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NON-METRIZABLE UNIFORMITIES AND PROXIMITIES ON METRIZABLE SPACES

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In the literature there exist examples of metrizable spaces admitting non-metrizable uniformities (e.g., see [**3**, Problem C, p. 204]). In this paper, this phenomenon is presented more coherently by showing that every non-compact metrizable space admits at least one non-metrizable proximity and uncountably many non-metrizable uniformities. It is also proved that the finest compatible uniformity (proximity) on a non-compact non-semidiscrete space is always non-metrizable.

The closure of a set A in a topological space is denoted by $\bar{A}$, and by c we denote the cardinal number of the real line. We follow the terminology of [**3**] and [**4**] throughout.

Let a completely regular space X be expressible as a disjoint topological sum of two of its subspaces X_1 and X_2 and let δ_1 and δ_2 be compatible proximities on X_1 and X_2 respectively. Define a binary relation $\delta_1 \oplus \delta_2$ on the power-set of X as follows:

For any two subsets A, B of X, let $(A, B) \in \delta_1 \oplus \delta_2$ if and only if

$$(A \cap X_1, B \cap X_1) \in \delta_1 \text{ or } (A \cap X_2, B \cap X_2) \in \delta_2.$$

It is easy to verify that $\delta_1 \oplus \delta_2$ is a compatible proximity on X. The proximity $\delta_1 \oplus \delta_2$ defined as above shall be called the *disjoint proximity sum* of the proximities δ_1 and δ_2. It is obvious that the subspace proximity induced on X_i by the proximity $\delta_1 \oplus \delta_2$ is δ_i, $i = 1, 2$. It follows that if $\delta_1 \oplus \delta_2$ is metrizable then so is each δ_i.

LEMMA A. *Every countably infinite discrete space has exactly 2^c compatible non-metrizable proximities and an equal number of compatible non-metrizable uniformities.*

Proof. Consider the discrete space N, the space of natural numbers. It is well-known (see [**2**]) that $\beta N - N$ has 2^c points and thus N has 2^c distinct compatifications [**5**]. It follows that (see Chapter III of [**4**]) N admits 2^c distinct proximities. Also there cannot be more that c distinct pseudometrics on N and so the number of compatible metrizable proximities on N is at most c. As $2^c - c = 2^c$, it follows that N has 2^c compatible non-metrizable proximities. Trivially each uniformity in the p-class of a non-metrizable proximity is non-metrizable and further there can be at most 2^c uniformities on a countable

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set. It follows that there are exactly 2^c compatible non-metrizable uniformities on N, and the lemma is proved.

A topological space X shall be called *semi-discrete* if and only if it is Tychonoff and has an infinite closed subset A such that for each $a \in A$, $\{a\}$ is open in X.

We remark that if a space X has an infinite closed subset A each of whose element is open in X then X also has a countably infinite closed subset B with each element open in X.

LEMMA B. *Every semi-discrete space has at least 2^c compatible non-metrizable proximities (uniformities).*

Proof. Let A be a countably infinite closed subset of a semi-discrete space X, such that each element of A is open in X. There exists a set $\{\delta_\lambda : \lambda \in \Lambda\}$ of 2^c compatible non-metrizable proximities on A. Let δ be any compatible proximity on $X - A$. The proximities $\{\delta \oplus \delta_\lambda : \lambda \in \Lambda\}$ are all distinct and non-metrizable and each is compatible with the topology of X. Thus the lemma is proved.

We recall that every completely regular space admits a finest proximity and also a finest uniformity and, further, the finest compatible uniformity lies in the p-class of the finest compatible proximity. For a normal Hausdorff space X the finest compatible proximity δ is defined by (see [**4**])

$$(*) \qquad (A, B) \in \delta \text{ if and only if } \bar{A} \cap \bar{B} \neq \phi.$$

THEOREM 1. *Let X be a completely regular Hausdorff space. If X is not semi-discrete nor compact then the finest compatible proximity (uniformity) on X is non-metrizable.*

Proof. Let δ be the finest compatible proximity on X. If possible, suppose δ is metrizable. Then X is metrizable, and there exists a metric d on X such that (by (*) above):

$$\text{(a)} \qquad \bar{A} \cap \bar{B} = \phi \text{ if and only if } d(A, B) \neq 0.$$

The given conditions on X assure the existence of a countably infinite subset $A = \{a_n : n = 1, 2, \ldots\}$ of X such that (i) A has no limit point in X, and (ii) for no $a \in A$, $\{a\}$ is open in X. Let $A_n = \{a_i \in A : i \geqq n\}$. The closed sets $\{a_1\}$ and A_2 are disjoint and so there exists a number $\epsilon_1 > 0$ such that $d(\{a_1\}, A_2) = \epsilon_1$. Put $\delta_1 = \epsilon_1/2$ and let $S_1 = \{y \in X : d(a_1, y) < \delta_1\}$. Having defined S_n, we define S_{n+1} inductively as follows: the sets

$$P_{n+1} = \bar{S}_1 \cup \ldots \cup \bar{S}_n \cup A_{n+2} \text{ and } Q_{n+1} = \{a_{n+1}\}$$

are disjoint, closed in X, and so there is a number $\epsilon_{n+1} > 0$ such that $d(P_{n+1}, Q_{n+1}) = \epsilon_{n+1}$. Let

$$\delta_{n+1} = \min\left\{\frac{\epsilon_1}{n+2}, \frac{\epsilon_{n+1}}{2}\right\}$$

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and define $S_{n+1} = \{y \in X : d(y, a_{n+1}) < \delta_{n+1}\}$. By condition (ii) above, we can take a $y_n \in S_n$ such that $y_n \neq a_n$. Let $B = \{y_n : n = 1, 2, \ldots\}$. Now it is easy to check that A and B are disjoint closed sets and $d(A, B) = 0$, contrary to (a) above. Thus we conclude that δ must be non-metrizable, and the theorem is proved.

THEOREM 2. *In a completely regular Hausdorff space, the statements* (1) *through* (3) *given below are equivalent.*

(1) *Each compatible uniformity on X is metrizable.*

(2) *Each compatible proximity on X is metrizable.*

(3) *X is a compact metric space.*

Proof. Obviously (1) implies (2) and (3) implies (1). Also Theorem 1 and Lemma B together show that (2) implies (3). Thus the theorem is proved.

The finest compatible uniformity on a metric space being always complete, it follows that the p-class of uniformities for the finest compatible proximity on a non-compact metrizable space must be non-trivial (a p-class having at least two members is called non-trivial). A very elegant result by Reed and Thron [**7**] states that any non-trivial p-class must have at least c members. Thus we get the following

COROLLARY. *Every non-compact metrizable space admits uncountably many non-metrizable uniformities.*

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