supseteq K_n(G) \dots$ be the lower central series of a group G, where $K_2(G) = [G, G]$ and inductively $K_{n+1}(G) = [K_n(G), G]$. A theorem of Blackburn ([1], Hilfssatz) states that

THEOREM 1. *The exponent of* $K_{n+1}(G) / K_{n+2}(G)$ *divides the exponent of* $K_n(G) / K_{n+1}(G)$.

In this note we shall establish

THEOREM 2. *Let* G_1, G_2 *be subgroups of a group* G, *and* $L = [G_1, G_2]$. *Define* $L_i = G_i \cap L$, $G_i' = [G_i, G_i]$ *and* $e_i = \exp.(G_i / G_i' \cdot L_i)$, $(i = 1, 2)$; *then for* $e = (e_1, e_2)$ (*the greatest common divisor of* e_1 *and* e_2) *we have*

$$L^e = [G_1, G_2]^e \subseteq [L, \langle G_1, G_2 \rangle]$$

where $\langle G_1, G_2 \rangle$ *denotes the subgroup of* G *generated by* G_1 *and* G_2.

We claim that Theorem 2 implies Theorem 1. Indeed suppose H is a normal subgroup of G and let us apply Theorem 2 with $G_1 = G$ and $G_2 = H$. Then clearly

$$L_1 = L_2 = [G, H] = L.$$

Also $e_1 = \exp.(G / G' \cdot [G, H]) = \exp.(G / G')$ and $e_2 = \exp.(H / [G, H])$. We have thus proved:

COROLLARY. *If* $H \Delta G$, *then the exponent of* $[G, H] / [G, [G, H]]$ *divides both*

(1) *the exponent of* G / G'

(2) *the exponent of* $H / [G, H]$.

In particular if there is no common divisor for the three integers

$$|\,[G,H]/[G,[G,H]]\,|, \quad |G/G'|, \quad |H/[G,H]\,|,$$

then $[G,H] = [G,[G,H]]$.

If we set now $H = K_n(G)$, then $[G,H] = K_{n+1}(G)$ and $[G,[G,H]] = K_{n+2}(G)$, so the second part of the corollary reproduces the theorem of Blackburn.

From the corollary, it follows trivially, for example, that for $G = S_3$, S_4 (and of course the other symmetric groups), A_4, or $|G|$ square free, $K_2(G) = K_3(G) = \dots$.

Proof of Theorem 2. We first recall that $L = [G_1, G_2]$ is normal in $\langle G_1, G_2\rangle$. Replacing G by $\langle G_1, G_2\rangle$ we may suppose, from now on, that L and hence $W = [L, \langle G_1, G_2\rangle] = [L, G]$ are normal in G. Write $\overline{L} = L/W$. For x_1 in G_1 consider a map $\beta(x_1) : G_2 \to \overline{L}$ defined by $(\beta(x_1))(x_2) = [x_1, x_2]W \in \overline{L}$. In virtue of the formula

$$[x, yz] = [x,z]\cdot z^{-1}\cdot[x,y]\cdot z$$

$\beta(x_1)$ is a homomorphism. This homomorphism clearly vanishes on $G_2 \cap L = L_2$. Moreover it vanishes on G_2' since the range $\overline{L}$ is abelian. $\beta(x_1)$ thus induces a homomorphism $G_2/G_2'\cdot L_2 \to \overline{L}$ which we again denote by $\beta(x_1)$. Now using the formula

$$[xy, z] = y^{-1}\cdot[x,z]\cdot y\cdot[y,z]$$

we see that $\beta : G_1 \to \mathrm{Hom}(G_2/G_2'\cdot L_2, \overline{L})$, sending x_1 to $\beta(x_1)$, is again a homomorphism. This likewise induces an element

$$\beta \in \mathrm{Hom}(G_1/G_1'\cdot L_1, \ \mathrm{Hom}(G_2/G_2'\cdot L_2, \overline{L})) = X.$$

The exponent of the group X clearly divides both $e_1 = \exp.(G_1/G_1'\cdot L_1)$ and $e_2 = \exp.(G_2/G_2'\cdot L_2)$. Consequently $\beta^e = 1$ where $e = (e_1, e_2)$.

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This means that $[x_1, x_2]^e \in W$ for all x_1 in G_1 and all x_2 in G_2, and hence that $[G_1, G_2]^e \subseteq W$, which is the desired conclusion.

REFERENCE

1. N. Blackburn, Über das Produkt von zwei zyklischen 2-Gruppen. Math. Z. 68 (1958) 422-427.

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