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FROBENIUS SYMBOLS AND THE GROUPS $\mathbf{S}_s$, $GL(n)$, $O(n)$ AND $Sp(n)$

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1. Introduction. Frobenius [**2**; **3**] introduced the symbols

$$(\mathbf{a} * \mathbf{b})_s = (a_1 a_2 \ldots a_r * b_1 b_2 \ldots b_r)$$

to specify partitions and the corresponding irreducible representations of the symmetric group $\mathbf{S}_s$. In terms of these symbols an interesting theorem has been proved [**10**, p. 49] which takes the form:

THEOREM 1.

$$(\mathbf{a} * \mathbf{b})_s = |(a_i * b_j)|^{\cdot},$$

where $(a_i * b_j)$ *is the* (ij)th *element in a determinant to be expanded in accordance with the outer product multiplication rules signified by the notation* $|\ |^{\cdot}$.

From this theorem it is easy to derive the formula: [**10**, p. 44]

$$f^{(\mathbf{a}*\mathbf{b})_s} = s!/H(\mathbf{a} * \mathbf{b})_s \tag{1.1}$$

for the degree of the representation $(\mathbf{a} * \mathbf{b})_s$ of $\mathbf{S}_s$, in terms of the hook length factor, $H(\mathbf{a} * \mathbf{b})_s$, associated with the corresponding Young tableau.

Using the well-known duality between the symmetric group, $\mathbf{S}_s$, and the general linear group, $GL(n)$, one can immediately write down a theorem analogous to Theorem 1, but which applies to irreducible representations of $GL(n)$, namely [**7**, p. 112]:

THEOREM 2.

$$\{\mathbf{a} * \mathbf{b}\}_s = |\{a_i * b_j\}|^{\cdot},$$

where the notation $|\ |^{\cdot}$ *indicates the use of the Kronecker product multiplication rules appropriate to* $GL(n)$ *in the expansion of the determinant.*

From this theorem it is once again easy to derive a dimensionality formula, this time for the dimension of the irreducible representation $\{\mathbf{a} * \mathbf{b}\}_s$ of $GL(n)$. This takes the form [**10**, p. 60]:

$$D_n\{\mathbf{a} * \mathbf{b}\}_s = N_n\{\mathbf{a} * \mathbf{b}\}_s/H(\mathbf{a} * \mathbf{b})_s, \tag{1.2}$$

where $N_n\{\mathbf{a} * \mathbf{b}\}_s$ is a factored polynomial in n of degree s.

The aim of this paper is to show that the Kronecker product multiplication rules appropriate to the orthogonal group, $O(n)$, and the symplectic group,

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$Sp(n)$, are such that the corresponding irreducible representations of these groups satisfy the theorems:

THEOREM 3.

$$[\mathbf{a} * \mathbf{b}]_s = |[a_i * b_j]|^{\times}$$

and

THEOREM 4.

$$\langle \mathbf{a} * \mathbf{b} \rangle_s = |\langle a_i * b_j \rangle|^{\times}$$

respectively, where the notation $| \, |^{\times}$ *indicates the use of the Kronecker product multiplication rules appropriate to* $O(n)$ *and* $Sp(n)$.

Just as the brackets ( ) and { } are used to denote irreducible representations of $\mathbf{S}_s$ and $GL(n)$, so the brackets [ ] and $\langle \; \rangle$ are used to denote irreducible representations of $O(n)$ and $Sp(n)$ respectively.

From these theorems dimensionality formulae completely analogous to (1.2) are derived, i.e.,

$$D_n[\mathbf{a} * \mathbf{b}]_s = N_n[\mathbf{a} * \mathbf{b}]_s / H(\mathbf{a} * \mathbf{b})_s \tag{1.3}$$

and

$$D_n\langle \mathbf{a} * \mathbf{b} \rangle_s = N_n\langle \mathbf{a} * \mathbf{b} \rangle_s / H(\mathbf{a} * \mathbf{b})_s, \tag{1.4}$$

where $N_n[\mathbf{a} * \mathbf{b}]_s$ and $N_n\langle \mathbf{a} * \mathbf{b} \rangle_s$ are once more factored polynomials in n of degree s. These formulae represent a considerable improvement on the only n-dependent formulae obtained previously [**5**] in which the expressions obtained for $N_n[\mathbf{a} * \mathbf{b}]_s$ and $N_n\langle \mathbf{a} * \mathbf{b} \rangle_s$ took the form of quotients of factored polynomials, in which the pattern of cancellation of the factors common to both numerator and denominator was by no means obvious in the general case.

In the following section the notation is established more precisely and Theorems 1 and 2 together with their important corollaries (1.1) and (1.2) are discussed. In Section 3, Theorem 3 is proved by induction. Since the Kronecker product multiplication rules appropriate to $O(n)$ and $Sp(n)$ are identical [**6**; **8**; **9**] the validity of Theorem 4 then follows immediately. In Section 4 expressions for the numerators of the formulae (1.3) and (1.4) are derived and an illustrative example of their use is given.

2. Frobenius symbols and the groups $\mathbf{S}_s$ and $GL(n)$. A regular partition of the integer s into p parts can be denoted by

$$(\lambda)_s = (\lambda_1, \lambda_2, \ldots, \lambda_p) \tag{2.1}$$

with $\lambda_1 + \lambda_2 + \ldots + \lambda_p = s$ and $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_p > 0$. This partition may be represented by a regular Young tableau with p rows, such that the kth row contains λ_k boxes or nodes. The weight of the partition and the corresponding tableau is said to be s.

Alternatively, the partition and the tableau may both be specified by means of the Frobenius symbol [**2**; **3**]

$$(\mathbf{a} * \mathbf{b})_s = (a_1 a_2 \ldots a_r * b_1 b_2 \ldots b_r) \tag{2.2}$$

with $a_1 + a_2 + \ldots + a_r + b_1 + b_2 + \ldots + b_r + r = s$ and

$$a_1 > a_2 > \ldots > a_r \geqq 0 \quad \text{and} \quad b_1 > b_2 > \ldots > b_r \geqq 0.$$

The quantities a_k and b_k denote the numbers of boxes in the Young tableau to the right of and below, respectively, the box at the intersection of the kth row and the kth column. The rank of the Frobenius symbol and the corresponding tableau is said to be r, which is clearly the number of boxes in the principal diagonal of the tableau.

The relationship between $(\lambda)_s$ and $(\mathbf{a} * \mathbf{b})_s$ is such that $a_k = \lambda_k - k$ and $b_k = \tilde{\lambda}_k - k$ for $k = 1, 2, \ldots, r$, where $(\tilde{\lambda})_s = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ is the partition conjugate to $(\lambda)_s$, i.e. the partition specifying the tableau obtained from that specified by $(\lambda)_s$ by interchanging the rows and columns.

There is a one-to-one correspondence between partitions

$$(\lambda)_s = (\lambda_1, \lambda_2, \ldots, \lambda_p)$$

and the irreducible representations of the symmetric group, $\mathbf{S}_s$, denoted by the same symbol, and also with the irreducible representations of the general linear group, $GL(n)$, denoted by $\{\lambda\}_s = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. The basis of such an irreducible representation of $GL(n)$ is a tensor whose index symmetry is specified by the partition $(\lambda)_s$.

In Frobenius notation the representations $(\lambda)_s$ and $\{\lambda\}_s$ are denoted by $(\mathbf{a} * \mathbf{b})_s$ and $\{\mathbf{a} * \mathbf{b}\}_s$.

Using the raising operator techniques developed by Young, Robinson [**10**, p. 44] proved Theorem 1, which written out in full takes the form:

$$(a_1 a_2 \ldots a_r * b_1 b_2 \ldots b_r) = \begin{vmatrix} (a_1 * b_1) & (a_1 * b_2) & \ldots & (a_1 * b_r) \\ (a_2 * b_1) & (a_2 * b_2) & \ldots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ (a_r * b_1) & \cdot & \ldots & (a_r * b_r) \end{vmatrix}^{\cdot} \tag{2.3}$$

where the notation $|\ |^{\cdot}$ indicates that the determinant is to be evaluated using the outer product reduction rules appropriate to the symmetric group. Each element, $(a_i * b_j)$, of the determinant is a Frobenius symbol of rank 1, and corresponds to a tableau consisting of a single hook. In the more conventional notation for partitions

$$(a_i * b_j) = (1 + a_i, 1^{b_j}). \tag{2.4}$$

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Earlier, using the algebra of S-functions, Littlewood [**7**, p. 112] proved Theorem 2, i.e.

(2.5) $$\{a_1a_2 \ldots a_r * b_1b_2 \ldots b_r\} = |\{a_i * b_j\}|^{\cdot}$$

where, in this case, the notation $| \;|^{\cdot}$ indicates that the determinant is to be evaluated using the multiplication rules appropriate to the reduction of Kronecker products of representations of $GL(n)$. These rules are the well-known rules [**4**; **7**] which give the coefficients $m_{\mu\nu,\lambda}$ in the product:

(2.6) $$\{\mu\}_a \cdot \{\nu\}_b = \sum_{\lambda} m_{\mu\nu,\lambda}\{\lambda\}_c,$$

where the summation is over all irreducible representations $\{\lambda\}_c$ of weight $c = a + b$. Of course these same coefficients $m_{\mu\nu,\lambda}$ also appear in the reduction formula appropriate to symmetric group representations i.e.,

(2.7) $$(\mu)_a \cdot (\nu)_b = \sum_{\lambda} m_{\mu\nu,\lambda}(\lambda)_c.$$

Thus the multiplication rules involved in Theorems 1 and 2 are identical and the validity of one of these theorems implies the validity of the other.

A very beautiful application of Theorems 1 and 2 is to the determination of the dimensions of the irreducible representations of $\mathbf{S}_s$ and $GL(n)$. From these theorems it is clear that

(2.8) $$f^{(\mathbf{a}*\mathbf{b})_s} = s! \left| \frac{f^{(a_i*b_j)}}{(a_i + b_j + 1)!} \right|,$$

and

(2.9) $$D_n\{\mathbf{a} * \mathbf{b}\}_s = |D_n\{a_i * b_j\}|,$$

since in general [**10**, p. 53] for an outer product:

(2.10) $$f^{(\mu)_a \cdot (\nu)_b} = \frac{(a + b)!}{a!b!} f^{(\mu)_a} f^{(\nu)_b},$$

and for a Kronecker product:

(2.11) $$D_n(\{\mu\}_a \cdot \{\nu\}_b) = D_n\{\mu\}_a D_n\{\nu\}_b.$$

The dimensions of the irreducible representations of $\mathbf{S}_s$ and $GL(n)$ corresponding to a given Young tableau are given by the number of distinct ways in which the numbers $1, 2, \ldots, s$ and $1, 2, \ldots, n$ may be inserted into the boxes of the tableau in a standard fashion. In the case of $\mathbf{S}_s$ the arrangement of numbers is standard if the numbers, all distinct, increase from left to right in each row and from top to bottom in each column [**4**, p. 199] whilst for $GL(n)$ the corresponding restriction is that the numbers, not necessarily all distinct, are non-decreasing from left to right in each row and are increasing from top to bottom in each column [**4**, p. 384]. It is then simply a combinatorial problem to show that for a tableau taking the form of a single hook:

(2.12) $$f^{(a_i*b_j)} = \frac{(a_i + b_j)!}{a_i!b_j!},$$

and

$$D_n\{a_i * b_j\} = \frac{(n + a_i)!}{(n - b_j - 1)!} \frac{1}{(a_i + b_j + 1)a_i!b_j!}. \tag{2.13}$$

Using (2.12) and (2.13) in (2.8) and (2.9) respectively and taking out factors common to every element of a row or a column of the determinants yields [**2**]

$$f^{(\mathbf{a}*\mathbf{b})_s} = s! \left|\frac{1}{h_{ij}}\right| \Big/ \prod_{k=1}^{r} (a_k!b_k!) \tag{2.14}$$

and

$$D_n\{\mathbf{a} * \mathbf{b}\}_s = \prod_{k=1}^{r} \frac{(n + a_k)!}{(n - b_k - 1)!} \frac{1}{a_k!b_k!} \left|\frac{1}{h_{ij}}\right| \tag{2.15}$$

where

$$h_{ij} = a_i + b_j + 1. \tag{2.16}$$

Clearly the determinant with elements $1/h_{ij}$ vanishes if either $a_i = a_j$ or $b_i = b_j$ for any $i \neq j$, for then two rows or two columns, respectively, of the determinant would be identical. Moreover, the common denominator of the expanded determinant must be $\prod_{i,j=1}^{r} h_{ij}$, whilst the numerator contains factors linear in a_i and b_j to a maximum total power of $(r^2 - r)$. The leading term in this numerator is given by $\prod_{i=1}^{r} (a_i^{r-i} b_i^{r-i})$ and this term can arise in one and only one way from the expansion of the determinant, in fact from the term $\prod_{i=1}^{r} (1/h_{ii})$. These considerations are sufficient to prove that:

$$\left|\frac{1}{h_{ij}}\right| = \prod_{i,j=1;\, i<j}^{r} (a_i - a_j)(b_i - b_j) \Big/ \prod_{i,j=1}^{r} (a_i + b_j + 1). \tag{2.17}$$

It is then not difficult to see that

$$\left|\frac{1}{h_{ij}}\right| \Big/ \prod_{k=1}^{r} a_k!b_k! = 1/H(\mathbf{a} * \mathbf{b})_s, \tag{2.18}$$

where $H(\mathbf{a} * \mathbf{b})_s$ is the product of the hook lengths [**1**; **10**, p. 44] associated with each box of the tableau corresponding to $\{\lambda\}_s$ and $\{\mathbf{a} * \mathbf{b}\}_s$. The (ij)th hook of this tableau consists of the box in the ith row and jth column of the tableau, together with the $(\lambda_i - j)$ boxes to the right and the $(\tilde{\lambda}_j - i)$ boxes beneath this box. (Note the misprints in [**10**, p. 44].) The length of the hook is then:

$$h_{ij} = (\lambda_i - j) + (\tilde{\lambda}_j - i) + 1. \tag{2.19}$$

This is consistent with the notation of (2.16) if both $i \leqq r$ and $j \leqq r$, but in other cases represents a generalization of (2.16). The denominator of (2.17) is the product of just r^2 hook lengths whilst $H(\mathbf{a} * \mathbf{b})_s$ is the product of the s hook lengths associated with the s boxes of the tableau, i.e.

$$H(\mathbf{a} * \mathbf{b})_s = \prod_{(i,j)} h_{ij} \tag{2.20}$$

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where, in this case, the product is taken over all pairs of values of i and j specifying the position of a box of the tableau. It is then a trivial matter to write down $H(\mathbf{a} * \mathbf{b})_s$ as a product of an array of s numbers. For example,

$$(2.21) \qquad H(310 * 420)_{13} = \begin{array}{|c|c|c|c|} \hline 8 & 6 & 4 & 1 \\ \hline 6 & 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$= 8 \cdot 6^2 \cdot 5 \cdot 4^2 \cdot 3^2 \cdot 2 \cdot 1^4 = 414,720.$$

The validity of (2.18) may be checked by using (2.17) in the left hand side of (2.18) to give in this case:

$$(2.22) \qquad \left|\frac{1}{h_{ij}}\right| \Big/ \prod_{k=1}^{r} a_k! b_k!$$

$$= 2 \cdot 3 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot / 8 \cdot 6 \cdot 4 \cdot 6 \cdot 4 \cdot 2 \cdot 5 \cdot 3 \cdot 1 \cdot 3!1!0!4!2!0!$$

$$= 1/8 \cdot 6^2 \cdot 5 \cdot 4^2 \cdot 3^2 \cdot 2$$

in agreement with (2.21).

It is unquestionably simpler to calculate the right hand side of (2.18) using (2.20), rather than the left hand side using (2.17). However, using (2.17) in (2.14) yields a result due to Frobenius [**2**], rederived as above by Littlewood [**7**, p. 112], whereas the use of (2.18) in (2.14) and (2.15) yields the formulae (1.1) and (1.2), i.e.

$$(2.23) \qquad f^{(\mathbf{a} * \mathbf{b})_s} = s!/H(\mathbf{a} * \mathbf{b})_s,$$

and

$$(2.24) \qquad D_n\{\mathbf{a} * \mathbf{b}\}_s = \prod_{k=1}^{r} \frac{(n + a_k)!}{(n - b_k - 1)!} \Big/ H(\mathbf{a} * \mathbf{b})_s.$$

These formulae were first established in essentially these forms by Frame, Robinson and Thrall [**1**], and by Robinson [**10**].

3. Frobenius symbols and the groups $O(n)$ and $Sp(n)$. In order to prove Theorems 3 and 4 it is necessary to consider in some detail the relationship between the Kronecker products of irreducible representations of $O(n)$ and $Sp(n)$ and the Kronecker products of irreducible representations of $GL(n)$ defined by (2.6). It has been shown [**6**] that

$$(3.1) \qquad [\mu]_a \times [\nu]_b = \sum_{\lambda} r_{\mu\nu,\lambda} [\lambda]_c$$

and

$$(3.2) \qquad \langle\mu\rangle_a \times \langle\nu\rangle_b = \sum_{\lambda} s_{\mu\nu,\lambda} \langle\lambda\rangle_c$$

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where the summations are over all irreducible representations $[\lambda]_c$ of $O(n)$ and $\langle\lambda\rangle_c$ of $Sp(n)$ of weight $c = a + b - 2t$, where t is an integer which can take all values from zero up to the minimum value of a and b. The coefficients appearing in (3.1) and (3.2) are identical, and are given by:

$$r_{\mu\nu,\lambda} = s_{\mu\nu,\lambda} = \sum_{\rho,\sigma,\tau} m_{\rho\sigma,\mu} m_{\rho\tau,\nu} m_{\sigma\tau,\lambda} \tag{3.3}$$

where $m_{\mu\nu,\lambda}$ is defined by (2.6) and the summation is carried out over all possible $\{\rho\}_t$, $\{\sigma\}_{a-t}$ and $\{\tau\}_{b-t}$.

It follows firstly, that any proof of Theorem 3 automatically implies the validity of Theorem 4, and secondly that:

$$[\mu]_a \times [\nu]_b = \sum_{\rho} [\mu/\rho]_{a-t} \cdot [\nu/\rho]_{b-t}, \tag{3.4}$$

where, as in (3.3), the summation is over all possible $\{\rho\}_t$, whilst the division symbol [**7**, p. 110] is such that

$$[\mu/\rho]_{a-t} = \sum_{\sigma} m_{\rho\sigma,\mu} [\sigma]_{a-t}. \tag{3.5}$$

The operation on the right hand side of (3.4) signified by the dot, $\cdot$, is associated with the factor $m_{\sigma\tau,\lambda}$ in (3.3), and corresponds to the reduction of a Kronecker product of irreducible representations of $GL(n)$. This operation, $\cdot$, is defined by the regular addition of boxes to a Young tableau, whereas the operation, /, is defined by the regular removal of boxes from a Young tableau, i.e., $\cdot$ and / denote mutually inverse operations.

The term in (3.4) corresponding to $\{\rho\}_t = \{0\}_0$ corresponds to the mutual symmetrisation of the tensor indices associated with $[\mu]_a$ and $[\nu]_b$, whilst all the other terms correspond to the mutual symmetrisation of tensors obtained from $[\mu]_a$ and $[\nu]_b$ after contractions have been carried out to produce a traceless tensor [**6**]. This provides the clue to the derivation of Theorems 3 and 4, for if it can be proved that in the expansion of the determinants

$$|[a_i * b_j]|^\times \quad \text{and} \quad |\langle a_i * b_j\rangle|^\times$$

the sum of all terms involving contractions is zero, then these theorems follow immediately from Theorem 2.

All the elements of the first of these determinants are representations of the form $[a * b]$ where the use of ordinary rather than bold-face type and the absence of the subscript s indicates that this symbol is a rank 1 Frobenius symbol rather than a shorthand notation for a general rank r Frobenius symbol. It is necessary to consider the possible divisors of $[a * b]$. Such a divisor must necessarily be a representation $[c * d]$ corresponding to a tableau consisting of a single hook, whilst the quotient $[a * b/c * d]$ must itself consist, when fully reduced, of representations corresponding to tableaux consisting of single hooks. In general it is found that

$$(3.6)\quad [a * b/c * d] = \begin{cases} [a - c * b - d - 1] + [a - c - 1 * b - d], & \\ & \text{if } a > c \text{ and } b > d \\ [0 * b - d - 1], & \text{if } a = c \text{ and } b > d \\ [a - c - 1 * 0], & \text{if } a > c \text{ and } b = d \\ \quad 1, & \text{if } a = c \text{ and } b = d \\ \quad 0, & \text{if } a < c \text{ or } b < d. \end{cases}$$

This result is exactly equivalent to the statement:

$$(3.7)\qquad [a * b/c * d] = \begin{cases} [a - c] \cdot [1^{b-d}], & \text{if } a \geqq c \text{ and } b \geqq d \\ \quad 0, & \text{if } a < c \text{ or } b < d \end{cases}$$

where of course in Frobenius notation $[a - c] = [a - c - 1 * 0]$ and $[1^{b-d}] = [0 * b - d - 1]$.

The result (3.6) may be generalized to include successive divisions. Thus

$$(3.8)\quad [a * b/c * d/e * f] = [a - c * b - d - 1/e * f] + [a - c - 1 * b - d/e * f] = [a - c - e * b - d - f - 2] + 2[a - c - e - 1 * b - d - f - 1] + [a - c - e - 2 * b - d - f].$$

Clearly

$$(3.9)\quad [a * b/c * d/e * f] = [a * b/e * f/c * d] = [a * b/c + e * d + f + 1] + [a * b/c + e + 1 * d + f].$$

Further generalization gives:

$$(3.10)\quad [a * b/c_1 * d_1/c_2 * d_2/ \ldots /c_m * d_m] = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} [a - c - k * b - d - m + k] = \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} [a * b/c + k * d + m - k - 1],$$

where $c = c_1 + c_2 + \ldots + c_m$ and $d = d_1 + d_2 + \ldots + d_m$.

Now it is convenient to return to the statement of Theorem 3 which will be proved by induction on the variable r, which gives the rank of the Frobenius symbol.

In the case $r = 1$ the theorem is just an identity involving no Kronecker products. In the case $r = 2$ it is necessary to evaluate the determinant:

$$\begin{vmatrix} [a_1 * b_1] & [a_2 * b_2] \\ [a_2 * b_1] & [a_2 * b_2] \end{vmatrix}^{\times}.$$

Clearly this determinant is given by

$$(3.11)\quad [a_1 * b_1] \times [a_2 * b_2] - [a_1 * b_2] \times [a_2 * b_1] = [a_1 * b_1] \cdot [a_2 * b_2] - [a_1 * b_2] \cdot [a_2 * b_1] + \sum_{c,d} \{[a_1 * b_1/c * d] \cdot [a_2 * b_2/c * d] - [a_1 * b_2/c * d] \cdot [a_2 * b_1/c * d]\}.$$

Using (3.7)

$$(3.12)\quad [a_1 * b_1/c * d]\cdot[a_2 * b_2/c * d] - [a_1 * b_2/c * d]\cdot[a_2 * b_1/c * d]$$
$$= [a_1 - c]\cdot[1^{b_1-d}]\cdot[a_2 - c]\cdot[1^{b_2-d}] - [a_1 - c]\cdot[1^{b_2-d}]\cdot[a_2 - c]\cdot[1^{b_1-d}]$$
$$= 0.$$

Thus every term in the double summation of (3.11) vanishes identically so that:

$$(3.13)\quad \begin{vmatrix} [a_1 * b_1] & [a_1 * b_2] \\ [a_2 * b_1] & [a_2 * b_2] \end{vmatrix}^{\times} = \begin{vmatrix} [a_1 * b_1] & [a_1 * b_2] \\ [a_2 * b_1] & [a_2 * b_2] \end{vmatrix}^{\cdot} = [a_1a_2 * b_1b_2],$$

where the last step involves the application of Theorem 1 or 2. This proves the validity of Theorem 3 in the rank 2 case.

At this stage it is convenient to assume that this result generalises to the rank $(m-1)$ case, i.e.

$$(3.14)\quad |[a_i * b_j]|^{\times} = |[a_i * b_j]|^{\cdot} = [a_1a_2\ldots a_{m-1} * b_1b_2\ldots b_{m-1}],$$

where once again the second equality is true by virtue of Theorem 1 or 2, whilst the complete equation corresponds to the rank (m − 1) statement of Theorem 3.

Expanding the corresponding determinant with respect to the elements in the pth row gives in the rank m case:

$$(3.15)\quad |[a_i * b_j]|^{\times} = \sum_{q=1}^{m} (-1)^{p+q}[a_p * b_q] \times [a_1\ldots\hat{a}_p\ldots a_m * b_1\ldots\hat{b}_q\ldots b_m]$$

where the notation is such that the Frobenius symbol on the right hand side is obtained from the minor of the pqth element, $[a_p * b_q]$, of the determinant by the use of (3.14). The product appearing in (3.15) is such that

$$(3.16)\quad |[a_i * b_j]|^{\times} = \sum_{q=1}^{m} (-1)^{p+q}[a_p * b_q]\cdot[a_1\ldots\hat{a}_p\ldots a_m * b_1\ldots\hat{b}_q\ldots b_m]$$
$$+\sum_{c,d}\sum_{q=1}^{m} (-1)^{p+q}[a_p * b_q/c * d]\cdot[a_1\ldots\hat{a}_p\ldots a_m * b_1\ldots\hat{b}_q\ldots b_m/c * d].$$

The first term involves no contractions whilst the second term corresponds to the sum of all possible contractions between the single hook representations, $[a_p * b_q]$, and those corresponding to Frobenius symbols of rank $m-1$. To enumerate these contractions in more detail it is helpful to use (3.14) to write

$$(3.17)\quad [a_1a_2\ldots\hat{a}_p\ldots a_m * b_1b_2\ldots\hat{b}_q\ldots b_m] = |[a_i * b_j]|^{\cdot}$$
$$= \sum_{\pi(k)} \epsilon_{k_1k_2\ldots\hat{k}_p\ldots k_m}[a_1 * b_{k_1}]\cdot[a_2 * b_{k_2}]\cdot\ldots\cdot[a_m * b_{k_m}]$$

where the summation is taken over all possible permutations $(k_1, k_2\ldots\hat{k}_p\ldots k_m)$ of the numbers $(1, 2, \ldots \hat{q}, \ldots m)$.

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With this notation it is clear that the divisions of $[a_p * b_q]$ by $[c * d]$ arises through multiple contractions of $[a_p * b_q]$ with the set of representations $[a_i * b_{k_i}]$. For each representation $[c * d]$ all possible sets of such contractions need to be considered. These contractions may be classified as being single, double, triple . . . contractions involving the factors:

$$(3.18)\quad \begin{aligned} &[a_p * b_q/c_s * d_s] \cdot [a_s * b_{k_s}/c_s * d_s] \\ &[a_p * b_q/c_s * d_s/c_t * d_t] \cdot [a_s * b_{k_s}/c_s * d_s] \cdot [a_t * b_{k_t}/c_t * d_t] \\ &[a_p * b_q/c_s * d_s/c_t * d_t/c_u * d_u] \cdot [a_s * b_{k_s}/c_s * d_s] \cdot [a_t * b_{k_t}/c_t * d_t] \\ &\qquad\qquad \cdot [a_u * b_{k_u}/c_u * d_u] \\ &\ldots \end{aligned}$$

respectively. In the notation of (3.16) it follows from (3.10) that

$$(3.19)\quad \begin{aligned} &c = c_s && d = d_s && \\ &c = c_s + c_t + k && d = d_s + d_t + 1 - k && k = 0, 1 \\ &c = c_s + c_t + c_u + k && d = d_s + d_t + d_u + 2 - k && k = 0, 1, 2 \\ &\ldots \end{aligned}$$

It should be stressed that for each given pair of values for c and d in (3.16) a summation will extend over all possible values of $c_s, c_t, c_u, \ldots d_s, d_t, d_u, \ldots, k$, and of course over all possible combinations of $s, t, u, \ldots$. Fortunately the restrictions (3.19) lead to a very great simplification. This can be seen most easily by observing that:

$$(3.20)\quad [a_s * b_{k_s}/c_s * d_s] \cdot [a_t * b_{k_t}/c_t * d_t] = [a_s - c_s] \cdot [1^{b_{k_s} - d_s}] \cdot [a_t - c_t] \cdot [1^{b_{k_t} - d_t}],$$

where use has been made of (3.7). Clearly, therefore, all contractions other than single contractions give rise to contributions whose explicit dependence on b_{k_s} and b_{k_t} is given by

$$(3.21)\quad [a_p * b_q/c_s + c_t + \ldots * d_s + d_t + \ldots] \cdot [1^{b_{k_s} - d_s}] \cdot [1^{b_{k_t} - d_t}].$$

If $d_s = d_t$ this expression is symmetric under the interchange of b_{k_s} and b_{k_t}. The antisymmetry under the interchange of k_s and k_t implied by (3.17) is then sufficient to prove that all terms containing the factor (3.20) give no contribution to (3.16). Furthermore if $d_s \neq d_t$ the fact that all possible contractions have to be included in the summation implies that for every term with a factor (3.21) there exists another, otherwise identical, containing the factor

$$(3.22)\quad [a_p * b_q/c_s + c_t + \ldots * d_t + d_s + \ldots] \cdot [1^{b_{k_s} - d_t}] \cdot [1^{b_{k_t} - d_s}].$$

The sum of (3.21) and (3.22) gives a factor which is symmetric under the interchange of b_{k_s} and b_{k_t}, and which therefore gives no contribution to (3.16). This then rules out any contribution from the sums of all possible double, triple, . . . contractions.

Slightly adapting the notation of (3.18) for the single contractions, and expanding the determinant of (3.17) with respect to the row containing the terms in a_s gives for these single contractions:

$$(3.23)\quad \sum_{q=1}^{m} (-1)^{p+q} \sum_{t=1,(t\neq q)}^{m} \eta_{st}^{pq} \sum_{c,d} \{[a_p * b_q/c * d] \cdot [a_s * b_t/c * d] \cdot [a_1 \ldots \hat{a}_p \ldots \hat{a}_s \ldots a_m * b_1 \ldots \hat{b}_q \ldots \hat{b}_t \ldots b_m]\},$$

where η_{st}^{pq} is a phase factor dependent upon the relative values of p and s, q and t as well as the explicit values of s and t. In fact

$$(3.24)\quad \eta_{st}^{pq} = \begin{cases} (-1)^{s+t}, & \text{if } p \gtrless s \text{ and } q \gtrless t, \\ (-1)^{s+t-1}, & \text{if } p \gtrless s \text{ and } q \lessgtr t. \end{cases}$$

The overall phase factor in (3.23) is therefore

$$(3.25)\quad (-1)^{p+q}\eta_{st}^{pq} = \begin{cases} (-1)^{p+q+s+t}, & \text{if } p \gtrless s \text{ and } q \gtrless t, \\ (-1)^{p+q+s+t-1}, & \text{if } p \gtrless s \text{ and } q \lessgtr t. \end{cases}$$

This factor is clearly antisymmetric under the interchange of q and t. However the dependence of the remainder of the expression (3.23) on q and t is just

$$(3.26)\quad [1^{b_q-d}] \cdot [1^{b_t-d}],$$

where use has again been made of (3.7). This term is symmetric under the interchange of q and t, so that the double summation of (3.23) gives zero. Thus the sum of all possible single contractions gives no contribution to (3.16).

This same conclusion may also be reached without examining the phase factors (3.24) in such detail, by reordering the rows of the determinant of (3.15) so that the p and sth rows become the first and second respectively, and then making a Laplace expansion into a sum of products of 2×2 and $(m-2) \times (m-2)$ determinants. The result (3.13) then ensures that the single contractions all vanish.

Whichever argument is used it follows that the sum of all the possible contractions in (3.16) vanishes identically, leaving

$$(3.27)\quad |[a_i * b_j]|^{\times} = \sum_{q=1}^{m} (-1)^{p+q}[a_p * b_q] \cdot [a_1 \ldots \hat{a}_p \ldots a_m * b_1 \ldots \hat{b}_q \ldots b_m]$$
$$= |[a_i * b_j]|^{\cdot} = [a_1a_2 \ldots a_m * b_1b_2 \ldots b_m]$$

where the last step involves the use of Theorem 2. This result is just the statement of Theorem 3 appropriate to the case of a rank m Frobenius symbol.

To summarize: the validity of Theorem 3 has been derived in the case $r = m$ from the assumption of its validity for $r = m - 1$. Moreover the theorem has been proven in the case $r = 2$, so that Theorem 3 is valid for all values of r, the rank of the Frobenius symbol.

To illustrate the cancellation of the contracted terms it is useful to consider the following rank 3 example associated with [310 * 420]. The corresponding determinant is:

$$(3.28)\qquad |[a_i * b_j]|^{\times} = \begin{vmatrix} [3*4] & [3*2] & [3*0] \\ [1*4] & [1*2] & [1*0] \\ [0*4] & [0*2] & [0*0] \end{vmatrix}^{\times}.$$

Expanding this determinant with respect to the third row, i.e. taking $p = 3$ in (3.15), gives

$$(3.29)\qquad |[a_i * b_j]|^{\times} = [0*4] \times [31*20] - [0*2] \times [31*40] + [0*0] \times [31*42]$$

where, for example, as in (3.13):

$$[31*20] = [3*2]\cdot[1*0] - [3*0]\cdot[1*2],$$

since all contractions of the form:

$$[3*2/c*d]\cdot[1*0/c*d] - [3*0/c*d]\cdot[1*2/c*d] = [3-c]\cdot[1-c]\cdot\{[1^{2-d}]\cdot[1^{0-d}] - [1^{0-d}]\cdot[1^{2-d}]\}$$

vanish as in (3.12).

A typical double contraction contribution to (3.29) is then given by:

$$(3.30)\qquad \sum_{c_1,c_2,d_1,d_2} [0*4/c_1*d_1/c_2*d_2]\cdot\{[3*2/c_1*d_1]\cdot[1*0/c_2*d_2] - [3*0/c_1*d_1]\cdot[1*2/c_2*d_2]\},$$

so that in the notation of (3.16)

$$[0*4/c*d] = [0*4/c_1*d_1/c_2*d_2].$$

Such a term vanishes unless $c \leqq 0$ and $d \leqq 4$ by virtue of (3.7), i.e. unless $c_1 + c_2 + k \leqq 0$ and $d_1 + d_2 + 1 - k \leqq 4$ in the notation of (3.10). The only non-vanishing contributions then arise when $k = 0$, $c_1 = c_2 = 0$ and $(d_1, d_2) = (3, 0), (2, 1), (1, 2), (0, 3), (2, 0), (1, 1), (0, 2), (1, 0), (0, 1)$ and $(0, 0)$. As emphasized before these values are such that either $d_1 = d_2$ or both (d_1, d_2) and (d_2, d_1) are included in the list. That such terms will mutually cancel can be seen in this example by taking, for instance, $(d_1, d_2) = (2, 0)$ and $(0, 2)$ and adding to give

$$\begin{aligned}
&[0*4/0*2/0*0]\cdot\{[3*2/0*2]\cdot[1*0/0*0] - [3*0/0*2]\cdot[1*2/0*0]\} \\
&+ [0*4/0*0/0*2]\cdot\{[3*2/0*0]\cdot[1*0/0*2] - [3*0/0*0]\cdot[1*2/0*2]\} \\
&= [0*4/0*2/0*0]\cdot[3-0]\cdot[1-0]\cdot\{[1^{2-2}]\cdot[1^{0-0}] - [1^{0-2}]\cdot[1^{2-0}] \\
&\quad + [1^{2-0}]\cdot[1^{0-2}] - [1^{0-0}]\cdot[1^{2-2}]\} \\
&= 0.
\end{aligned}$$

In the same way it is clear that all the double contractions (3.30) sum to give zero.

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Finally in the notation of (3.23), taking $p = 3$ and $s = 2$ in a typical example involving a single contraction gives:

$$[0 * 4/c * d] \cdot [1 * 0/c * d] \cdot [3 * 2] - [0 * 4/c * d] \cdot [1 * 2/c * d] \cdot [3 * 0]$$
$$- [0 * 2/c * d] \cdot [1 * 0/c * d] \cdot [3 * 4] + [0 * 2/c * d] \cdot [1 * 4/c * d] \cdot [3 * 0]$$
$$+ [0 * 0/c * d] \cdot [1 * 2/c * d] \cdot [3 * 4] - [0 * 0/c * d] \cdot [1 * 4/c * d] \cdot [3 * 2].$$

Taking the first and last terms together since these have a common factor yields

$$\{[0 * 4/c * d] \cdot [1 * 0/c * d] - [0 * 0/c * d] \cdot [1 * 4/c * d]\} \cdot [3 * 2]$$
$$= [3 * 2] \cdot [0 - c] \cdot [1 - c] \cdot \{[1^{4-d}] \cdot [1^{0-d}] - [1^{0-d}] \cdot [1^{4-d}]\}$$
$$= 0.$$

Similarly for the other pairs of terms, so that all the single contractions of (3.29) sum to give zero.

Thus in (3.29) the symbol $\times$ may be replaced by . , so that using Theorem 2

$$|[a_i * b_j]|^{\times} = [310 * 420],$$

as required.

4. Dimensions of irreducible representations of $O(n)$ and $Sp(n)$. From Theorem 3 it follows that:

$$D_n[\mathbf{a} * \mathbf{b}]_s = |D_n[a_i * b_j]|. \tag{4.1}$$

Application of a known formula [**5**] for the dimensions of an irreducible representation of $O(n)$ to the special case of a representation specified by a tableau consisting of a single hook yields:

$$D_n[a_i * b_j] = \frac{(n + a_i - 1)!}{(n - b_j - 2)!} \frac{(n + 2a_i)}{(n + a_i - b_j - 1)} \frac{1}{(a_i + b_j + 1)a_i!b_j!}, \tag{4.2}$$

which may be written in the form

$$D_n[a_i * b_j] = \frac{(n + a_i - 1)!}{(n - b_j - 2)!} \frac{1}{a_i!b_j!} \left(\frac{1}{h_{ij}} + \frac{1}{n_{ij}}\right), \tag{4.3}$$

where

$$h_{ij} = a_i + b_j + 1,$$

and

$$n_{ij} = n + a_i - b_j - 1.$$

Hence taking out the factors common to every element in each row and each column of the determinant (4.1) gives the formula

$$D_n[\mathbf{a} * \mathbf{b}]_s = \prod_{k=1}^{r} \left\{\frac{(n + a_k - 1)!}{(n - b_k - 2)!} \frac{1}{a_k!b_k!}\right\} \left|\frac{1}{h_{ij}} + \frac{1}{n_{ij}}\right|. \tag{4.4}$$

The determinant appearing in this expression may be evaluated by noting that either a row or column will consist entirely of zeros, or two rows or two columns of the determinant will be proportional, and the determinant will therefore vanish, if any one or more of the following conditions is satisfied:

(i) $h_{ij} = -n_{ij}$ for all j,
(ii) $h_{ij} = -n_{ij}$ for all i,
(iii) $h_{ij} = h_{ik}$ and $n_{ij} = n_{ik}$ for all i with $j \neq k$,
(iv) $h_{ij} = n_{ik}$ and $n_{ij} = h_{ik}$ for all i with $j \neq k$,
(v) $h_{ij} = h_{kj}$ and $n_{ij} = n_{kj}$ for all j with $i \neq k$,
(vi) $h_{ij} = -n_{kj}$ and $n_{ij} = -h_{kj}$ for all j with $i \neq k$.

In terms of n, a_i and b_j these conditions correspond to:

(i) $(n + 2a_i) = 0$ for all j,
(ii) $(n + 2a_i) = 0$ for all i,
(iii) $(b_j - b_k) = 0$ for all i with $j \neq k$,
(iv) $(n - b_j - b_k - 2) = 0$ for all i with $j \neq k$,
(v) $(a_i - a_k) = 0$ for all j with $i \neq k$,
(vi) $(n + a_i + a_k) = 0$ for all j with $i \neq k$.

In addition

$$\frac{1}{h_{ij}} + \frac{1}{n_{ij}} = \frac{(n + 2a_i)}{h_{ij}n_{ij}}, \tag{4.5}$$

so that

$$\left|\frac{1}{h_{ij}} + \frac{1}{n_{ij}}\right| = K \prod_{i,j=1}^{r} \frac{1}{h_{ij}n_{ij}} \prod_{i=1}^{r} (n + 2a_i) \cdot \prod_{i,j=1;\, i<j}^{r} (a_i - a_j)(b_i - b_j)(n + a_i + a_j)(n - b_i - b_j - 2). \tag{4.6}$$

It is easy to see that K is independent of n since the numerator of the expression on the right hand side of (4.6) is then a polynomial in n of degree r^2, as is required by the facts that the n-dependence of each element (4.5) of the determinant is just $(n + 2a_i)/n_{ij}$, and that each quantity n_{ij} appears in the denominator on the right hand side of (4.6).

Furthermore, taking the limit as n becomes large in (4.6) gives

$$\lim_{n\to\infty} \left|\frac{1}{h_{ij}} + \frac{1}{n_{ij}}\right| = \left|\frac{1}{h_{ij}}\right| = K \prod_{i,j=1}^{r} \frac{1}{h_{ij}} \prod_{i,j=1;\, i<j}^{r} (a_i - a_j)(b_i - b_j). \tag{4.7}$$

Comparison with (2.17) then gives

$$K = 1. \tag{4.8}$$

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Using this result, (4.8), and (4.6) in (4.4) together with (2.18) gives the formula

$$(4.9)\quad D_n[\mathbf{a} * \mathbf{b}]_s = \prod_{i=1}^{r} \left\{ \frac{(n + a_i - 1)!}{(n - b_i - 2)!} \frac{(n + 2a_i)}{(n + a_i - b_i - 1)} \cdot \prod_{j=i+1}^{r} \frac{(n + a_i + a_j)(n - b_i - b_j - 2)}{(n - b_i + a_j - 1)(n + a_i - b_j - 1)} \right\} \Big/ H(\mathbf{a} * \mathbf{b})_s.$$

It is possible to calculate $D_n\langle \mathbf{a} * \mathbf{b}\rangle_s$ in the same way from Theorem 4. However it is more convenient to use the conjugacy relationship [**5**] between $O(n)$ and $Sp(n)$ which implies that

$$(4.10)\qquad D_n\langle \mathbf{a} * \mathbf{b}\rangle_s = (-1)^s D_{-n}[\mathbf{b} * \mathbf{a}]_s.$$

Using this relationship and the identities:

$$\frac{(n + a_i - 1)!}{(n - b_i - 2)!} = \prod_{x=1}^{h_{ii}} (n + a_i - x),$$

$$\frac{(n + a_i + 1)!}{(n - b_i)!} = \prod_{x=1}^{h_{ii}} (n - b_i + x),$$

and

$$(-1)^{\Sigma_{i=1}^{r} h_{ii}} = (-1)^s$$

in (4.9) yields the formula:

$$(4.11)\quad D_n\langle \mathbf{a} * \mathbf{b}\rangle_s = \prod_{i=1}^{r} \left\{ \frac{(n + a_i + 1)!}{(n - b_i)!} \frac{(n - 2b_i)}{(n + a_i - b_i + 1)} \cdot \prod_{j=i+1}^{r} \frac{(n - b_i - b_j)(n + a_i + a_j + 2)}{(n + a_i - b_j + 1)(n - b_i + a_j + 1)} \right\} \Big/ H(\mathbf{a} * \mathbf{b})_s,$$

since

$$(4.12)\qquad H(\mathbf{a} * \mathbf{b})_s = H(\mathbf{b} * \mathbf{a})_s.$$

It was claimed in the introduction that formulae would be obtained of the type (1.3) and (1.4) in which $N_n[\mathbf{a} * \mathbf{b}]_s$ and $N_n\langle \mathbf{a} * \mathbf{b}\rangle_s$ were factored polynomials in n of degree s. The formulae (4.9) and (4.11) are not manifestly of this type. However it is clear that each of the denominator factors appearing within the curly brackets of (4.9) and (4.11), also appears in the series of terms $(n + a_i - 1)!/(n - b_i - 2)!$ and $(n + a_i + 1)!/(n - b_i)!$, respectively. Thus unlike the formulae obtained previously [**5**] these formulae (4.9) and (4.11) are such that the pattern of cancellations which lead to the required result is quite clear. To be precise:

$$(4.13)\quad D_n[\mathbf{a} * \mathbf{b}]_s = \prod_{i=1}^{r} \{(n + a_i - 1)(n + a_i - 2)$$
$$\ldots (n + a_i - b_r - 1 - h_{ii}) \ldots (n + a_i - b_{r-1} - 1 - h_{ii})$$
$$\ldots (n + a_i - b_{i+1} - 1 - h_{ii}) \ldots (n + a_i - b_i - 1 + h_{ii})$$
$$\ldots (n + a_{i+1} - b_i - 1 + h_{ii}) \ldots (n + a_{r-1} - b_i - 1 + h_{ii})$$
$$\ldots (n + a_r - b_i - 1 + h_{ii}) \ldots (n - b_i)(n - b_i - 1)\}/H(\mathbf{a} * \mathbf{b})_s,$$

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and

$$(4.14)\quad D_n\langle \mathbf{a} * \mathbf{b}\rangle_s = \prod_{i=1}^{r} \{(n + a_i + 1)(n + a_i) \ldots (n + a_i - b_r + 1 - h_{ii}) \ldots (n + a_i - b_{r-1} + 1 - h_{ii}) \ldots (n + a_i - b_{i+1} + 1 - h_{ii}) \ldots (n + a_i - b_i + 1 - h_{ii}) \ldots (n + a_{i+1} - b_i + 1 + h_{ii}) \ldots (n + a_{r-1} - b_i + 1 + h_{ii}) \ldots (n + a_r - b_i + 1 + h_{ii}) \ldots (n - b_i + 2)(n - b_i + 1)\}/H(\mathbf{a} * \mathbf{b})_s.$$

For each value of i the series in these formulae (4.13) and (4.14) contain exactly h_{ii} factors, which, apart from the addition and subtraction of the number h_{ii}, are consecutive integers. The additions occur in each term a distance $(a_j + 1)$ from the end of the series, and the subtractions in each term a distance $(b_j + 1)$ from the beginning of the series. If the series are presented as arrays in the shape of the ith hook of the appropriate Young tableau, with an arm length of a_i and a leg length of b_i, then $r - i$ additions of h_{ii} appear in the arm and $r - i$ subtractions of h_{ii} in the leg of the hook, whilst an extra addition or subtraction occurs in the diagonal element of the hook according as the group in question is $O(n)$ or $Sp(n)$ (see Figure 1). It is understood that in these arrays the numbers, apart from the additions and subtractions, decrease in steps of one up the leg and along the arm from left to right.

By way of illustration, the application of (4.13) to the case $[\mathbf{a} * \mathbf{b}]_s = [310 * 420]$ yields:

$$(4.15)\quad D_n[310 * 420] = \frac{n-2+8}{8} \cdot \frac{n-3}{6} \cdot \frac{n-4+8}{4} \cdot \frac{n-5+8}{1} \cdot \frac{n-1}{6} \cdot \frac{n-2+4}{4} \cdot \frac{n-3+4}{2} \cdot \frac{n+0-8}{5} \cdot \frac{n-1}{3} \cdot \frac{n-1+1}{1} \cdot \frac{n+1}{3} \cdot \frac{n+0-4}{1} \cdot \frac{n+2-8}{1}$$

where the crucial role played by the diagonal element hook lengths 8, 4 and 1 is made clear. Thus

$$(4.16)\quad D_n[310 * 420] = (n + 6)(n + 4)(n + 3)(n + 2)(n + 1)^2 n(n - 1)^2 \cdot (n - 3)(n - 4)(n - 6)(n - 8)/414720.$$

Then the use of (4.10) gives:

$$(4.17)\quad D_n\langle 420 * 310\rangle = (n + 8)(n + 6)(n + 4)(n + 3)(n + 1)^2 n(n - 1)^2 \cdot (n - 2)(n - 3)(n - 4)(n - 6)/414720.$$

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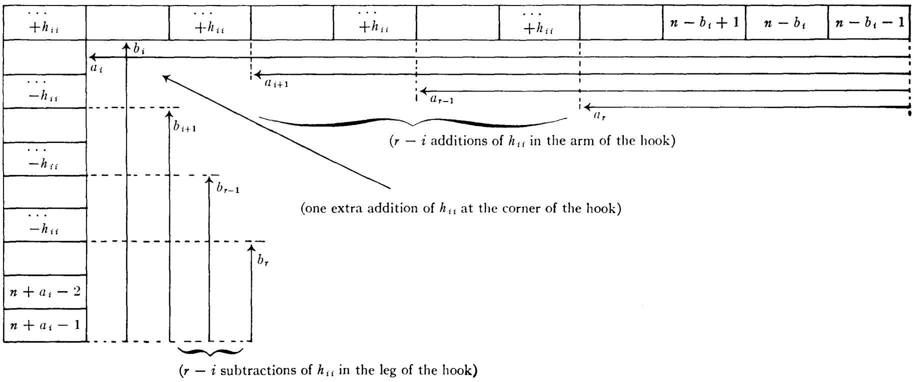

FIGURE 1 (a). Orthogonal Group.

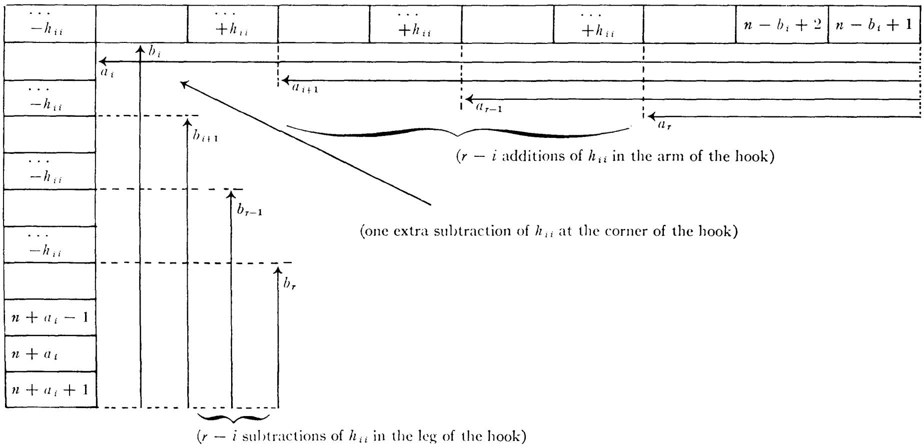

FIGURE 1 (b). Symplectic Group.

FIGURE 1. The arrays of numbers associated with (a) $N_n^i[\mathbf{a} * \mathbf{b}]_s$ and (b) $N_n^i\langle \mathbf{a} * \mathbf{b}\rangle_s$, where

$$D_n[\mathbf{a} * \mathbf{b}]_s = \prod_{i=1}^{r} N_n^i[\mathbf{a} * \mathbf{b}]_s / H(\mathbf{a} * \mathbf{b})_s$$

and

$$D_n\langle \mathbf{a} * \mathbf{b}\rangle_s = \prod_{i=1}^{r} N_n^i\langle \mathbf{a} * \mathbf{b}\rangle_s / H(\mathbf{a} * \mathbf{b})_s,$$

illustrating the scheme for additions and subtractions of h_{ii} to the h_{ii} consecutive integers inserted in the boxes of the ith hook of the Young tableau specified by $(\mathbf{a} * \mathbf{b})_s$ for (a) the orthogonal group and (b) the symplectic group. The h_{ii} are added and subtracted to particular members of a set of h_{ii} consecutive integers which decrease by unity from box to box up the leg and along the arm from left to right. The symplectic case, (b), differs from the orthogonal case, (a), only in the replacement of -1 by $+1$ at the starting position and in the replacement of $+h_{ii}$ by $-h_{ii}$ at the corner of each hook.

This same result can of course be obtained from (4.14) which yields an array such that:

$$(4.18)\quad D_n\langle 420 * 310\rangle = \frac{n+2-8}{8}\cdot\frac{n+1}{6}\cdot\frac{n+0+8}{5}\cdot\frac{n-1}{3}\cdot\frac{n-2+8}{1}$$

$$\cdot\frac{n+3}{6}\cdot\frac{n+2-4}{4}\cdot\frac{n+1}{3}\cdot\frac{n+0+4}{1}$$

$$\cdot\frac{n+4-8}{4}\cdot\frac{n+3-4}{2}\cdot\frac{n+1-1}{1}$$

$$\cdot\frac{n+5-8}{1}$$

in agreement with (4.17).

It is worth pointing out that just as the dimensionality formula (4.2) appropriate to a single hook representation of $O(n)$ can be derived directly from a previously known formula, so also can the general formula (4.9) or (4.13) be derived from a known expression. This can be done directly without the use of Theorem 3, and this fact serves to check the validity of this theorem.

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