J. Aust. Math. Soc. **98** (2015), 78–103
doi:10.1017/S1446788714000391

GLOBAL WELL-POSEDNESS AND INSTABILITY OF A NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL

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(Received 13 January 2014; accepted 6 August 2014; first published online 14 October 2014)

Communicated by A. Hassell

Abstract

This paper is concerned with the Cauchy problem for a nonlinear Schrödinger equation with a harmonic potential and exponential growth nonlinearity in two space dimensions. In the defocusing case, global well-posedness is obtained. In the focusing case, existence of nonglobal solutions is discussed via potential-well arguments.

2010 *Mathematics subject classification*: primary 35Q55.

Keywords and phrases: nonlinear Schrödinger equation, potential, well-posedness, blow-up, Moser–Trudinger inequality, ground state.

1. Introduction

Consider the initial value problem for a nonlinear Schrödinger (NLS) equation with quadratic potential

$$\begin{cases} i\dot{u} + \Delta u - |x|^2 u + \epsilon g(u) = 0; \\ u_{|t=0} = u_0, \end{cases} \tag{1.1}$$

where $\epsilon \in \{-1, 1\}$, u is a complex-valued function of the variable $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ and the nonlinearity takes the Hamiltonian form $g(u) := uG'(|u|^2)$ for some positive real function $G \in C^3(\mathbb{R}_+)$ satisfying $G(0) = G'(0) = G''(0) = 0$.

Equation (1.1) models Bose–Einstein condensates with attractive interparticle interactions under a magnetic trap [3, 9, 21, 33, 35]. The isotropic harmonic potential $|x|^2$ describes a magnetic field whose role is to confine the movement of particles [3, 9, 33].

T. Saanouni is grateful to the Laboratory of PDE and Applications at the Faculty of Sciences of Tunis.

A solution u of (1.1) satisfies formally conservation of the mass and the energy:

$$M(t) = M(u(t)) := \|u(t)\|_{L^2}^2 = M(0);$$
$$E(t) = E(u(t)) := \int_{\mathbb{R}^2} (|\nabla u(t)|^2 + |xu(t)|^2 - \epsilon G(|u(t)|^2))\, dx = E(0).$$

Moreover, such a solution enjoys the so-called virial identity [7],

$$\frac{1}{8}(\|xu(t)\|_{L^2}^2)'' = \|\nabla u\|_{L^2}^2 - \|xu\|_{L^2}^2 - \epsilon \int_{\mathbb{R}^2} (\bar{u}g(u) - G(|u|^2))\, dx. \qquad (1.2)$$

If $\epsilon = -1$, the energy is always positive and we say that (1.1) is defocusing. Otherwise, (1.1) is said to be focusing. Naturally, we would like to study the problem (1.1) in some space where energy and mass are well defined.

Definition 1.1. We define:

(1) the conformal space

$$\Sigma := \{u \in H^1,\ \text{s.t. } |x|u \in L^2\},$$

where here and hereafter s.t. stands for such that;

(2) the conformal norm

$$\|u\|_\Sigma := (\|u\|_{H^1}^2 + \|\,|x|u\|_{L^2}^2)^{1/2} = (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\,|x|u\|_{L^2}^2)^{1/2}.$$

In the monomial case $g(u) = u|u|^{p-1}$ for $1 < p < (n+2)/(n-2)$ if $n \geq 3$ and $1 < p < \infty$ if $n \in \{1, 2\}$, local well-posedness in the conformal space was established [7, 19]. By [6], when $p < 1 + (4/n)$ or $p \geq 1 + (4/n)$ and $\epsilon = -1$, the solution to the Cauchy problem (1.1) exists globally. For $p = 1 + (4/n)$, there exists a sharp condition [36] of the global existence for the Cauchy problem (1.1). When $p > 1 + (4/n)$, the solution to the Cauchy problem (1.1) blows up in a finite time for a class of sufficiently large data and globally exists for a class of sufficiently small data [4, 5, 33].

In two space dimensions, the initial value problem (1.1) in the monomial case is energy subcritical for all $p > 1$. So, it is natural to consider problems with exponential nonlinearities, which have several applications, such as for example self-trapped beams in plasmas [14]. Moreover, the two-dimensional case is interesting because of its relation to the critical Moser–Trudinger inequalities [1, 22].

The two-dimensional semilinear Schrödinger problem with exponential growth nonlinearity was studied, for small Cauchy data, by Nakamura and Ozawa [18]. They proved global well-posedness and scattering. Later on, Colliander *et al.* [8] obtained global well-posedness and scattering for small data. The author [25] obtained a decay result in a critical case.

Recently, the author [23, 27–29] proved global well-posedness and scattering in the energy space, without any condition on the data, of a Schrödinger equation with exponential nonlinearity. Moreover, scattering was proved in the conformal space [24] (similar results were proved for the corresponding wave equation [15, 16, 26, 30]).

It is the aim of this paper to obtain three results about the Cauchy problem (1.1) in the two-space-dimensional case. First, we prove global well-posedness in the

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defocusing case. Then, we establish existence of a ground state solution for the stationary associated problem. Third, we discuss, in the focusing case, either global well-posedness or finite time blow-up. It is worth pointing out that the present study uses the potential well method based on the concepts of invariant sets suggested by Payne and Sattinger [20].

The rest of the paper is organized as follows. The main results and some technical tools needed in the sequel are listed in the next section. The third section is devoted to prove well-posedness of (1.1). The goal of the fourth section is to study the stationary problem associated to (1.1). In the fifth section we prove either global well-posedness or finite time blow-up of solutions to (1.1) with energy under the ground state one. In the last section we establish the existence of infinitely many blowing-up solutions to (1.1), with data near the ground state.

In this paper, we are interested in the two-space-dimensions case, so, here and hereafter, we denote $\int . \, dx := \int_{\mathbb{R}^2} . \, dx$. For $p \geq 1$, $L^p := L^p(\mathbb{R}^2)$ is the Lebesgue space endowed with the norm $\| \cdot \|_p := \| \cdot \|_{L^p}$, $\| \cdot \| := \| \cdot \|_2$ and H^1 is the usual Sobolev space endowed with the norm $\| \cdot \|_{H^1} := (\| \cdot \|^2 + \| \nabla . \|^2)^{1/2}$.

For $T > 0$ and X, an abstract functional space, we denote $C_T(X) := C([0, T], X)$, the space of continuous functions with variable in $[0, T]$ and values in X and X_{rd}, the set of radial functions in X. We mention that C is an absolute positive constant, which may vary from line to line. If A and B are nonnegative real numbers, $A \lesssim B$ means that $A \leq CB$. Finally, we define the operator $(Df)(x) := xf'(x)$.

2. Background material

In this section, we give the main results and some technical tools needed in the sequel. First, let us fix the set of nonlinearities considered in this paper.

(i) Ground state condition

$$\exists \varepsilon_g > 0 \quad \text{s.t.} \ \min\{(D - 1 - \varepsilon_g)G, \ ((D - 1)^2 - \varepsilon_g)G\} \geq 0 \quad \text{on } \mathbb{R}_+. \tag{2.1}$$

(j) Strong ground state condition

$$\exists \varepsilon_g > 0 \quad \text{s.t.} \ \min\{(D - 2 - \varepsilon_g)G, ((D - 1)^2 - \varepsilon_g)G\} > 0 \quad \text{on } \mathbb{R}_+^*. \tag{2.2}$$

(k) Subcritical case

$$\forall \alpha > 0, \quad |G'''(r)| = o(e^{\alpha r}) \quad \text{as } r \to \infty. \tag{2.3}$$

(l) Critical case

$$\exists \alpha_0 > 0 \quad \text{s.t.} \ |G'''(r)| = O(e^{\alpha_0 r}) \quad \text{as } r \to \infty. \tag{2.4}$$

We say that the nonlinearity of the problem (1.1) is subcritical (respectively critical) if G satisfies (2.3) (respectively (2.4)). Moreover, we should assume (2.3) or (2.4) in order to prove well-posedness of (1.1) and use [(2.1) or (2.2)] with [(2.3) or (2.4)] in order to obtain existence of a ground state solution to the stationary problem associated to (1.1).

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REMARK 2.1. We give explicit examples.

(1) Subcritical case: $G(r) := e^{(1+r)^{1/2}} - er/2 - e$.

(2) Critical case: $G(r) = e^r - 1 - r - (1/2)r^2$.

PROOF. (1) For $t := \sqrt{r+1}$, we have $G(r) = e^t - (e/2)t^2 - (e/2)$. Thus, $DG(r) = ((t^2-1)/2t)(e^t - et)$. Then, for $\varepsilon > 0$,

$$\phi(t) = 2(D-1-\varepsilon)G(r) = e^t\Big(t - \frac{1}{t} - 2 - 2\varepsilon\Big) + e(\varepsilon t^2 + 2 + \varepsilon);$$
$$\phi'(t) = e^t\Big(t - \frac{1}{t} + \frac{1}{t^2} - 1 - 2\varepsilon\Big) + 2e\varepsilon t;$$
$$\phi''(t) = e^t\Big(t - \frac{1}{t} + \frac{2}{t^2} - \frac{2}{t^3} - 2\varepsilon\Big) + 2e\varepsilon \geq 0.$$

Since $\phi(1) = \phi'(1) = 0$, we have $\phi \geq 0$. Moreover,

$$D(D-1)G(r) = \frac{1}{4}e^t\Big(t - \frac{1}{t}\Big)\Big(t - 1 - \frac{1}{t} + \frac{1}{t^2}\Big);$$
$$[(D-1)^2 - \varepsilon]G(r)$$
$$= \frac{1}{4}\Big[e^t\Big(t^2 - 3t + 2 - 4\varepsilon + \frac{4}{t} + \frac{1}{t^2} - \frac{1}{t^3}\Big) + 2e\varepsilon t^2 + 2\varepsilon e - 4e\Big] = \frac{1}{4}\psi(t);$$
$$\psi'(t) = e^t\Big(t^2 - t - 1 - 4\varepsilon + \frac{4}{t} - \frac{3}{t^2} - \frac{3}{t^3} + \frac{3}{t^4}\Big) + 4e\varepsilon t;$$
$$\psi''(t) = e^t\Big(t^2 + t - 2 - 4\varepsilon + \frac{4}{t} - \frac{7}{t^2} + \frac{3}{t^3} + \frac{12}{t^4} - \frac{12}{t^5}\Big) + 4e\varepsilon \geq 0.$$

Since $\psi(1) = \psi'(1) = 0$, we have $\psi \geq 0$.

(2) Take $\varepsilon \in (0, 2)$ and $G(x) := e^x - 1 - x - x^2/2$. Then $DG(x) = x(e^x - 1 - x)$ and

$$(D-1-\varepsilon)G(x) = (x-1-\varepsilon)e^x + (\varepsilon-1)\frac{x^2}{2} + \varepsilon x + 1 + \varepsilon := \phi(x);$$
$$\phi'(x) = (x-\varepsilon)e^x + (\varepsilon-1)x + \varepsilon, \phi''(x) = (x-\varepsilon+1)e^x + \varepsilon - 1;$$
$$\phi''(x) = (x-\varepsilon+2)e^x \geq 0.$$

Since $\phi(0) = \phi'(0) = 0$, we have $\phi \geq 0$. Moreover,

$$(D-1)G(x) = (x-1)e^x - \frac{x^2}{2} + 1, D(D-1)G(x) = x(xe^x - x);$$
$$[(D-1)^2 - \varepsilon]G(x) = (x^2 - x + 1 - \varepsilon)e^x + (\varepsilon - 1) + (\varepsilon-1)\frac{x^2}{2} + \varepsilon x := \psi(x);$$
$$\psi'(x) = (x^2 + x - \varepsilon)e^x + (\varepsilon-1)x + \varepsilon, \psi''(x) = (x^2 + 3x - \varepsilon + 1)e^x + \varepsilon - 1;$$
$$\psi'''(x) = (x^2 + 5x - \varepsilon + 2)e^x \geq 0.$$

Since $\psi(0) = \psi'(0) = \psi''(0) = 0$, we have $\psi \geq 0$. This finishes the proof. □

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The results proved in this paper are listed in the following subsection.

2.1. Main results. The first result deals with well-posedness of (1.1). Assuming that the nonlinearity satisfies (2.4), we obtain existence of a unique global solution for small data.

THEOREM 2.2. *Assume that g satisfies (2.4). Let $u_0 \in \Sigma$ be such that $\|\nabla u_0\|^2 < 4\pi/\alpha_0$. Then there exist $T > 0$ and a unique solution u to the Cauchy problem (1.1) in the class $C_T(\Sigma)$. Moreover:*

(1) *$u, \nabla u, xu \in L^4_T(L^4(\mathbb{R}^2))$;*

(2) *u satisfies conservation of the energy and the mass;*

(3) *u is global if $E(u_0) \leq 4\pi/\alpha_0$.*

REMARK 2.3. Note that if g satisfies (2.3), then global well-posedness holds without any condition on the data size [28].

Next, we are interested in the focusing case of the Schrödinger problem (1.1). This case is related to the associated stationary problem. Indeed, under the condition (2.1), we prove existence of a ground state ϕ, meaning that ϕ is a solution of the stationary problem

$$-\Delta\phi + \phi - |x|^2\phi = g(\phi), \quad 0 \neq \phi \in \Sigma, \tag{2.5}$$

which minimizes the problem

$$m_{\alpha,\beta} := \inf_{0\neq v\in\Sigma}\{S(v), \text{ s.t. } K_{\alpha,\beta}(v) = 0\}, \tag{2.6}$$

where $\alpha, \beta \in \mathbb{R}$ and

$$S(v) := E(v) + M(v) = \|v\|^2_\Sigma - \int G(|v|^2)\,dx;$$

$$K_{\alpha,\beta}(v) := 2\int [\alpha|\nabla v|^2 + (\alpha+\beta)|v|^2 + (\alpha+2\beta)|xv|^2 - \alpha|v|g(|v|) - \beta G(|v|^2)]\,dx.$$

Precisely, we obtain the next result, where we denote some set depending on the nonlinearity:

$$\mathcal{A}_g := \{(a,b) \in \mathbb{R}^*_+ \times \mathbb{R}_+, \text{ s.t. } b < a\varepsilon_g\}.$$

THEOREM 2.4. *Assume that g satisfies (2.1) and [(2.3) or (2.4)]. Let two real numbers $(\alpha, \beta) \in \mathcal{A}_g$. Then:*

(1) *there is a minimizer of (2.6), which satisfies (2.5);*

(2) *$m := m_{\alpha,\beta}$ is nonnegative and independent of (α, β).*

Following the potential well theory [13, 20], we are interested in the focusing case of the Schrödinger problem (1.1) with data in some stable sets. Here and hereafter, we denote for $(\alpha, \beta) \in \mathbb{R}^2_+$ the sets

$$A^+_{\alpha,\beta} := \{v \in \Sigma \text{ s.t. } S(v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(v) \geq 0\};$$

$$A^-_{\alpha,\beta} := \{v \in \Sigma \text{ s.t. } S(v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(v) < 0\};$$

$$A_{1,-1} := \{v \in \Sigma \text{ s.t. } S(v) < m_{1,0},\ K_{1,0}(v) < 0 \text{ and } K_{1,-1}(v) < 0\}.$$

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Global existence and finite time blow-up are now discussed.

Theorem 2.5. *Assume that $\epsilon = 1$ and g satisfies (2.3) or (2.4). Let $u \in C_{T^*}(\Sigma)$ be the maximal solution to (1.1).*

(1) *Suppose that (2.2) holds. If there exists $t_0 \in [0, T^*)$ such that $u(t_0) \in A_{1,-1}$, then u blows up in finite time.*
(2) *Suppose that (2.1) holds. If there exist $(\alpha, \beta) \in \mathcal{A}_g$ and $t_0 \in [0, T^*)$ such that $u(t_0) \in A^+_{\alpha,\beta}$, then u is global.*

The last result concerns the instability by blow-up for the standing waves of the Schrödinger problem (1.1).

Theorem 2.6. *Assume that $\epsilon = 1$ and g satisfies (2.2) with [(2.3) or (2.4)]. Let ϕ be a ground state solution to (2.5). Then, for any $\varepsilon > 0$, there exists $u_0 \in \Sigma$ such that $\|u_0 - \phi\|_\Sigma < \varepsilon$ and the maximal solution to (1.1) given by Theorem 2.2 is not global.*

We list in what follows some intermediate results.

2.2. Tools. This subsection is devoted to give some estimates needed in this paper. First, let us recall some known results [10, 11] about the free propagator associated to (1.1). The following result holds [6].

Proposition 2.7. *There exists a family of operators $U := U(t, s)$, $U(t) := U(t, 0)$ such that $u(t, x) := U(t, s)\phi(x)$ is a solution to the linear problem*

$$i\dot{u} + \Delta u = |x|^2 u, \quad u(s, \cdot) = \phi.$$

Moreover, we have the following elementary properties:

(1) $U(t, t) = \mathrm{Id}$;
(2) $(t, s) \mapsto U(t, s)$ *is continuous;*
(3) $U(t, s)^* = U(t, s)^{-1}$;
(4) $U(t, \tau)U(\tau, s) = U(t, s)$;
(5) $U(t, s)$ *is unitary of* L^2.

The Duhamel formula yields the following result.

Proposition 2.8. *If u is a solution to the inhomogeneous Schrödinger problem*

$$i\dot{u} + \Delta u - |x|^2 u = h, \quad u(0, \cdot) = 0,$$

then:

(1) $u(t) = -i \int_0^t U(t - s)h(s, x)\, ds$;
(2) $\nabla u(t) = -i \int_0^t U(t - s)[\nabla h + 2xu]\, ds$;
(3) $xu(t) = -i \int_0^t U(t - s)[xh + 2\nabla u]\, ds$.

Remark 2.9. Taking the derivative of the equation satisfied by u, we obtain the second point. For the last one, we multiply the same equation with x.

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A classical tool to study Schrödinger problems is the so-called Strichartz-type estimate.

Definition 2.10. A pair (q, r) of positive real numbers is admissible if

$$2 \leq r < \infty \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{2}.$$

In order to control an eventual solution to (1.1), we will use the following Strichartz estimate [6].

Proposition 2.11. *For any time slab I and any admissible pairs (q, r) and (α, β):*

(1) $\|U(t)\phi\|_{L^q(I,L^r)} \leq C_q\|\phi\| \; \forall \phi \in L^2$;

(2) $\|\int_0^t U(t-s)h(s,x)ds\|_{L^q(I,L^r)} \leq C_{\alpha,|I|}\|h\|_{L^{\alpha'}(I,L^{\beta'})} \; \forall h \in L^{\alpha'}(I, L^{\beta'})$.

In order to estimate the quantity $\int G(|u|^2)\,dx$, which is a part of the energy, we will use Moser–Trudinger-type inequalities [1, 17, 32].

Proposition 2.12. *Let $\alpha \in (0, 4\pi)$; a constant C_α exists such that for all $u \in H^1$ satisfying $\|\nabla u\| \leq 1$,*

$$\int (e^{\alpha|u(x)|^2} - 1)\,dx \leq C_\alpha\|u\|^2.$$

Moreover, this inequality is false if $\alpha \geq 4\pi$.

Remark 2.13. The number $\alpha = 4\pi$ becomes admissible if we take $\|u\|_{H^1} \leq 1$ rather than $\|\nabla u\| \leq 1$. In this case,

$$\sup_{\|u\|_{H^1} \leq 1} \int (e^{4\pi|u(x)|^2} - 1)\,dx < \infty$$

and this is false for $\alpha > 4\pi$. See [22] for more details.

Despite the lack of injection of H^1 on the bounded functions set, we can control the L^∞ norm by the H^1 norm and some Hölder norm with a logarithmic growth.

Proposition 2.14. *Let $\beta \in (0, 1)$. For any $\lambda > 1/2\pi\beta$ and any $0 < \mu \leq 1$, a constant C_λ exists such that, for any function $u \in (H^1 \cap C^\beta)(\mathbb{R}^2)$,*

$$\|u\|_{L^\infty}^2 \leq \lambda\|u\|_\mu^2 \log\Big(C_\lambda + \frac{8^\beta\|u\|_{C^\beta}}{\mu^\beta\|u\|_\mu}\Big), \tag{2.7}$$

where

$$\|u\|_\mu^2 := \|\nabla u\|^2 + \mu^2\|u\|^2.$$

Recall that C^β denotes the space of β-Hölder continuous functions endowed with the norm

$$\|u\|_{C^\beta} := \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta}.$$

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We refer the reader to [12] for the proof of this proposition and for more details. We just point out that the condition $\lambda > 1/2\pi\beta$ in (2.7) is optimal.

In the next section, we will use the L^∞ logarithmic estimate for $\beta = 1/2$, coupled with the continuous Sobolev injection

$$W^{1,4}(\mathbb{R}^2) \hookrightarrow C^{1/2}(\mathbb{R}^2).$$

Let us recall some standard Sobolev embeddings [2, 34].

PROPOSITION 2.15.

(1) *Whenever* $1 < p < q < \infty$, $s > 0$ *and* $1/p \le 1/q + s/d$, *we have the continuous injection*

$$W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d).$$

(2) *The compact injection holds:*

$$H^1_{rd}(\mathbb{R}^2) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^2) \quad \forall p \in (2, \infty).$$

(3) *The following embedding is compact:*

$$\Sigma_{rd}(\mathbb{R}^2) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^2) \quad \forall p \in [2, \infty). \tag{2.8}$$

We close this subsection with the following absorption result [31].

LEMMA 2.16. *Let* $T > 0$ *and* $X \in C([0, T], \mathbb{R}_+)$ *be such that*

$$X \le a + bX^\theta \quad \text{on } [0, T],$$

where $a, b > 0, \theta > 1, a < (1 - 1/\theta)1/(\theta b)^{1/\theta}$ *and* $X(0) \le 1/(\theta b)^{1/(\theta-1)}$. *Then*

$$X \le \frac{\theta}{\theta - 1} a \quad \text{on } [0, T].$$

3. Well-posedness

This section is devoted to prove Theorem 2.2 about well-posedness of the nonlinear Schrödinger problem (1.1). In this section, we assume that $\alpha_0 = 1$.

REMARK 3.1. Note that in all of this section, if we omit the condition $\alpha_0 = 1$, the spirit of proof is the same.

Let us identify g with a function defined on $\mathbb{R}^2$ and denote by $\mathcal{D}g$ the $\mathbb{R}^2$ derivative of the identified function. Then, using (2.4), the mean-value theorem and the convexity of the exponential function, we derive the following property.

LEMMA 3.2. *For any* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$|g(z_1) - g(z_2)| \le C_\varepsilon |z_1 - z_2| \sum_{i=1}^{2} (e^{(1+\varepsilon)|z_i|^2} - 1) \quad \forall z_1, z_2 \in \mathbb{C},$$

$$|\mathcal{D}g(z_1) - \mathcal{D}g(z_2)| \le C_\varepsilon |z_1 - z_2| \sum_{i=1}^{2} (|z_i| + e^{2(1+\varepsilon)|z_i|^2} - 1) \quad \forall z_1, z_2 \in \mathbb{C}.$$

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The next auxiliary result will be useful.

Lemma 3.3. *Let $u \in C_T(H^1) \cap L^4_T(W^{1,4})$ be a solution to (1.1) satisfying $\|\nabla u\|^2_{L^\infty_T(L^2)} < 4\pi$; then there exist two real numbers $\alpha < 4$ near to 4 and $\varepsilon > 0$ near to zero such that, for any Hölder couple (p, p'),*

$$\|e^{(1+\varepsilon)|u|^2} - 1\|_{L^{p'}_T(L^p)} \lesssim T^{1-(1/p)} + \|u\|^\alpha_{L^4_T(W^{1,4})} T^{(1-(1/p))(1-(\alpha/4))}.$$

Proof. By the Hölder inequality, for any $\varepsilon > 0$,

$$\|e^{(1+\varepsilon)|u|^2} - 1\|_{L^{p'}_T(L^p)} \lesssim \|e^{1/p'(1+\varepsilon)\|u\|^2_{L^\infty_x}}\|_{L^{p'}(0,T)} \|e^{(1+\varepsilon)|u|^2} - 1\|^{1/p}_{L^\infty_T(L^1)}.$$

We can find $\varepsilon > 0$ small such that $(1 + \varepsilon)\|\nabla u\|^2 < 4\pi$. So, by the Moser–Trudinger inequality,

$$\int (e^{(1+\varepsilon)|u|^2} - 1)\, dx \le \int (e^{(1+\varepsilon)\|\nabla u\|^2(|u|/\|\nabla u\|)^2} - 1)\, dx \lesssim \|u\|^2 \lesssim 1.$$

For any $\lambda > 1/\pi$ and $\omega \in (0, 1]$, by the logarithmic inequality in Proposition 2.14,

$$e^{(1+\varepsilon)\|u\|^2_{L^\infty_x}} \le \Big(C + 2\sqrt{\frac{2}{\omega}}\frac{\|u\|_{C^{1/2}}}{\|u\|_\omega}\Big)^{\lambda(1+\varepsilon)\|u\|^2_\omega}.$$

Since $\|u\|^2_\omega = \omega^2\|u\|^2 + \|\nabla u\|^2$, we may take $0 < \omega, \varepsilon$ near to zero and $\alpha < 4$ near to 4 such that $(1 + \varepsilon)\|u\|^2_\omega < \alpha\pi < 4\pi$. Thus, for λ near $1/\pi$,

$$\begin{aligned} e^{(1+\varepsilon)\|u\|^2_{L^\infty_x}} &\le \Big(C + 2\sqrt{\frac{2}{\omega}}\frac{\|u\|_{C^{1/2}}}{\|u\|_\omega}\Big)^{\lambda(1+\varepsilon)\|u\|^2_\omega} \\ &\lesssim (1 + \|u\|_{C^{1/2}})^\alpha \lesssim 1 + \|u\|^\alpha_{W^{1,4}}. \end{aligned}$$

It follows that

$$\begin{aligned} \|e^{(1+\varepsilon)|u|^2} - 1\|_{L^{p'}_T(L^p)} &\lesssim \|e^{1/p'(1+\varepsilon)\|u\|^2_{L^\infty_x}}\|_{L^{p'}(0,T)} \|e^{(1+\varepsilon)|u|^2} - 1\|^{1/p}_{L^\infty_T(L^1)} \\ &\lesssim \|e^{1/p'(1+\varepsilon)\|u\|^2_{L^\infty_x}}\|_{L^{p'}(0,T)} \\ &\lesssim \|1 + \|u\|^\alpha_{W^{1,4}}\|^{1/p'}_{L^1(0,T)} \\ &\lesssim T^{1-(1/p)} + \|u\|^\alpha_{L^4_T(W^{1,4})} T^{(1-(1/p))(1-(\alpha/4))}. \end{aligned}$$

□

The proof of Theorem 2.2 contains three steps. First, we prove local well-posedness, second we show uniqueness and third we obtain global well-posedness. In the two next subsections, we assume that $\epsilon = 1$. The sign of ϵ has no local effect.

3.1. Local well-posedness. We use a standard fixed-point argument. For $T > 0$, denote $I_T := (0, T)$ and the space

$$X_T := \{u \in C(I_T, \Sigma) \text{ s.t. } u, \nabla u, xu \in L^4(I_T, L^4)\}$$

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endowed with the complete norm

$$\|u\|_T := \|u\|_{L^\infty(I_T,H^1)} + \|u\|_{L^4(I_T,W^{1,4})} + \|xu\|_{L^\infty(I_T,L^2)} + \|xu\|_{L^4(I_T,L^4)}.$$

Define the map

$$\phi :\ v \longmapsto -i\int_0^t U(t-s)g(v+w)(s)\,ds,$$

where $w := U(t)u_0$ is the solution to the associated free problem to (1.1), namely, for $V(x) := |x|^2$,

$$i\dot{w} + \Delta w = |x|^2 w\ w(0,\cdot) = u_0.$$

With a continuity argument, there exists a positive time $T_0 > 0$ such that $\|\nabla w\|_{L^\infty(I_{T_0},L^2)} < 4\pi$. We shall prove the existence of $T_0 > T > 0$ such that ϕ is a contraction on some closed ball of X_T. Using Strichartz estimates in Proposition 2.11 with the facts that $\nabla\phi(v) = -i\int_0^t U(t-s)[\nabla g(v+w)(s) + \nabla V\phi(v)]\,ds$ and $x\phi(v) = -i\int_0^t U(t-s)[xg(v+w)(s) + 2\nabla\phi(v)]\,ds$,

$$\begin{aligned}
\|\phi(v)\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} &\lesssim \|g(v+w)\|_{L^1(I_T,L^2)}\\
\|\nabla(\phi(v))\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} &\lesssim \|\nabla(g(v+w))\|_{L^1(I_T,L^2)} + \|\phi(v)\nabla V\|_{L^1(I_T,L^2)}\\
&\lesssim \|\nabla(g(v+w))\|_{L^1(I_T,L^2)} + T\|x\phi(v)\|_{L^\infty(I_T,L^2)}\\
\|x\phi(v)\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} &\lesssim \|xg(v+w)\|_{L^1(I_T,L^2)} + T\|\nabla(\phi(v))\|_{L^\infty(I_T,L^2)}.
\end{aligned}$$

Thus,

$$\|\phi(v)\|_T \lesssim \|g(v+w)\|_{L^1(I_T,\Sigma)} + T(\|\nabla(\phi(v))\|_{L^\infty(I_T,L^2)} + \|x\phi(v)\|_{L^\infty(I_T,L^2)}).$$

Let $v \in B_T(r)$ be the closed ball of X_T centered on zero and with radius $r > 0$. Since

$$\|\nabla(v+w)\| \le r + \|\nabla w\| \le r + \|\nabla w\|_{L^\infty(I_{T_0},L^2)},$$

we can find two small positive numbers denoted by r and ε such that $(1+\varepsilon)\|\nabla(v+w)\|^2 < 4\pi$. By the Hölder inequality,

$$\begin{aligned}
\|xg(v+w)\|_{L^1(I_T,L^2)} &\lesssim \|x(v+w)\|_{L^4(I_T,L^4)}\|e^{(1+\varepsilon)|v+w|^2} - 1\|_{L^{4/3}(I_T,L^4)}\\
&\lesssim \|v+w\|_T\|e^{(1+\varepsilon)|v+w|^2} - 1\|_{L^{4/3}(I_T,L^4)}.
\end{aligned}$$

Now, thanks to Lemma 3.3,

$$\begin{aligned}
\|xg(v+w)\|_{L^1(I_T,L^2)} &\lesssim [T^{3/4} + \|v+w\|^\alpha_{L^4_T(W^{1,4})}T^{(1-(1/p))(1-(\alpha/4))}]\|v+w\|_T\\
&\lesssim [T^{3/4} + \|v+w\|^\alpha_T T^{(1-(1/p))(1-(\alpha/4))}]\|v+w\|_T.
\end{aligned}$$

It remains to control $\|g(v+w)\|_{L^1(I_T,H^1)}$. For any $\varepsilon > 0$,

$$\begin{aligned}
\|\nabla g(v+w)\|_{L^1(I_T,L^2)} &\lesssim \|\nabla(v+w)(e^{(1+\varepsilon)|v+w|^2} - 1)\|_{L^1(I_T,L^2)}\\
&\lesssim \|\nabla(v+w)\|_{L^4(I_T,L^4)}\|e^{(1+\varepsilon)|v+w|^2} - 1\|_{L^{4/3}(I_T,L^4)}\\
&\lesssim \|v+w\|_T\|e^{(1+\varepsilon)|v+w|^2} - 1\|_{L^{4/3}(I_T,L^4)}.
\end{aligned}$$

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The previous computations imply that

$$\|\nabla g(v+w)\|_{L^1(I_T,L^2)} \lesssim [T^{3/4} + \|v+w\|_T^{\alpha} T^{(1-(1/p))(1-(\alpha/4))}]\, \|v+w\|_T.$$

Similarly,

$$\begin{aligned}
\|g(v+w)\|_{L^1(I_T,L^2)} &\lesssim \|(v+w)(e^{(1+\varepsilon)|v+w|^2}-1)\|_{L^1(I_T,L^2)}\\
&\lesssim \|v+w\|_{L^4(I_T,L^4)} \|e^{(1+\varepsilon)|v+w|^2}-1\|_{L^{4/3}(I_T,L^4)}\\
&\lesssim \|v+w\|_T \|e^{(1+\varepsilon)|v+w|^2}-1\|_{L^{4/3}(I_T,L^4)}\\
&\lesssim [T^{3/4} + \|v+w\|_T^{\alpha} T^{(1-(1/p))(1-(\alpha/4))}]\, \|v+w\|_T.
\end{aligned}$$

Therefore, for $0 < T < T_0$ small enough,

$$\begin{aligned}
\|\phi(v)\|_T &\lesssim \|g(v+w)\|_{L^1(I_T,\Sigma)} + T\|\phi(v)\|_{L^\infty(I_T,\Sigma)}\\
&\lesssim \|g(v+w)\|_{L^1(I_T,\Sigma)} + T\|\phi(v)\|_T\\
&\lesssim [T^{3/4} + \|v+w\|_T^{\alpha} T^{(1-(1/p))(1-(\alpha/4))}]\, \|v+w\|_T + T\|\phi(v)\|_T\\
&\lesssim \frac{[T^{3/4} + \|v+w\|_T^{\alpha} T^{(1-(1/p))(1-(\alpha/4))}]}{1-T} \|v+w\|_T\\
&\lesssim \frac{[T^{3/4} + (1+\|u_0\|_\Sigma)^{\alpha} T^{(1-(1/p))(1-(\alpha/4))}]}{1-T} (1+\|u_0\|_\Sigma).
\end{aligned}$$

Thus, for $r, T > 0$ small enough, ϕ maps $B_T(r)$ into itself. It remains to prove that ϕ is a contraction. Let $v_1, v_2 \in B_T(r)$ be solutions to (1.1) and $u_i := w + v_i$ for $i \in \{1, 2\}$. Then

$$\phi(v_1) - \phi(v_2) = -i \int_0^t U(t-s)(g(u_1) - g(u_2))(s)\, ds.$$

Using Strichartz estimates in Proposition 2.11 and arguing as previously,

$$\begin{aligned}
&\|\phi(v_1) - \phi(v_2)\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} \lesssim \|g(u_1) - g(u_2)\|_{L^1(I_T,L^2)};\\
&\|\nabla(\phi(v_1) - \phi(v_2))\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)}\\
&\qquad \lesssim \|\nabla(g(u_1) - g(u_2))\|_{L^1(I_T,L^2)} + T\|x(\phi(v_1) - \phi(v_2))\|_{L^\infty(I_T,L^2)};\\
&\|x(\phi(v_1) - \phi(v_2))\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)}\\
&\qquad \lesssim \|x(g(u_1) - g(u_2))\|_{L^1(I_T,L^2)} + T\|\nabla(\phi(v_1) - \phi(v_2))\|_{L^\infty(I_T,L^2)}.
\end{aligned}$$

Thus, for small $T > 0$,

$$\begin{aligned}
\|\phi(v_1) - \phi(v_2)\|_T &\lesssim \|g(u_1) - g(u_2)\|_{L^1(I_T,\Sigma)} + T\|\phi(v_1) - \phi(v_2)\|_{L^\infty(I_T,\Sigma)}\\
&\lesssim \frac{1}{1-T} \|g(u_1) - g(u_2)\|_{L^1(I_T,\Sigma)}.
\end{aligned}$$

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By the Hölder inequality, via Lemma 3.2, for all $\varepsilon > 0$,

$$\begin{aligned}\|x(g(u_1) - g(u_2))\|_{L^1(I_T,L^2)} &\lesssim \sum_{i=1}^{2} \|x(v_1 - v_2)(e^{(1+\varepsilon)|u_i|^2} - 1)\|_{L^1(I_T,L^2)} \\ &\lesssim \|x(v_1 - v_2)\|_{L^4(I_T,L^4)} \sum_{i=1}^{2} \|e^{(1+\varepsilon)|u_i|^2} - 1\|_{L^{4/3}(I_T,L^4)} \\ &\lesssim \|v_1 - v_2\|_T \sum_{i=1}^{2} \|e^{(1+\varepsilon)|u_i|^2} - 1\|_{L^{4/3}(I_T,L^4)}.\end{aligned}$$

By a continuity argument, we can find some small real numbers $\varepsilon, r, T > 0$ such that $(1 + \varepsilon)(r + \|\nabla w\|_{L^\infty(I_T,L^2)})^2 < 4\pi$. Lemma 3.3 implies that

$$\|x(g(v_1 + w) - g(v_2 + w))\|_{L^1(I_T,L^2)} \lesssim \|v_1 - v_2\|_T [T^{3/4} + \|u_1\|^{\alpha}_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}].$$

Compute

$$\begin{aligned}&\|\nabla(g(u_1) - g(u_2))\|_{L^1(I_T,L^2)} \\ &\quad = \|(\mathcal{D}g(u_1) - \mathcal{D}g(u_2))\nabla u_1 + \mathcal{D}g(u_2)(\nabla v_1 - \nabla v_2)\|_{L^1(I_T,L^2)} \\ &\quad \le \|(\mathcal{D}g(u_1) - \mathcal{D}g(u_2))\nabla u_1\|_{L^1(I_T,L^2)} + \|\mathcal{D}g(u_2)\nabla(v_1 - v_2)\|_{L^1(I_T,L^2)} \\ &\quad := (I) + (II).\end{aligned}$$

By Lemma 3.2, for any $\varepsilon > 0$,

$$\begin{aligned}(II) &\lesssim \|\nabla(v_1 - v_2)(e^{(1+\varepsilon)|u_2|^2} - 1)\|_{L^1(I_T,L^2)} \\ &\lesssim \|\nabla(v_1 - v_2)\|_{L^4(I_T,L^4)} \|e^{(1+\varepsilon)|u_2|^2} - 1\|_{L^{4/3}(I_T,L^4)} \\ &\lesssim \|v_1 - v_2\|_T \|e^{(1+\varepsilon)|u_2|^2} - 1\|_{L^{4/3}(I_T,L^4)} \\ &\lesssim \|v_1 - v_2\|_T [T^{3/4} + \|u_2\|^{\alpha}_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}].\end{aligned}$$

In the last inequality, we used Lemma 3.3. It remains to estimate (I). Write, using Lemma 3.2, via Sobolev and Hölder inequalities,

$$\begin{aligned}(I) &\lesssim \sum_{i=1,2} \|\nabla u_1(v_2 - v_1)(|u|_i + e^{(1+\varepsilon)|u_i|^2} - 1)\|_{L^1(I_T,L^2)} \\ &\lesssim \sum_{i=1,2} [\|\nabla u_1(v_2 - v_1)|u_i|\,\|_{L^1(I_T,L^2)} + \|\nabla u_1(v_2 - v_1)(e^{(1+\varepsilon)|u_i|^2} - 1)\|_{L^1(I_T,L^2)}] \\ &\lesssim \sum_{i=1,2} \|v_2 - v_1\|_{L^\infty(I_T,H^1)} \|\nabla u_1\|_{L^4(I_T,L^4)} [\|u_i\|_{L^\infty(I_T,H^1)} T^{3/4} + \|e^{(1+\varepsilon)\|u_i|^2} - 1\|_{L^{4/3}(I_T,L^{4+\varepsilon})}] \\ &\lesssim \|v_2 - v_1\|_T \|\nabla u_1\|_T [\|u_1\|_T T^{3/4} + T^{3/4} + \|u_1\|^{\alpha}_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}] \\ &\lesssim \|v_2 - v_1\|_T (1 + \|u_0\|_\Sigma)[(1 + \|u_0\|_\Sigma) T^{3/4} + \|u_1\|^{\alpha}_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}].\end{aligned}$$

Thus, for some $\alpha < 4$ near to 4,

$$\|\phi(v_1) - \phi(v_2)\|_T \le C(1 + \|u_0\|_\Sigma)[(1 + \|u_0\|_\Sigma) T^{3/4} + (1 + \|u_0\|_\Sigma)^{\alpha} T^{3/4(1-(\alpha/4))}]\, \|v_1 - v_2\|_T.$$

So, ϕ is a contraction of $B_T(r)$ for some $T, r > 0$ small enough. Its fixed point v satisfying $u := v + w$ is a solution to (1.1). The existence is proved.

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3.2. Uniqueness in the conformal space. We prove uniqueness of solutions to (1.1) in the conformal space. Letting $u_1, u_2 \in C_T(\Sigma)$ be two solutions to (1.1) and $u := u_1 - u_2$,

$$i\dot{u} + \Delta u = |x|^2 u + g(u_1) - g(u_2), \quad u(0, \cdot) = 0.$$

With a continuity argument, there exists $0 < T < 1$ such that

$$\max_{i\in\{1,2\}} \|\nabla u_i\|^2_{L^\infty(I_T,L^2)} < 4\pi \quad \text{and} \quad \max_{i\in\{1,2\}} \|u_i\|_{L^\infty(I_T,\Sigma)} \leq 1 + \|u_0\|_\Sigma.$$

With the Strichartz estimate,

$$\|u\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} \lesssim \|g(u_1) - g(u_2)\|_{L^1(I_T,L^2)};$$
$$\|\nabla u\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} \lesssim \|\nabla(g(u_1) - g(u_2))\|_{L^1(I_T,L^2)} + T\|xu\|_{L^\infty(I_T,L^2)}.$$

Via the previous calculation,

$$\begin{aligned}\|\nabla u_1\|_{L^4(I_T,L^4)} &\lesssim \|u_0\|_{H^1} + \|\nabla u_1\|_{L^4(I_T,L^4)}[T^{3/4} + \|u\|^\alpha_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}] + T\|xu_1\|_{L^\infty(I_T,L^2)}\\ &\lesssim \|u_0\|_\Sigma + \|u_1\|^{1+\alpha}_{L^4(I_T,W^{1,4})} T^{3/4(1-(\alpha/4))}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|u_1\|_{L^4(I_T,L^4)} &\lesssim \|u_0\| + \|u_1\|_{L^4(I_T,L^4)}[T^{3/4} + \|u\|^\alpha_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}]\\ &\lesssim \|u_0\|_\Sigma + \|u_1\|^{1+\alpha}_{L^4(I_T,W^{1,4})} T^{3/4(1-(\alpha/4))}.\end{aligned}$$

Finally, with the absorption lemma (Lemma 2.16),

$$\|u_1\|_{L^4(I_T,W^{1,4})} \lesssim \|u_0\|_\Sigma.$$

Arguing as previously and using the Moser–Trudinger inequality,

$$\begin{aligned}\|u\|_{L^\infty(I_T,L^2)\cap L^4(I_T,L^4)} &\lesssim \|g(u_1) - g(u_2)\|_{L^1(I_T,L^2)}\\ &\lesssim \sum_{i=1}^{2} \|u(e^{(1+\varepsilon)|u_i|^2} - 1)\|_{L^1(I_T,L^2)}\\ &\lesssim \sum_{i=1}^{2} \|u\|_{L^4(I_T,L^4)} \|e^{(1+\varepsilon)|u_i|^2} - 1\|_{L^{4/3}(I_T,L^4)}\\ &\lesssim \|u\|_{L^4(I_T,L^4)}[T^{3/4} + \|u\|^\alpha_{L^4_T(W^{1,4})} T^{3/4(1-(\alpha/4))}]\\ &\lesssim \|u\|_{L^4(I_T,L^4)}[T^{3/4} + (1 + \|u_0\|_\Sigma)^\alpha T^{3/4(1-(\alpha/4))}].\end{aligned}$$

Finally, for $T > 0$ small enough,

$$\|u\|_{L^4(I_T,L^4)} = 0.$$

So, for small times, $u = 0$. The proof of uniqueness is achieved via a translation argument.

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3.3. Global well-posedness in the defocusing case. This subsection is devoted to prove that the maximal solution of (1.1) is global in the defocusing case and where $E(u_0) \leq 4\pi$. Recall an important fact, that is, the time of local existence depends only on the quantity $\|u_0\|_\Sigma$. Let u be the unique maximal solution of (1.1) in the space $\mathcal{E}_T$ for any $0 < T < T^*$ with initial data u_0, where $0 < T^* \leq +\infty$ is the lifespan of u. We shall prove that u is global. By contradiction, suppose that $T^* < +\infty$. Consider, for $0 < s < T^*$, the problem

$$(\mathcal{P}_s)\ \begin{cases} i\partial_t v + \Delta v - |x|^2 v = g(v); \\ v(s,\cdot) = u(s,\cdot). \end{cases}$$

First, let us treat the simplest case $E(u_0) < 4\pi$. In this case,

$$\sup_{[0,T^*]} \|\nabla u(t)\|^2 \leq E(u_0) < 4\pi.$$

Using the same arguments used in the local existence, we can find a real $\tau > 0$ and a solution v to $(\mathcal{P}_s)$ on $[s, s+\tau]$. According to the section on local existence, and using the conservation of energy, τ does not depend on s. Thus, if we let s be close to T^* such that $s + \tau > T^*$, we can extend v for times higher than T^*. This fact contradicts the maximality of T^*. We obtain the claimed result.

Second, let us treat the limiting case

$$E = 4\pi \quad \text{and} \quad \sup_{[0,T^*]} \|\nabla u(t)\|^2 = \limsup_{T^*} \|\nabla u(t)\|^2 = 4\pi.$$

Then, since $x^2 \lesssim G(x)$,

$$\liminf_{T^*} \|G(|u(t)|^2)\|_1 = \liminf_{T^*} \|u(t)\|_4 = \liminf_{T^*} \|xu(t)\| = 0.$$

Global well-posedness is a consequence of the following result.

Lemma 3.4. *Let $T > 0$ and $u \in C([0,T],\Sigma)$ be a solution to the Schrödinger equation (1.1) with $\epsilon = -1$ such that $E(u_0) + M(u_0) < \infty$. Then a positive constant C_0 depending on u_0 exists such that, for any $R, R' > 0$ and any $0 < t < T$,*

$$\int_{B_{R+R'}} |u(t)|^2\,dx \geq \int_{B_R} |u_0|^2\,dx - C_0 \frac{t}{R'}. \tag{3.1}$$

Proof of Lemma 3.4. Let $R, R' > 0$, $d_R(x) := d(x, B_R)$ and a cut-off function $\phi := h(1 - (d_R/R'))$, where $h \in C^\infty(\mathbb{R})$, $0 \leq h \leq 1$, $h(t) = 1$ for $t \geq 1$ and $h(t) = 0$ for $t \leq 0$. So, $\phi(x) = 1$ for $x \in B_R$ and $\phi(x) = 0$ for $x \notin B_{R+R'}$. Moreover,

$$\nabla\phi(x) = -\frac{x - R}{R'|x - R|} h'\Big(1 - \frac{d_R(x)}{R'}\Big) \mathbf{1}_{\{R<|x|<R+R'\}};$$

$$\|\nabla\phi\|_{L^\infty} \leq \frac{\|h'\|_{L^\infty([0,1])}}{R'} \lesssim \frac{1}{R'}.$$

Multiplying (1.1) by $\phi^2\bar{u}$,

$$\phi^2\bar{u}(iu_t + \Delta u - |x|^2 u) = \phi^2|u|^2 G'(|u|^2).$$

Integrating over space and then taking the imaginary part yields

$$\begin{aligned}\partial_t\|\phi u\|^2 &= -2\Im\int \phi^2\bar{u}\Delta u\,dx\\ &= 2\Im\int \nabla(\phi^2\bar{u})\nabla u\,dx\\ &= 4\Im\int(\phi\nabla\phi\bar{u}\nabla u)\,dx \geq -\frac{C_0}{R'}.\end{aligned}$$

An integration over time achieves the proof. □

We return to the proof of global well-posedness. With the Hölder inequality, via (3.1),

$$\begin{aligned}\sqrt{\pi}(R+R')\Big(\int_{B_{R+R'}}|u(t)|^4\,dx\Big)^2 &\geq \int_{B_R}|u_0|^2\,dx - C_0\frac{t}{R'}\\ &\geq \int_{B_R}|u_0|^2\,dx - C_0\frac{T^*}{R'}.\end{aligned}$$

Taking the lower limit when t tends to T^* and then $R' \to \infty$ yields the contradiction $u_0 = 0$. This ends the proof.

4. The stationary problem

The goal of this section is to prove Theorem 2.4 about existence of a ground state solution to the stationary problem associated to (1.1).

Remark 4.1. If ϕ is a solution to the stationary problem (2.5), then $e^{it}\phi$ is a solution to the Schrödinger problem (1.1) with data ϕ. This particular global solution said standing wave does not scatter.

For $\alpha, \beta, \lambda \in \mathbb{R}$ and $v \in \Sigma$, we denote the quantities

$$S(v) := \|\nabla v\|^2 + \|v\|^2 + \|xv\|^2 - \int G(|v|^2)\,dx;$$

$$v^\lambda_{\alpha,\beta} := e^{\alpha\lambda}v(e^{-\beta\lambda}.),\quad \mathcal{L}_{\alpha,\beta}S(v) := \partial_\lambda(S(v^\lambda_{\alpha,\beta}))_{|\lambda=0};$$

$$K_{\alpha,\beta}(v) = 2\int[\alpha|\nabla v|^2 + (\alpha+\beta)|v|^2 + (\alpha+2\beta)|xv|^2 - \alpha|v|g(|v|) - \beta G(|v|^2)]\,dx;$$

$$K^Q_{\alpha,\beta}(v) := 2\int[\alpha|\nabla v|^2 + (\alpha+\beta)|v|^2 + (\alpha+2\beta)|xv|^2]\,dx;$$

$$K^N_{\alpha,\beta}(v) := -2\int[\alpha|v|g(|v|) + \beta G(|v|^2)]\,dx;$$

$$H_{\alpha,\beta}(v) := \Big(1 - \frac{1}{2(\alpha+2\beta)}\mathcal{L}_{\alpha,\beta}\Big)S(v).$$

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A direct computation gives $K_{\alpha,\beta} = \mathcal{L}_{\alpha,\beta} S$ and

$$H_{\alpha,\beta}(v) = \frac{1}{\alpha + 2\beta} \int [\beta(2|\nabla v|^2 + |v|^2) + \alpha |v| g(|v|) - (\alpha + \beta) G(|v|^2)]\, dx.$$

Let us start with a useful classical result about the solution to (2.5).

Proposition 4.2 (Generalized Pohozaev identity). *If ϕ is a solution to (2.5), then, for any $\alpha, \beta \in \mathbb{R}$,*

$$K_{\alpha,\beta}(\phi) = 0.$$

Proof. Since $S'(v) = 2\langle -\Delta v + v + |x|^2 v - vG'(|v|^2), \cdot\rangle$ and ϕ is a solution to (2.5), then $S'(\phi) = 0$. Now, because $\partial_\lambda (S(\phi^\lambda_{\alpha,\beta}))_{|\lambda=0} = \langle S'(\phi), \partial_\lambda(\phi^\lambda_{\alpha,\beta})_{|\lambda=0}\rangle = 0$, we have $K_{\alpha,\beta}(\phi) = 0$. □

We assume in the rest of this section that $\epsilon = 1$ and (2.1) is satisfied. Our aim is to prove that (2.5) has a ground state, meaning that it has a nontrivial positive radial solution which minimizes the action S when $K_{\alpha,\beta}$ vanishes. The proof of Theorem 2.4 is based on several lemmas.

Lemma 4.3. *Let $(\alpha, \beta) \in \mathcal{A}_g$ and $\phi \in \Sigma$. Then:*

(1) $\min(\mathcal{L}_{\alpha,\beta} H_{\alpha,\beta}(\phi), H_{\alpha,\beta}(\phi)) \geq 0$;
(2) *if $\alpha\phi \neq 0$, then* $\min(\mathcal{L}_{\alpha,\beta} H_{\alpha,\beta}(\phi), H_{\alpha,\beta}(\phi)) > 0$;
(3) $\lambda \mapsto H_{\alpha,\beta}(\phi^\lambda_{\alpha,\beta})$ *is increasing.*

Proof. Denote $\mathcal{L} := \mathcal{L}_{\alpha,\beta}$. With (2.1),

$$\begin{aligned} H_{\alpha,\beta}(\phi) &= \frac{1}{\alpha + 2\beta}\Big[\beta(2\|\nabla\phi\|^2 + \|\phi\|^2) + \alpha \int \Big(|\phi| g(|\phi|) - \Big(1 + \frac{\beta}{\alpha}\Big) G(|\phi|^2)\Big)\, dx\Big] \\ &= \frac{1}{(\alpha + 2\beta)}\Big[\beta(2\|\nabla\phi\|^2 + \|\phi\|^2) + 2\alpha \int \Big(D - \Big(1 + \frac{\beta}{\alpha}\Big)\Big) G(|\phi|^2)\, dx\Big] \geq 0. \end{aligned}$$

Moreover, with a direct computation,

$$\begin{aligned} \mathcal{L} H_{\alpha,\beta}(\phi) &= \mathcal{L}\Big(1 - \frac{1}{2(\alpha + 2\beta)}\mathcal{L}\Big) S(\phi) \\ &= -\frac{1}{2(\alpha + 2\beta)}(\mathcal{L} - 2\alpha)(\mathcal{L} - 2(\alpha + 2\beta)) S(\phi) + 2\alpha\Big(1 - \frac{1}{2(\alpha + 2\beta)}\mathcal{L}\Big) S(\phi) \\ &= -\frac{1}{2(\alpha + 2\beta)}(\mathcal{L} - 2\alpha)(\mathcal{L} - 2(\alpha + 2\beta)) S(\phi) + 2\alpha H_{\alpha,\beta}(\phi). \end{aligned}$$

Now, since $(\mathcal{L} - 2\alpha)\|\nabla\phi\|^2 = (\mathcal{L} - 2(\alpha + 2\beta))\|x\phi\|^2 = 0$, we have $(\mathcal{L} - 2\alpha)(\mathcal{L} - 2(\alpha + 2\beta))[\|\nabla\phi\|^2 + \|x\phi\|^2] = 0$. Moreover, $\mathcal{L} G(|\phi|^2) = 2[(\alpha D + \beta) G](|\phi|^2)$, so

$$\begin{aligned} \mathcal{L} H_{\alpha,\beta}(\phi) &\geq \frac{1}{2(\alpha + 2\beta)} \int (\mathcal{L} - 2\alpha)(\mathcal{L} - 2(\alpha + 2\beta)) G(|\phi|^2)\, dx \\ &= \frac{2}{\alpha + 2\beta} \int [\alpha(D - 1) + \beta][\alpha(D - 1) - \beta] G(|\phi|^2)\, dx \\ &= \frac{2\alpha^2}{\alpha + 2\beta} \int \Big(\Big[\Big((D - 1)^2 - \Big(\frac{\beta}{\alpha}\Big)^2\Big) G\Big](|\phi|^2)\Big)\, dx \geq 0. \end{aligned}$$

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The last inequality comes from (2.1). The two first points of the lemma follow. The last point is a consequence of the equality $\partial_\lambda H_{\alpha,\beta}(\phi^\lambda) = \mathcal{L}H_{\alpha,\beta}(\phi^\lambda)$. □

The next intermediate result is the following lemma.

Lemma 4.4. *Let $\alpha > 0, \beta \geq 0$ and (ϕ_n) be a bounded sequence of $\Sigma - \{0\}$ such that $\lim_n K^Q_{\alpha,\beta}(\phi_n) = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $K_{\alpha,\beta}(\phi_n) > 0$ for all $n \geq n_0$.*

Proof of Lemma 4.4. We start with the subcritical case.

(1) Subcritical case. Using (2.1) and (2.3), there exists $p > 4$ such that $\sup_{r\geq 0}(|rg(r) + G(r^2)|)/(r^p e^{r^2}) \lesssim 1$. Thus, for any $q \geq 1$,

$$\begin{aligned}K^N_{\alpha,\beta}(\phi_n) &\lesssim \int |\phi_n|^p (e^{|\phi_n|^2} - 1)\, dx + \|\phi_n\|_p^p\\ &\lesssim \|\phi_n\|^p_{qp}\|e^{|\phi_n|^2} - 1\|_{q'} + \|\phi_n\|_p^p\\ &\lesssim \|\phi_n\|^p_{qp}\|e^{q'|\phi_n|^2} - 1\|_1^{1/q'} + \|\phi_n\|_p^p.\end{aligned}$$

Now, if ${q'}^2\|\phi_n\|^2_{H^1} < 2\pi$, thanks to the Moser–Trudinger inequality, via the interpolation inequality,

$$\|\cdot\|_r \lesssim \|\cdot\|^{2/r}\|\nabla .\|^{1-(2/r)} \quad \forall r \in [2,\infty),$$

$$K^N_{\alpha,\beta}(\phi_n) \lesssim \|\phi_n\|^p_{qp} + \|\phi_n\|^p_p \lesssim \|\phi_n\|^{2/q}\|\nabla\phi_n\|^{p-(2/q)} + \|\phi_n\|^2\|\nabla\phi_n\|^{p-2}. \tag{4.1}$$

The proof is achieved by the fact that $\|\nabla\phi_n\|^2 \lesssim K^Q_{\alpha,\beta}(\phi_n)$ and taking q such that $p > 2 + (2/q)$.

(2) Critical case. By (2.1) and (2.4), there exist $p > 4$ and $a > 0$ such that $\sup_{r\geq 0}(|g(r) + G(r^2)|)/(r^p e^{ar^2}) \lesssim 1$. Thus, for any $q \geq 1$,

$$\begin{aligned}K^N_{\alpha,\beta}(\phi_n) &\lesssim \int |\phi_n|^p (e^{a|\phi_n|^2} - 1)\, dx + \|\phi_n\|_p^p\\ &\lesssim \|\phi_n\|^p_{qp}\|e^{a|\phi_n|^2} - 1\|_{q'} + \|\phi_n\|_p^p\\ &\lesssim \|\phi_n\|^p_{qp}\|e^{q'a|\phi_n|^2} - 1\|_1^{1/q'} + \|\phi_n\|_p^p.\end{aligned}$$

The rest is similar to the previous proof via the Moser–Trudinger inequality, since $\|\nabla\phi_n\|^2 \lesssim K^Q_{\alpha,\beta}(\phi_n) \to 0$. □ □

We have the last lemma of this section.

Lemma 4.5. *Let $\alpha > 0$ and $\beta \geq 0$. Then*

$$m_{\alpha,\beta} = \inf_{0\neq\phi\in\Sigma}\{H_{\alpha,\beta}(\phi),\ s.t.\ K_{\alpha,\beta}(\phi) \leq 0\}.$$

Proof of Lemma 4.5. Let m_1 be the right-hand side; it is sufficient to prove that $m_{\alpha,\beta} \leq m_1$. Take $\phi \in \Sigma$ such that $K_{\alpha,\beta}(\phi) < 0$; then, by Lemma 4.4 and the facts that $\lim_{\lambda\to-\infty} K^Q_{\alpha,\beta}(\phi^\lambda_{\alpha,\beta}) = 0$ and $\lambda \mapsto H_{\alpha,\beta}(\phi^\lambda)$ is increasing, there exists $\lambda < 0$ such that

$$K_{\alpha,\beta}(\phi^\lambda) = 0, \quad H_{\alpha,\beta}(\phi^\lambda) \leq H_{\alpha,\beta}(\phi). \tag{4.2}$$

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Then

$$m_{\alpha,\beta} \le S(\phi^\lambda) = H_{\alpha,\beta}(\phi^\lambda) \le H_{\alpha,\beta}(\phi).$$

The proof is completed. □

Proof of Theorem 2.4.

PROOF. *First case (2.3).*

Let (ϕ_n) a minimizing sequence, namely

$$0 \ne \phi_n \in \Sigma, \quad K_{\alpha,\beta}(\phi_n) = 0 \quad \text{and} \quad \lim_n H_{\alpha,\beta}(\phi_n) = \lim_n S(\phi_n) = m. \tag{4.3}$$

With a rearrangement argument via (4.2), we can assume that (ϕ_n) is radial decreasing and satisfies (4.3). Then $\alpha[\|\phi_n\|_\Sigma^2 - \int |\phi_n| g(|\phi_n|)\,dx] = \beta[\int G(|\phi_n|^2)\,dx - \|\phi_n\|^2 - 2\|x\phi_n\|^2]$. Denoting $\lambda := \beta/\alpha$ yields $\|\phi_n\|_\Sigma^2 - \int |\phi_n| g(|\phi_n|)\,dx = \lambda[\|\nabla\phi_n\|^2 - \|x\phi_n\|^2 - \|\phi_n\|_\Sigma^2 + \int G(|\phi_n|^2)\,dx]$. Thus,

$$\lambda\Big[\|\phi_n\|_\Sigma^2 - \int G(|\phi_n|^2)\,dx\Big] = \lambda[\|\nabla\phi_n\|^2 - \|x\phi_n\|^2] - \|\phi_n\|_\Sigma^2 + \int |\phi_n| g(|\phi_n|)\,dx.$$

So, the following sequence: $\lambda[\|\nabla\phi_n\|^2 - \|x\phi_n\|^2] + \int(|\phi_n| g(|\phi_n|) - G(|\phi_n|^2))\,dx$ is bounded. Since $(\|\phi_n\|_\Sigma^2 - \int G(|\phi_n|^2)) \to m$, the sequence $\lambda\|\nabla\phi_n\|^2 + \int(|\phi_n| g(|\phi_n|) - (1+\lambda)G(|\phi_n|^2))\,dx - \lambda(\|\phi_n\|_\Sigma^2 - \int G(|\phi_n|^2)\,dx) + \lambda\|\phi_n\|_{H^1}^2$ is also bounded. Then

$$\sup_n\Big[\lambda\|\nabla\phi_n\|^2 + \int(|\phi_n| g(|\phi_n|) - (1+\lambda)G(|\phi_n|^2))\,dx + \lambda\|\phi_n\|_{H^1}^2\Big] < \infty.$$

Suppose that $\beta \ne 0$. Thus, taking account of the assumption $(D - 1 - \lambda)G \ge 0$, we have $\|\phi_n\|_{H^1}^2 \lesssim 1$. This implies that (ϕ_n) is bounded in Σ; in fact, if $\|\phi_n\|_{H^1} \lesssim 1$ and $\|x\phi_n\| \to \infty$,

$$\int G(|\phi_n|^2)\,dx \ge -2m - 1 + \|\phi_n\|^2 + \|x\phi_n\|^2 \ge C(\|\phi_n\|^2 + \|x\phi_n\|^2).$$

By the Moser–Trudinger inequality, we obtain the absurdity

$$\infty \leftarrow \int G(|\phi_n|^2)\,dx \lesssim \|\phi_n\|^2.$$

So, (ϕ_n) is bounded in Σ. Assume now that $\beta = 0$; then

$$\|\phi_n\|_\Sigma^2 = \int |\phi_n| g(|\phi_n|)\,dx, \quad \Big(\|\phi_n\|_\Sigma^2 - \int G(|\phi_n|^2)\,dx\Big) \to m.$$

Thus, for any real number $a \ne 0$,

$$\begin{aligned}&\Big((1-a)\|\phi_n\|_\Sigma^2 + a\int\Big[|\phi_n| g(|\phi_n|) - \frac{1}{a}G(|\phi_n|^2)\Big]\,dx\Big)\\ &\quad = \Big((1-a)\|\phi_n\|_\Sigma^2 + a\int\Big[D - \frac{1}{a}\Big]G(|\phi_n|^2)\,dx\Big) \to m.\end{aligned}$$

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Taking $(1/1+\varepsilon_g) < a < 1$, we conclude that (ϕ_n) is bounded in Σ. This implies, via the compact injection $H^1_{rd} \hookrightarrow L^p$, for any $2 < p < \infty$, that, for some subsequence also denoted (ϕ_n),

$$\phi_n \rightharpoonup \phi \quad \text{in } \Sigma \quad \text{and} \quad \phi_n \to \phi \quad \text{in } L^p \ \forall p \in (2,\infty).$$

Assume that $\phi = 0$. There exist $p > 2$ and $a > 0$ small enough such that

$$\max\{G(r^2), rg(r)\} \lesssim r^p(e^{ar^2} - 1).$$

Since (ϕ_n) is bounded in H^1 and using the Moser–Trudinger inequality,

$$\begin{aligned}\max\Big\{\int G(|\phi_n|^2)\,dx, \int |\phi_n| g(|\phi_n|)\,dx\Big\} &\lesssim \|\phi_n^p(e^{a|\phi_n|^2} - 1)\|_1 \\ &\lesssim \|\phi_n\|_{2p}^p \|e^{2a|\phi_n|^2} - 1\|_1^{1/2} \\ &\lesssim \|\phi_n\|_{2p}^p \|\phi_n\| \to 0. \end{aligned} \tag{4.4}$$

By Lemma 4.4, $K_{\alpha,\beta}(\phi_n) > 0$ for large n, which is absurd. So,

$$\phi \neq 0.$$

With lower semicontinuity of the conformal norm, we have $K_{\alpha,\beta}(\phi) \leq 0$ and $H_{\alpha,\beta}(\phi) \leq m$. Using (4.2), we can assume that $K_{\alpha,\beta}(\phi) = 0$ and $S(\phi) = H_{\alpha,\beta}(\phi) \leq m$, so that ϕ is a minimizer satisfying $0 \neq \phi \in \Sigma_{rd}$, $K_{\alpha,\beta}(\phi) = 0$ and $S(\phi) = H_{\alpha,\beta}(\phi) = m$. Since

$$H_{\alpha,\beta}(\phi) = \frac{1}{\alpha + 2\beta}\Big[\beta(\|\nabla\phi\|^2 + \|\phi\|^2) + \alpha \int \Big(D - \Big(1 + \frac{\beta}{\alpha}\Big)\Big) G(|\phi|^2)\,dx\Big]$$

and $(\beta/\alpha) \leq \varepsilon_g$,

$$m > 0.$$

Now there is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'(\phi)$. So, recalling that $\mathcal{L}S(\phi) = (\partial_\lambda S(\phi^\lambda_{\alpha,\beta}))_{|\lambda=0}$,

$$\begin{aligned} 0 = K_{\alpha,\beta}(\phi) = \mathcal{L}S(\phi) &= \langle S'(\phi), (\partial_\lambda \phi^\lambda_{\alpha,\beta})_{|\lambda=0}\rangle \\ &= \eta\langle K'(\phi), (\partial_\lambda \phi^\lambda_{\alpha,\beta})_{|\lambda=0}\rangle \\ &= \eta\mathcal{L}K(\phi) = \eta\mathcal{L}^2 S(\phi). \end{aligned}$$

With a previous computation and taking account of (2.1),

$$\begin{aligned} -(\mathcal{L} - 2(\alpha+2\beta))(\mathcal{L} - 2\alpha)S(\phi) &= 2\alpha^2 \int \Big(\Big[\Big((D-1)^2 - \Big(\frac{\beta}{\alpha}\Big)^2\Big) G\Big](\phi^2)\Big)\,dx \\ &= -\mathcal{L}^2 S(\phi) - 4\alpha(\alpha+2\beta)S(\phi) \\ &> 0. \end{aligned}$$

Thus, $\eta = 0$ and $S'(\phi) = 0$. So, ϕ is a ground state and m is independent of α, β.

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Second case (2.4).

The proof is similar to the first case; the only point to change is (4.4). Let, for $\lambda \in (0, (1/(\alpha_0 \sup_n \|\phi_n\|_{H^1})))$, $\phi_{n,\lambda} := \lambda\phi_n$. Thanks to the Moser–Trudinger inequality,

$$\begin{aligned}\max\Big\{\int G(|\phi_n^\lambda|^2)\,dx, \int |\phi_n^\lambda| g(|\phi_n^\lambda|)\,dx\Big\} &\lesssim \|(\phi_n^\lambda)^p(e^{\lambda^2\alpha_0|\phi_n|^2}-1)\|_1\\ &\lesssim \|\phi_n^\lambda\|_{2p}^p \|e^{2\lambda^2\alpha_0|\phi_n|^2}-1\|_1^{1/2}\\ &\lesssim \|\phi_n^\lambda\|_{2p}^p\|\phi_n^\lambda\| \to 0.\end{aligned}$$

Thus, $K^N(\phi_{n,\lambda}) \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned}K_{\alpha,\beta}(\phi_{n,\lambda}) &= K^N_{\alpha,\beta}(\phi_{n,\lambda}) + K^Q_{\alpha,\beta}(\phi_{n,\lambda})\\ &= K^N_{\alpha,\beta}(\phi_{n,\lambda}) - \lambda^2 K^Q_{\alpha,\beta}(\phi_n)\\ &\leq 0 \quad \text{for large } n.\end{aligned}$$

Now, since $\lambda \mapsto H_{\alpha,\beta}(\phi_{n,\lambda})$ is increasing, we have, for large n,

$$K_{\alpha,\beta}(\phi_{n,\lambda}) \leq 0, \quad H_{\alpha,\beta}(\phi_{n,\lambda}) \leq m.$$

So,

$$K_{\alpha,\beta}(\phi_{n,\lambda}) \leq 0, \quad H_{\alpha,\beta}(\phi_{n,\lambda}) = m.$$

By Lemma 4.4, $K_{\alpha,\beta}(\phi_{n,\lambda}) > 0$ for large n, which is absurd. □

5. Invariant sets and applications

This section is devoted to prove either global well-posedness or finite time blow-up of the solution to (1.1) with data in some stable sets. In all of this section, we assume that $\epsilon = 1$. Our aim is to prove Theorem 2.5. Denote, for $v \in \Sigma$ and $\lambda \in \mathbb{R}$, the quantities

$$K(v) = K_{1,-1}(v) = 2\|\nabla v\|^2 - 2\|xv\|^2 - 2\int [|v|g(|v|) - G(|v|^2)]\,dx;$$

$$I(v) = K_{1,0}(v) = 2\|v\|_\Sigma^2 - 2\int |v|g(|v|)\,dx;$$

$$m_{1,-1} := \inf_{0\neq v\in\Sigma}\{S(v),\ \text{s.t. } K(v) = 0 \text{ and } I(v) \leq 0\}.$$

First, let us prove existence of a ground state to (2.5) for $(\alpha, \beta) = (1, -1)$.

Proposition 5.1. *Assume that g satisfies (2.2) with [(2.3) or (2.4)] and take ϕ to be a ground state solution to (2.5). Then*

$$m_{1,-1} = S(\phi) = m_{1,0}.$$

Proof of Proposition 5.1. Let (ϕ_n) be a minimizing sequence, supposed to be radial with a classical rearrangement argument, namely

$$0 \neq \phi_n \in \Sigma_{rd}, \quad K(\phi_n) = 0, \quad I(\phi_n) \leq 0 \quad \text{and} \quad \lim_n S(\phi_n) = m_{1,-1}.$$

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Then

$$\|\nabla\phi_n\|^2 - \|x\phi_n\|^2 = \int (D-1)G(|\phi_n|^2)\,dx \quad \text{and} \quad \Big(\|\phi_n\|_\Sigma^2 - \int G(|\phi_n|^2)\,dx\Big) \to m_{1,-1}.$$

So, for any real number $a \neq 0$,

$$\Big((1-a)\|\nabla\phi_n\|^2 + (1+a)\|x\phi_n\|^2 + \|\phi_n\|^2 + a\int\Big[D-1-\frac{1}{a}\Big]G(|\phi_n|^2)\,dx\Big) \to m_{1,-1}.$$

Taking $a := 1/(1+\varepsilon_g)$,

$$\Big(\frac{\varepsilon_g}{1+\varepsilon_g}\|\nabla\phi_n\|^2 + \frac{2+\varepsilon_g}{1+\varepsilon_g}\|x\phi_n\|^2 + \|\phi_n\|^2 + \frac{1}{1+\varepsilon_g}\int[D-2-\varepsilon_g]G(|\phi_n|^2)\,dx\Big) \to m_{1,-1}.$$

We conclude, via (2.2), that (ϕ_n) is bounded in Σ_{rd}. Taking account of the compact injection (2.8), we take $\phi \in \Sigma_{rd}$ satisfying

$$\phi_n \rightharpoonup \phi \quad \text{in } \Sigma \quad \text{and} \quad \phi_n \to \phi \quad \text{in } L^p \ \forall p \in [2,\infty).$$

We have $I(\phi_n) \leq 0$ and $K(\phi_n) = 0$; then

$$\|x\phi_n\|^2 = \|\nabla\phi_n\|^2 - \int (D-1)G(|\phi_n|^2)\,dx;$$
$$2\|\nabla\phi_n\|^2 \leq \int [(2D-1)G(|\phi_n|^2) - |\phi_n|^2]\,dx.$$

Assume, by contradiction, that $\phi = 0$. Using the Moser–Trudinger inequality and arguing as previously,

$$\int (2D-1)G(|\phi_n|^2)\,dx \to 0 \quad \text{as } n \to \infty.$$

Then, following (4.1),

$$\|\nabla\phi_n\|^2 \lesssim \int (2D-1)G(|\phi_n|^2)\,dx = o(\|\nabla\phi_n\|^2) \quad \text{as } n \to \infty.$$

This contradiction implies that

$$\phi \neq 0.$$

With lower semicontinuity of the Σ norm,

$$K(\phi) \leq 0, \quad I(\phi) \leq 0 \quad \text{and} \quad S(\phi) \leq m_{1,-1}.$$

Using Lemma 4.5,

$$m_{1,0} \leq H_{1,0}(\phi) \leq S(\phi) \leq m_{1,-1} \leq m_{1,0}.$$

Finally, taking account of Theorem 2.4, we get $S(\phi) = m_{1,-1} = m_{1,0}$ and ϕ is a ground state solution to (2.5). Then $K(\phi) = 0$ and the proof is finished. □

We now have the last auxiliary result of this section.

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PROPOSITION 5.2. *Let $(\alpha,\beta)\in\mathcal{A}_g$. Then:*

(1) *$m_{\alpha,\beta}$ is independent of (α,β);*
(2) *the sets $A^+_{\alpha,\beta}$, $A_{1,-1}$ and $A^-_{\alpha,\beta}$ are invariant under the flow of (1.1);*
(3) *the sets $A^+_{\alpha,\beta}$ and $A^-_{\alpha,\beta}$ are independent of (α,β).*

PROOF. Let (α,β) and (α',β') in $\mathcal{A}_g$.

(1) Let ϕ be a minimizing of (2.6) which satisfies (2.5); then $K_{\alpha',\beta'}(\phi)=0$. Thus, $m_{\alpha,\beta}\le m_{\alpha',\beta'}$. In the same way, we have the opposite inequality.

(2) Let $u_0\in A^+_{\alpha,\beta}$ and $u\in C_{T^*}(\Sigma)$ be the maximal solution to (1.1). Assume that for some time $t_0\in(0,T^*)$, $u(t_0)\notin A^+_{\alpha,\beta}$. Since the energy is conserved, $K_{\alpha,\beta}(u(t_0))\le 0$. So, with a continuity argument, there exists a positive time $t_1\in(0,t_0)$ such that $K_{\alpha,\beta}(u(t_1))=0$. This contradicts the definition of $m_{\alpha,\beta}$. The proof is similar for $A^-_{\alpha,\beta}$.

Let $u_0\in A_{1,-1}$ and $u\in C_{T^*}(\Sigma)$ be the maximal solution to (1.1). Assume that for some time $t_0\in(0,T^*)$, $u(t_0)\notin A_{1,-1}$. Since the energy is conserved, $K(u(t_0))>0$ or $I(u(t_0))>0$. With a continuity argument, there exists a positive time $t_1\in(0,t_0)$ such that either $K(u(t_1))=0$ and $I(u(t_1))<0$, which contradicts the definition of $m_{1,-1}$, or $K(u(t_1))<0$ and $I(u(t_1))=0$, which contradicts the definition of $m_{1,0}$.

(3) By the first point, the reunion $A^+_{\alpha,\beta}\cup A^-_{\alpha,\beta}$ is independent of (α,β). So, it is sufficient to prove that $A^+_{\alpha,\beta}$ is independent of (α,β). If $S(\phi)<m$ and $K_{\alpha,\beta}(\phi)=0$, then $\phi=0$. So, $A^+_{\alpha,\beta}$ is open. The rescaling $\phi^\lambda := e^{\alpha\lambda}\phi(e^{-\beta\lambda})$ implies that a neighborhood of zero is in $A^+_{\alpha,\beta}$. Moreover, this rescaling with $\lambda\to-\infty$ gives that $A^+_{\alpha,\beta}$ is contracted to zero and so is connected. Now, by the definition, $A^-_{\alpha,\beta}$ is open, and $0\in A^+_{\alpha,\beta}\cap A^+_{\alpha',\beta'}$. Writing

$$A^+_{\alpha,\beta}=A^+_{\alpha,\beta}\cap(A^+_{\alpha',\beta'}\cup A^-_{\alpha',\beta'})=(A^+_{\alpha,\beta}\cap A^+_{\alpha',\beta'})\cup(A^+_{\alpha,\beta}\cap A^-_{\alpha',\beta'}),$$

we have $A^+_{\alpha,\beta}=A^+_{\alpha',\beta'}$. The proof is achieved. □

Finally, we are ready to prove the main result of this section.

Proof of Theorem 2.5. Using a time-translation argument, we can assume that $t_0=0$.

(1) With Proposition 5.2, $u(t)\in A_{1,-1}$ for any $t\in[0,T^*)$. By contradiction, assume that $T^*=\infty$. With the virial identity (1.2),

$$\frac{1}{8}(\|xu(t)\|^2)''=\|\nabla u\|^2-\|xu\|^2-\int(\bar{u}g(u)-G(|u|^2))\,dx=\frac{1}{2}K(u(t))<0.$$

We infer that there exists $\delta>0$ such that $K(u(t))<-\delta$ for large time. Otherwise, there exists a sequence of positive real numbers $t_n\to+\infty$ such that $K(u(t_n))\to 0$. By the definition of $m_{1,-1}$,

$$m_{1,-1}\le(S-\tfrac{1}{2}K)(u(t_n))=S(u_0)-\tfrac{1}{2}K(u(t_n))\to S(u_0)<m_{1,-1}.$$

This absurdity finishes the proof of the claim. Thus, $(\|xu\|^2)''<-4\delta$. Integrating twice, $\|xu(t)\|$ becomes negative for some positive time. This contradiction closes the proof.

(2) By Proposition 5.2, $u(t) \in A^+_{\alpha,\beta}$ for any $t \in [0, T^*)$. Then

$$\begin{aligned} m &> S(u(t)) \\ &> H_{\alpha,\beta}(u(t)) \\ &= \frac{\alpha}{\alpha+2\beta}\int\Big[\frac{\beta}{\alpha}(2|\nabla u(t)|^2 + |u(t)|^2) + |u(t)|g(|u(t)|) - \Big(1+\frac{\beta}{\alpha}\Big)G(|u(t)|^2)\,dx\Big] \\ &\gtrsim \|u(t)\|^2_{H^1} + \int G(|u(t)|^2)\,dx. \end{aligned}$$

If $\beta \neq 0$, with the energy conservation,

$$\sup_{t\in[0,T^*]} \|u(t)\|_\Sigma < \infty.$$

Assume now that $\beta = 0$. Then

$$(1+\varepsilon_g)\int G(|u|^2)\,dx < \int |u|g(|u|)\,dx < \|u\|^2_\Sigma < m_{\alpha,\beta} + \int G(|u|^2)\,dx.$$

The first inequality is by (2.1) and the two other inequalities follow from the definition of $A^+_{\alpha,\beta}$. This implies that

$$\sup_{t\in[0,T^*]} \|u(t)\|_\Sigma < \infty \quad \text{and} \quad T^* = \infty. \Lambda$$

6. Instability in the focusing case

In this section, existence of infinitely many nonglobal solutions to (1.1) in the focusing case is proved. We say that the problem (1.1) is strongly unstable. In the rest of this paper, we assume that $\epsilon = 1$ and (2.2) is satisfied. We keep the notation of the previous section, namely $K = (1/2)K_{1,-1}$, $I = (1/2)K_{1,0}$ and denote for $v \in \Sigma$ the scaling $v_\lambda := \lambda v(\lambda.)$, where λ is a nonnegative real number. Let us prepare the proof of Theorem 2.6.

Lemma 6.1. *Let $v \in \Sigma$ be such that $I(v) \le 0$ and $K(v) \le 0$. Then, for any $\lambda > 1$:*

(1) $(\partial/\partial\lambda)S(v_\lambda) = (2/\lambda)K(v_\lambda)$;

(2) $K(v_\lambda) < 0$ *when λ is close to 1.*

Proof. (1) Compute

$$\begin{aligned} \frac{\lambda}{2}\frac{\partial}{\partial\lambda}S(v_\lambda) &= \frac{\lambda}{2}\frac{\partial}{\partial\lambda}\Big(\lambda^2\|\nabla v\|^2 + \|v\|^2 + \lambda^{-2}\|xv\|^2 - \lambda^{-2}\int G(|\lambda v|^2)\,dx\Big) \\ &= \lambda^2\|\nabla v\|^2 - \lambda^{-2}\|xv\|^2 - \lambda^{-2}\int (D-1)G(|\lambda v|^2)\,dx \\ &= K(v_\lambda). \end{aligned}$$

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(2) We have $I(v) \leq 0$ and $K(v) \leq 0$; then

$$\|xv\|^2 \geq \|\nabla v\|^2 - \int (|v|g(|v|) - G(|v|^2))\,dx := \|\nabla v\|^2 - \int h(|v|)\,dx;$$
$$2\|\nabla v\|^2 \leq \int [h(|v|) + |v|g(|v|) - |v|^2]\,dx.$$

Moreover,

$$\begin{aligned}
K(v_\lambda) &= \lambda^2\|\nabla v\|^2 - \lambda^{-2}\|xv\|^2 - \lambda^{-2}\int (|\lambda v|g(|\lambda v|) - G(|\lambda v|^2))\,dx \\
&\leq \Big(\lambda^2 - \frac{1}{\lambda^2}\Big)\|\nabla v\|^2 + \int \frac{h(|v|) - h(|\lambda v|)}{\lambda^2}\,dx \\
&\leq \frac{1}{\lambda^2}\Big[(\lambda^4 - 1)\int\Big(|v|g(|v|) - \frac{G(|v|^2)}{2} - \frac{|v|^2}{2}\Big)dx + \int h(|v|) - h(|\lambda v|)\,dx\Big] \\
&\leq \frac{1}{\lambda^2}\Big[(\lambda^4 - 1)\int\Big(h(|v|) + \frac{G(|v|^2)}{2} - \frac{|v|^2}{2}\Big)dx + \int h(|v|) - h(|\lambda v|)\,dx\Big] \\
&\leq \frac{1}{\lambda^2}\int\Big[(\lambda^4 - 1)\Big(\frac{G(|v|^2)}{2} - \frac{|v|^2}{2}\Big) + \lambda^4 h(|v|) - h(|\lambda v|)\Big]\,dx.
\end{aligned}$$

Take, for $t > 0$, the real function defined on $(1, \infty)$ by

$$\begin{aligned}
f(r) &:= (r^2 - 1)(G(t) - t) + 2r^2 h(\sqrt{t}) - 2h(\sqrt{rt}) \\
&:= (r^2 - 1)(G(t) - t) + 2r^2(D - 1)G(t) - 2(D - 1)G(rt).
\end{aligned}$$

Then the derivative satisfies the following equation when r tends to 1:

$$\begin{aligned}
f'(r) &= 2r(G(t) - t + 2(D - 1)G(t) - t^2 G''(rt)) \\
&\simeq 2(G(t) - t + 2(D - 1)G(t) - t^2 G''(t)) \\
&\simeq -2((D^2 - 3D + 1)G(t) + t).
\end{aligned}$$

This implies, via (2.2), that for $r > 1$ and close to 1, $f'(r) < -2(D^2 - 3D + 1)G(t) = -2[(D - 1)(D - 2) - 1]G(t) < 0$ and f is decreasing near to 1. Since $f(1) = 0$, we get $f < 0$ near to 1. The proof of the second point of the lemma is finished. □

We have the next intermediate result.

Lemma 6.2. *Let ϕ to be a ground state solution to (2.5), $\lambda > 1$ a real number close to 1 and u the solution to (1.1) with data ϕ_λ. Then, for any $t \in (0, T^*)$,*

$$S(u(t)) < S(\phi), \quad I(u(t)) < 0 \quad \textit{and} \quad K(u(t)) < 0.$$

Proof. By Lemma 6.1,

$$S(\phi_\lambda) < S(\phi) \quad \text{and} \quad K(\phi_\lambda) < 0.$$

Moreover, since $I(\phi) = K(\phi) = 0$,

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$$\begin{aligned} I(\phi_\lambda) &= S(\phi_\lambda) + K(\phi_\lambda) - \|\nabla\phi_\lambda\|^2 + \|x\phi_\lambda\|^2 \\ &= S(\phi_\lambda) + K(\phi_\lambda) - \lambda^2\|\nabla\phi\|^2 + \lambda^{-2}\|x\phi\|^2 \\ &\leq S(\phi) + K(\phi) - I(\phi) - \lambda^2\|\nabla\phi\|^2 + \lambda^{-2}\|x\phi\|^2 \\ &\leq (1-\lambda^2)\|\nabla\phi\|^2 + (-1+\lambda^{-2})\|x\phi\|^2 < 0. \end{aligned}$$

Thanks to the conservation laws, it follows that, for any $t > 0$,

$$S(u(t)) = S(\phi_\lambda(t)) < S(\phi).$$

Then $K(u(t)) \neq 0$ and $I(u(t)) \neq 0$ because ϕ is a ground state. Finally, $K(u(t)) < 0$ and $I(u(t)) < 0$ with a continuity argument. □

Now we are ready to prove the instability result.

Proof of Theorem 2.6. Take $u_\lambda \in C_{T^*}(\Sigma)$, the maximal solution to (1.1) with data ϕ_λ, where $\lambda > 1$ is close to 1 and ϕ is a ground state solution to (2.5). With the previous lemma,

$$u_\lambda(t) \in A_{1,-1} \quad \text{for any } t \in (0, T^*).$$

Then, using Theorem 2.5,

$$\lim_{t \to T^*} \|u_\lambda(t)\|_\Sigma = \infty.$$

The proof is finished via the fact that

$$\lim_{\lambda \to 1} \|\phi_\lambda - \phi\|_\Sigma = 0.$$

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