CORRELATION FUNCTIONS FOR THE GRAVITATIONAL FORCE

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Abstract. Spatial and time correlations of the force acting on a star are derived for finite gravitational systems. It is shown that their behavior is qualitatively different than that for infinite mediums.

The dissolution time for a binary system is considered. We explain why Chandrasekhar's dissolution time differs from that given by Ambartsumian and Oort in that it does not depend on the velocities of the field stars. We show that the difference lies in the definition of what constitutes 'relative change in velocity' of the two stars in the binary. Indeed, using the general approach of Chandrasekhar and von Neumann (appropriately modified) we derive a velocity dependent dissolution time.

1. Introduction

The total force on a star may be expressed as the sum of two forces which have different behavior. One force, due to the smoothed out distribution of the system as a whole, changes very slowly and can be expressed as the gradient of the smoothed out potential. The random part of the force is due to the rapid change of the positions of the nearby stars and its description must be stochastic. Chandrasekhar and Von Neumann (Chandrasekhar, 1941, 1944a, b, c; Chandrasekhar and Von Neumann, 1942, 1943) have developed a statistical theory of the random force under certain simplifying assumptions. They assumed that the field stars are not correlated with themselves or the test star, and that the field stars in the neighborhood of the test star are uniformly distributed. Furthermore, the number of field stars were taken to be infinite (i.e., an infinite medium) in such a manner as to keep the density constant.

Numerical experiments performed to test various aspects of the theory (Ahmad and Cohen, 1972, 1973, 1974) have shown excellent agreement with theory for the distribution of random force (Holtsmark distribution), the time rate of change of the random force and dynamical friction. The experiments to verify the two time autocorrelation function showed that the experimental curve decreased faster for large times than that predicted by theory.

Chandrasekhar's result for the time autocorrelation for force is that the decrease with time, for large times, is extremely slow, namely as $1/t$. Similarly, for large separation distance, the force correlation acting at two different points at the same time was shown to decrease only as its inverse.

We shall calculate below space and time autocorrelation functions for force for bounded gravitational systems and show that its behavior is qualitatively different from that for infinite systems. Also, we shall calculate the correlations of the forces acting at two different points at two different times and discuss its application to the problem of the stability of binaries.

* Work supported by a grant from The City University Faculty Research Award Program.

Hayli (ed.), Dynamics of Stellar Systems, 33–45.

2. Force Correlations

We consider a finite spherical system of radius λ, density n, and assume that the field stars move in linear orbits. To keep the density constant, we shall introduce stars into the system at the same rate they are leaving it. To accomplish this, we distribute stars over all space with a density n, and with a velocity probability distribution, $\tau(\mathbf{v})$, as the stars in the real system. It is clear that as long as the velocity distribution is not a function of position, the density within λ will be constant. A field star will be 'counted' in any averaging only if at that time it is inside the sphere λ. If two times are involved, then it must be within the sphere at both times. That is, at time t a field star whose original position and velocity is $\mathbf{r}$, $\mathbf{v}$, must satisfy

$$|\mathbf{r}_i + \mathbf{v}_i t| < \lambda. \tag{1}$$

Also, we shall impose a lower limit, ε_i, restricting the distance with which a field star can approach the test star which is at position $\mathbf{r}_1$, say;

$$\varepsilon_i < |\mathbf{r}_i + \mathbf{v}_i t - \mathbf{r}_1|. \tag{2}$$

The lower limit ε_i, is a function of the relative velocity of the field and test star and is to be obtained by requiring the relative energy to be positive. Otherwise it would be a binary.

In the following we shall take all the masses to be equal and consider the case where the test stars are stationary. Further we shall not take the ε_i's as function of the relative velocities but take them to be constants and estimate them by

$$\varepsilon_i \equiv \varepsilon \sim \frac{2Gm}{\langle v^2 \rangle}. \tag{3}$$

The force correlation function at two points $\mathbf{r}_1$, $\mathbf{r}_2$, at two different times, t_1, t_2 is then

$$\begin{aligned} \langle \mathbf{F}(\mathbf{r}_1, t_1) \cdot \mathbf{F}(\mathbf{r}_2, t_2) \rangle = {} & G^2 m^2 \sum_{\substack{ij \\ i \neq j}} \left\langle \frac{\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1}{|\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1|^3} \cdot \frac{\mathbf{r}_j + \mathbf{v}_j t_2 - \mathbf{r}_2}{|\mathbf{r}_j + \mathbf{v}_j t_2 - \mathbf{r}_2|^3} \right\rangle + \\ & + G^2 m^2 \sum_i \left\langle \frac{\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1}{|\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1|^3} \cdot \frac{\mathbf{r}_i + \mathbf{v}_i t_2 - \mathbf{r}_2}{|\mathbf{r}_i + \mathbf{v}_i t_2 - \mathbf{r}_2|^3} \right\rangle ; \\ & |\mathbf{r}_i + \mathbf{v}_i t_1| < \lambda \\ & |\mathbf{r}_i + \mathbf{v}_i t_2| < \lambda \\ & \varepsilon < |\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1| \\ & \varepsilon < |\mathbf{r}_i + \mathbf{v}_i t_2 - \mathbf{r}_2| . \end{aligned} \tag{4}$$

Since the field stars are uncorrelated, the first part of (4) vanishes.

$$\langle \mathbf{F}(\mathbf{r}_1, t_1) \cdot F(\mathbf{r}_2, t_2) \rangle = G^2 m^2 \sum \left\langle \frac{\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1}{|\mathbf{r}_i + \mathbf{v}_i t_1 - \mathbf{r}_1|^3} \cdot \frac{\mathbf{r}_i + \mathbf{v}_i t_2 - \mathbf{r}_2}{|\mathbf{r}_i + \mathbf{v}_i t_2 - \mathbf{r}_2|^3} \right\rangle$$

$$= G^2 m^2 n \int \frac{\mathbf{r}+\mathbf{v}t_1-\mathbf{r}_1}{|\mathbf{r}+\mathbf{v}t_1-\mathbf{r}_1|^3} \cdot \frac{\mathbf{r}+\mathbf{v}t_2-\mathbf{r}_2}{|\mathbf{r}+\mathbf{v}t_2-\mathbf{r}_2|^3} \tau(\mathbf{v})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{v} \quad (5)$$

$$|\mathbf{r}+\mathbf{v}t_1| < \lambda, \qquad |\mathbf{r}+\mathbf{v}t_2| < \lambda$$

$$\varepsilon < |\mathbf{r}+\mathbf{v}t_1-\mathbf{r}_1|, \quad \varepsilon < |\mathbf{r}+\mathbf{v}t_2-\mathbf{r}_2|.$$

Performing the transformation

$$\boldsymbol{\varrho} = \mathbf{r}+\mathbf{v}t_1; \qquad \mathrm{d}\boldsymbol{\varrho} = \mathrm{d}\mathbf{r} \quad (6)$$

$$\langle \mathbf{F}(\mathbf{r}_1, t_1) \cdot \mathbf{F}(\mathbf{r}_2, t_2) \rangle = G^2 m^2 n \int \frac{\boldsymbol{\varrho}-\mathbf{r}_1}{|\boldsymbol{\varrho}-\mathbf{r}_1|^3} \cdot \frac{\boldsymbol{\varrho}+\mathbf{v}t-\mathbf{r}_2}{|\boldsymbol{\varrho}+\mathbf{v}t-\mathbf{r}_2|^3} \tau(\mathbf{v})\, \mathrm{d}\boldsymbol{\varrho}\, \mathrm{d}\mathbf{v} \quad (7)$$

$$|\boldsymbol{\varrho}| < \lambda,\ |\boldsymbol{\varrho}+\mathbf{v}t| < \lambda,\ |\boldsymbol{\varrho}-\mathbf{r}_1| > \varepsilon,\ |\boldsymbol{\varrho}+\mathbf{v}t-\mathbf{r}_2| > \varepsilon$$

where $t = t_2 - t_1$. Hence,

$$\langle \mathbf{F}(\mathbf{r}_1, t_1) \cdot \mathbf{F}(\mathbf{r}_2, t_2) \rangle = \langle \mathbf{F}(\mathbf{r}_1, 0) \cdot \mathbf{F}(\mathbf{r}_2, t_2 - t_1) \rangle. \quad (8)$$

Evaluation of (7) in its full generality is quite involved although one can find series expansions when a particular parameter (e.g., the separation $|\mathbf{r}_2 - \mathbf{r}_1|$) is small. This will be done in the last section where it will be applied to the problem of the stability of binaries. In the next two sections, we consider special cases of (7), namely the correlations at two different points at the same time and the correlation at the same point at two different times.

But in the case of infinite systems and $\varepsilon = 0$, (7) can be evaluated explicitly. Taking

$$\mathbf{r}_1 = -\mathbf{r}_2 = \mathbf{s}$$
$$\mathbf{a} = 2\mathbf{s} \quad (9)$$

we have

$$\langle \mathbf{F}(-\mathbf{s}, 0) \cdot \mathbf{F}(\mathbf{s}, t) \rangle = G^2 m^2 n \int \frac{\boldsymbol{\varrho}-\mathbf{s}}{|\boldsymbol{\varrho}-\mathbf{s}|^3} \cdot \frac{\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{s}}{|\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{s}|^3} \tau(\mathbf{v})\, \mathrm{d}\boldsymbol{\varrho}\, \mathrm{d}\mathbf{v} =$$
$$= G^2 m^2 n \int \frac{\boldsymbol{\varrho}}{\varrho^3} \cdot \frac{\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{a}}{|\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{a}|^3} \tau(\mathbf{v})\, \mathrm{d}\boldsymbol{\varrho}\, \mathrm{d}\mathbf{v}. \quad (10)$$

The spatial integration is straightforward

$$\int \frac{\boldsymbol{\varrho}}{\varrho^3} \cdot \frac{\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{a}}{|\boldsymbol{\varrho}+\mathbf{v}t+\mathbf{a}|^3} \mathrm{d}\varrho = \frac{4\pi}{|\mathbf{v}t+\mathbf{a}|} \quad (11)$$

and accordingly

$$\langle \mathbf{F}(-\mathbf{s}, 0) \cdot \mathbf{F}(\mathbf{s}, t) \rangle = 4\pi G^2 m^2 n \int \frac{\tau(\mathbf{v})}{|\mathbf{v}t+\mathbf{a}|} \mathrm{d}\mathbf{v}. \quad (12)$$

If we specialize to the Gaussian distribution

$$\tau(\mathbf{v}) = \frac{j^3}{\pi^{3/2}} \exp(-j^2 \mathbf{v}^2) \quad (13)$$

and note that the angular integration gives

$$2\pi \int_{-1}^{1} \frac{d\mu}{(a^2+avt\mu+v^2t^2)^{1/2}} = 4\pi \begin{cases} \dfrac{1}{a} & vt<a \\ \dfrac{1}{vt} & vt>a \end{cases} \tag{14}$$

we have after simplification

$$\langle \mathbf{F}(0,0)\cdot\mathbf{F}(\mathbf{a},t)\rangle = \frac{4\pi G^2m^2n}{a}\,\varphi\left(\frac{ja}{t}\right), \tag{15}$$

where $\varphi(x)$ is the error function

$$\varphi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x^2)\,dx. \tag{16}$$

Asymptotically the behavior of (15) is

$$\langle \mathbf{F}(0,0)\cdot\mathbf{F}(\mathbf{a},t)\rangle \sim \frac{8\sqrt{\pi}\,G^2m^2nj}{t} \qquad t\to\infty \tag{17}$$

$$\sim \frac{4\pi G^2m^2n}{a} \qquad a\to\infty. \tag{18}$$

We thus see that for infinite systems the behavior of the correlation function is inversely proportional to both the separation distance and the time when one of them approaches infinity.

2.1. Correlation at two different times at the same spatial point

We now consider the correlation of the force, for finite systems, at two different times but at the same point. A detailed discussion of the two time autocorrelation has been given by Cohen and Ahmad (1974). We shall derive here the same result, for the case of a Gaussian distribution, in a more direct way.

Taking $r_1=r_2=0$, (7) becomes

$$\langle \mathbf{F}(0)\cdot\mathbf{F}(t)\rangle = G^2m^2n \int \frac{\boldsymbol{\varrho}}{\varrho^3}\cdot\frac{\boldsymbol{\varrho}+\mathbf{v}t}{|\boldsymbol{\varrho}+\mathbf{v}t|^3}\,\tau(\mathbf{v})\,d\boldsymbol{\varrho}\,d\mathbf{v} \qquad \begin{matrix} \varepsilon<\varrho<\lambda \\ \varepsilon<|\boldsymbol{\varrho}+\mathbf{v}t|<\lambda. \end{matrix} \tag{19}$$

One could perform the $\boldsymbol{\varrho}$ integration first and thus keep the velocity distribution arbitrary (Cohen and Ahamd, 1974). This procedure is quite cumbersome and algebraically tedious. Specializing immediately to a Gaussian distribution and performing the transformation

$$\mathbf{x}=\boldsymbol{\varrho}+\boldsymbol{\eta}, \qquad d\mathbf{x}=d\boldsymbol{\eta}, \qquad \boldsymbol{\eta}\equiv\mathbf{v}t \tag{20}$$

we have

$$\langle \mathbf{F}(0)\cdot\mathbf{F}(t)\rangle = \frac{G^2m^2n\omega^3}{\pi^{3/2}} \int \frac{\boldsymbol{\varrho}\cdot\mathbf{x}}{\varrho^3x^3}\exp[-\omega^2(\boldsymbol{\varrho}-\mathbf{x})^2]\,d\mathbf{x} \qquad \begin{matrix} \varepsilon<x<\lambda \\ \varepsilon<\varrho<\lambda. \end{matrix} \tag{21}$$

$\omega\equiv j/t$

Using the direction of **x** as the z axis and performing the angular integration of both **x** and **ϱ**, we obtain

$$\langle F(0)\cdot F(t)\rangle = 8\sqrt{\pi}\, G^2 m^2 n \int_\varepsilon^\lambda \int_\varepsilon^\lambda \int_{-1}^{1} \exp[-\omega^2(\mathbf{x}-\boldsymbol{\varrho})^2]\, \mathrm{d}\mu\, \mathrm{d}x\, \mathrm{d}\varrho \tag{22}$$

$$= 8\sqrt{\pi}\, G^2 m^2 n \left\{ \int_\varepsilon^\lambda \int_\varepsilon^\lambda \left(\frac{1}{2\omega^2 x\varrho} - \frac{1}{(2\omega^2 x\varrho)^2}\right) \times \right.$$

$$\times \exp[-\omega^2(x-\varrho)^2]\, \mathrm{d}x\, \mathrm{d}\varrho + \int_\varepsilon^\lambda \int_\varepsilon^\lambda \left(\frac{1}{2\omega^2 x\varrho} + \frac{1}{(2\omega^2 x\varrho)^2}\right) \times$$

$$\left. \times \exp[-\omega^2(x+\varrho)^2]\, \mathrm{d}x\, \mathrm{d}\varrho \right\}. \tag{23}$$

To proceed further, we shall integrate by parts the second term in each of the integrals of (23). We note that

$$\int_\varepsilon^\lambda \frac{\exp[-\omega^2(x\pm\varrho)^2]}{\varrho^2}\, \mathrm{d}\varrho = \frac{1}{\varepsilon} \exp[-\omega^2(x\pm\varepsilon)^2] - \frac{1}{\lambda} \exp[-\omega^2(x\pm\lambda)^2] -$$

$$-2\omega^2 \int_\varepsilon^\lambda \frac{\varrho\pm x}{\varrho} \exp[-\omega^2(\varrho\pm x)^2]\, \mathrm{d}\varrho \tag{24}$$

and repeated use of (24) yields

$$\int_\varepsilon^\lambda \int_\varepsilon^\lambda \frac{\exp[-\omega^2(x+\varrho)^2]}{x^2\varrho^2}\, \mathrm{d}x\, \mathrm{d}\varrho = \frac{1}{\varepsilon^2} \exp[-\omega^2(2\varepsilon)^2] + \frac{1}{\lambda^2} \exp[-\omega^2(2\lambda)^2] -$$

$$-\frac{2}{\varepsilon\lambda} \exp[-\omega^2(\lambda+\varepsilon)^2] -$$

$$-\frac{2\omega}{\varepsilon} \sqrt{\pi}(\varphi(\omega(\lambda+\varepsilon)) - \varphi(2\omega\varepsilon)) +$$

$$+\frac{2\omega}{\lambda} \sqrt{\pi}(\varphi(2\omega\lambda) - \varphi(\omega(\lambda+\varepsilon)) -$$

$$-2\omega^2 \int_\varepsilon^\lambda \int_\varepsilon^\lambda \frac{\exp[-\omega^2(x+\varrho)^2]}{x\varrho}\, \mathrm{d}x\, \mathrm{d}\varrho \tag{25}$$

$$\int \int \frac{\exp[-\omega^2(x-\varrho)^2]}{x^2\varrho^2}\, \mathrm{d}x\, \mathrm{d}\varrho = \frac{1}{\varepsilon^2} + \frac{1}{\lambda^2} - \frac{2}{\varepsilon\lambda} \exp[-\omega^2(\lambda-\varepsilon)^2] -$$

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$$-\frac{2\omega}{\varepsilon}\sqrt{\pi}\,\phi(\omega(\lambda-\varepsilon))+$$
$$+\frac{2\omega}{\lambda}\sqrt{\pi}\,\varphi(\omega(\lambda-\varepsilon))+$$
$$+2\omega^2\int\limits_{\varepsilon}^{\lambda}\int\limits_{\varepsilon}^{\lambda}\frac{\exp[-\omega^2(\varrho-x)^2]}{x\varrho}\,\mathrm{d}\varrho\,\mathrm{d}x. \quad (26)$$

Substituting (25) and (26) into (23)

$$\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle=16\sqrt{\pi}\,G^2m^2nj\left\{\tfrac{1}{2}\left[\frac{1}{t_1}\,S(t_1/t)+\frac{1}{t_4}\,S(t_4/t)\right]+\right.$$
$$\left.+\frac{1}{t_1t_4}\left[t_2S(t_2/t)-t_3S(t_3/t)\right]-\tfrac{1}{2}t\left[\frac{1}{t_1^2}+\frac{1}{t_4^2}\right]\right\}, \quad (27)$$

where we have defined

$$t_1=2\varepsilon j;\qquad t_2=(\lambda-\varepsilon)\,j;\qquad t_3=(\lambda+\varepsilon)\,j;\qquad t_4=2\lambda j$$
$$S(x)=\sqrt{\pi}\,\varphi(x)+\frac{\exp(-x^2)}{x}. \quad (28)$$

Asymptotic expansions can be obtained

$$\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle=4\pi G^2m^2n$$
$$\frac{1}{\varepsilon}-\frac{1}{\lambda}-\frac{t}{2\sqrt{\pi}\,j}\left(\frac{1}{\varepsilon^2}+\frac{1}{\lambda^2}\right)\qquad t\ll t_1 \quad (29)$$
$$\frac{2j}{\sqrt{\pi}}\frac{1}{t}-\frac{1}{\lambda}-\frac{t}{2\sqrt{\pi}\,\lambda^2j}\qquad t_1\ll t<t_4 \quad (30)$$
$$\frac{2j^5}{3\sqrt{\pi}}\frac{(\lambda^2-\varepsilon^2)^2}{t^5}\qquad t_4\ll t. \quad (31)$$

If λ is taken to be infinite, we recover the $1/t$ dependence obtained by Chandrasekhar (1944b) and Lee (1968). But as long as λ is kept finite, the dependence for large times is $1/t^5$.

The two time autocorrelation function affords a straightforward method of calculating the mean square velocity change. Under the assumption that the autocorrelation function is an even function of the difference in the two times, then the mean square change in the velocity within a time T can be obtained from

$$\langle(\Delta\mathbf{v})^2\rangle=2\int\limits_0^T(T-t)\,\langle\mathbf{F}(0)\cdot\mathbf{F}(t)\rangle\,\mathrm{d}t. \quad (32)$$

As these conditions are met in our case, we can substitute (27) into (32) and obtain.

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after some algebra

$$\langle(\Delta\mathbf{v})^2\rangle = 16\sqrt{\pi}\, G^2m^2nj\Big\{t_1H(t_1/T)+t_4H(t_4/T)+ + \frac{2}{t_1t_4}\left[t_2^3H(t_2/T)-t_3^3H(t_3/T)\right]-\frac{T^3}{6}\left(\frac{1}{t_1^2}+\frac{1}{t_4^2}\right)\Big\}, \tag{33}$$

where

$$H(x)=\tfrac{1}{3}\frac{\exp(-x^2)}{x}\left(\frac{1}{2x^2}-1\right)+\frac{\sqrt{\pi}}{3}+\frac{1}{2x}\,\mathscr{E}_i(x^2)+\sqrt{\pi}\,\varphi(x)\left(\frac{1}{2x^2}-\tfrac{1}{3}\right) \tag{34}$$

and $\mathscr{E}_i(x)$ is the exponential integral.

$$\mathscr{E}_i(x)=\int_x^\infty \frac{\exp(-u)}{u}\,du. \tag{35}$$

Asymptotically

$$\langle(\Delta\mathbf{v})^2\rangle = 16\pi G^2m^2n$$

$$\left(\frac{1}{\varepsilon}-\frac{1}{\lambda}\right)\frac{T^2}{4}, \qquad T\ll t_1, \tag{36}$$

$$\frac{j}{\sqrt{\pi}}\,T\,\ln\frac{T}{2\varepsilon j}, \qquad t_1\ll T\ll t_4 \tag{37}$$

$$-\frac{(\lambda-\varepsilon)^2}{3\lambda}j^2+\frac{j}{\sqrt{\pi}}\,T\left(\ln\frac{(\lambda+\varepsilon)^2}{4\varepsilon\lambda}+\frac{(\lambda-\varepsilon)^2}{2\varepsilon\lambda}\ln\frac{\lambda+\varepsilon}{\lambda-\varepsilon}\right), \qquad t_4\ll T. \tag{38}$$

The infinite system case can be obtained from (37) and we note that it has a $T\ln T$ dependence for all time. This has been derived by other methods by Hénon (1958), Ostriker and Davidson (1968), Lee (1968), and Prigogine and Severne (1960).

For finite systems $\langle(\Delta\mathbf{v})^2\rangle$ changes from a $T\ln T$ dependence for times up to t_4 into a T dependence for long times. This behavior has been discussed by Hénon (1958).

2.2. The Correlation in the Force at Two Points at the Same Time

Taking $t=0$ and $\mathbf{r}_1=-\mathbf{r}_2=\mathbf{s}$, we have from (7)

$$\langle\mathbf{F}(-\mathbf{s})\cdot\mathbf{F}(\mathbf{s})\rangle = G^2m^2n \int \frac{\boldsymbol{\varrho}-\mathbf{s}}{|\boldsymbol{\varrho}-\mathbf{s}|^3}\cdot\frac{\boldsymbol{\varrho}+\mathbf{s}}{|\boldsymbol{\varrho}+\mathbf{s}|^3}\,d\boldsymbol{\varrho} \qquad \begin{array}{l}\varrho<\lambda\\ \varepsilon<|\boldsymbol{\varrho}-\mathbf{s}|\\ \varepsilon<|\boldsymbol{\varrho}+\mathbf{s}|.\end{array} \tag{39}$$

We shall not evaluate (39) but only give appropriate asymptotic expansions to illustrate the behavior for large separation distances in the case of finite systems.

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$$\langle F(-s)\cdot F(s)\rangle = 4\pi G^2 m^2 n$$

$$\frac{1}{\varepsilon}-\frac{1}{\lambda}+\tfrac{1}{6}\left(\frac{1}{\lambda^3}-\frac{1}{\varepsilon^3}\right)a^2, \qquad a\ll\varepsilon \tag{40}$$

$$\frac{1}{a}-\frac{1}{\lambda}+\tfrac{1}{6}\frac{a^2}{\lambda^3}, \qquad \varepsilon\ll a\ll\lambda \tag{41}$$

$$-\tfrac{16}{3}\frac{\lambda^3}{a^4}+\tfrac{64}{5}\frac{\lambda^5}{a^6}, \qquad \lambda\ll a. \tag{42}$$

Again, for infinite λ we recover the $1/a$ dependence but as in the time correlation case the decrease is much faster when λ is kept finite.

3. Dissolution of Binary Systems

An application which Chandrasekhar (1944c) made of his theory is to the problem of the dissolution time of binary systems. The expression he obtained is fundamentally different from that obtained by Ambartsumian (1937) in that the dissolution time did not depend on the mean velocity of the field stars. Oort (1950) has also obtained an expression for the dissolution time and although it differs somewhat from Ambartsumian's, it does depend on the mean velocity of the field stars. Heggie (1974) has also considered the problem.

Cruz-Gonzalez and Poveda (1972) performed numerical experiments to test for agreement with theory. They found that none of the three expressions of the dissolution time was in conformity with experiment although they did find dependence on the velocity of the field stars. (But see note added in proof.)

We shall explain the reason as to why Chandrasekhar's expression is independent of the mean velocity. In particular, we shall show that the reason is not due to the statistical theory but in the definition of the 'relative velocity change' of the two components of the binary system. Indeed, we shall use the general approach of Chandrasekhar, as modified above, to obtain a velocity dependent dissolution time.

Essentially, in all three approaches the dissolution time is obtained by finding the relative absolute velocity change of the binary component within a time T and defining the dissolution time as the amount of time needed for the relative velocity change of the two components to be of the same order as the initial relative velocity. Or, equivalently, the square of the velocity change is equated to twice the mean kinetic energy of the binary. Chandrasekhar calculates the relative change, $\Delta\mathbf{v}_1-\Delta\mathbf{v}_2\equiv\Delta\mathbf{v}_{12}$ between stars 1 and 2, constituting the binary, by considering the component of $\mathbf{F}_1-\mathbf{F}_2$ in the direction of one of the two forces. This is appropriate since the average relative velocity change in the perpendicular direction is zero. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are due to the field stars only.

$$\langle\Delta\mathbf{v}_{12}\rangle=\int\left\langle\frac{(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1}{|\mathbf{F}_1|}\right\rangle \mathrm{d}t. \tag{43}$$

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It is clear that if we assume field stars to be distributed over all space, $\langle(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1\rangle/|\mathbf{F}_1|$ will be independent of time and

$$\langle\Delta\mathbf{v}_{12}\rangle=\left\langle\frac{(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1}{|\mathbf{F}_1|}\right\rangle T. \tag{44}$$

In a previous paper, Chandrasekhar (1944b) obtained*, for small separation, a,

$$\left\langle\frac{(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1}{|\mathbf{F}_1|}\right\rangle\sim 4\pi Gmna \tag{45}$$

where m is the average mass of a field star. Equating (45) to

$$\left(\frac{G(m_1+m_2)}{a}\right)^{1/2}, \tag{46}$$

where m_1 and m_2 are the masses of the stars forming the binary, the dissolution time is then

$$\tau=\frac{(m_1+m_2)^{1/2}}{4\pi G^{1/2}\,mna^{3/2}}. \tag{47}$$

On the other hand, Oort and Ambartsumian (using a theory developed by Bohr for the ionization of hydrogen) calculate $\langle(\Delta\mathbf{v}_{12})^2\rangle$, which, as will be clear from the considerations below, brings in the field velocities. We shall now proceed to calculate in the context of Chandrasekhar's Theory with the modifications described above. The introduction of ε, the cut off at small distances is essential; otherwise the integrals appearing would diverge.

We remark that we will keep the two stars forming the binary stationary. A more refined derivation would allow for the motion and take into account the interaction between them.

Also we shall assume that the separation distance, a, is much smaller than ε, in which case we can neglect $\mathbf{r}_1$ and $\mathbf{r}_2$ appearing in the constraints in (7). Placing the two components of the binary at positions $-\mathbf{s}$ and $\mathbf{s}$, $(a=2s)$, we have for the mean square change in the relative velocity,

$$\langle(\Delta\mathbf{v}_{12})^2\rangle=\left\langle\left(\int_0^T\mathbf{F}_1\,\mathrm{d}t-\int_0^T\mathbf{F}_2\,\mathrm{d}t\right)^2\right\rangle=$$

$$=\int_0^T\int_0^T\langle\mathbf{F}(-\mathbf{s},t_1)\cdot\mathbf{F}(-\mathbf{s},t_2)\rangle\,\mathrm{d}t_1\,\mathrm{d}t_2+$$

* It may be of interest to point out that $\langle(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1/|\mathbf{F}_1|\rangle$ can be estimated from the usual tidal force argument. If d is the distance to a field star, the difference in force on the two components in the direction of the field star is $(2Gm/d^3)\,a$. Inserting 4π to take into account averaging over the sphere of radius a and estimating d by the interparticle distance $d\sim n^{-1/3}$, we have

$$\langle(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{F}_1/|\mathbf{F}_1|\rangle\sim 8\pi Gmna.$$

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$$+\int\int \langle \mathbf{F}(\mathbf{s}, t_1)\cdot\mathbf{F}(\mathbf{s}, t_2)\rangle \, dt_1 \, dt_2 -$$
$$-2\int\int \langle \mathbf{F}(-\mathbf{s}, t_1)\cdot\mathbf{F}(\mathbf{s}, t_2)\rangle \, dt_1 \, dt_2. \tag{48}$$

Using (32) and remembering that we will take $a \ll \varepsilon$

$$\langle(\Delta \mathbf{v}_{12})^2\rangle = 4\int_0^T \langle \Delta F_{12}\rangle (T-t)\, dt, \tag{49}$$

where, for convenience, we have defined

$$\langle \Delta F_{12}\rangle = \langle \mathbf{F}(s, 0)\cdot\mathbf{F}(s, t)\rangle - \langle \mathbf{F}(-s, 0)\cdot\mathbf{F}(s, t)\rangle \tag{50}$$

$$= \frac{G^2 m^2 n j^3}{\pi^{3/2}}\int\left(\frac{\mathbf{r}+\mathbf{s}}{|\mathbf{r}+\mathbf{s}|^3} - \frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\right)\cdot\frac{\mathbf{r}+\mathbf{s}+\mathbf{v}t}{|\mathbf{r}+\mathbf{s}+\mathbf{v}t|^3}\exp(-j^2\mathbf{v}^2)\, d\mathbf{r}\, d\mathbf{v}$$
$$\varepsilon < r < \lambda; \quad \varepsilon < |\mathbf{r}+\mathbf{v}t| < \lambda. \tag{51}$$

Following the same procedure as in Section 2.1., we have

$$\langle \Delta F_{12}\rangle = \frac{G^2 m^2 n \omega^3}{\pi^{3/2}}\int\left(\frac{\boldsymbol{\varrho}+\mathbf{s}}{|\boldsymbol{\varrho}-\mathbf{s}|^3} - \frac{\boldsymbol{\varrho}-\mathbf{s}}{|\boldsymbol{\varrho}-\mathbf{s}|^3}\right)\cdot\frac{\mathbf{x}+\mathbf{s}}{|\mathbf{x}+\mathbf{s}|^3}\times$$
$$\times \exp[-\omega^2(\mathbf{x}-\boldsymbol{\varrho})^2]\, d\mathbf{x}\, d\boldsymbol{\varrho} \tag{52}$$
$$\varepsilon < x < \lambda; \qquad \varepsilon < \varrho < \lambda.$$

As s is small in comparison to ε, we can expand (52) as a power series in s.

$$\left(\frac{\boldsymbol{\varrho}+\mathbf{s}}{|\boldsymbol{\varrho}+\mathbf{s}|^3} - \frac{\boldsymbol{\varrho}-\mathbf{s}}{|\boldsymbol{\varrho}-\mathbf{s}|^3}\right)\cdot\frac{\mathbf{x}+\mathbf{s}}{|\mathbf{x}+\mathbf{s}|^3} \sim \frac{2\mu_1 - 6\mu\mu_2}{\varrho^3 x^2} s +$$
$$+ \frac{2-6\mu_2^2-6\mu_1^2+18\mu\mu_1\mu_2}{\varrho^3 x^3} s^2 \cdots, \tag{53}$$

where the cosine of the angles are defined as follows

$$\begin{aligned} \boldsymbol{\varrho}\cdot\mathbf{s} &= \varrho s \mu_2 \\ \mathbf{x}\cdot\mathbf{s} &= x s \mu_1 \\ \mathbf{x}\cdot\boldsymbol{\varrho} &= x \varrho \mu. \end{aligned} \tag{54}$$

If we take a spherical coordinate system with the z axis in the direction of $\mathbf{x}$ and place $\mathbf{s}$ in the y, z plane, then

$$\mu_2 = \mu_1\mu + \sqrt{1-\mu_1^2}\sqrt{1-\mu^2}\cos\varphi, \tag{55}$$

where φ is the azimuthal angle.

https://doi.org/10.1017/S0074180900015369 Published online by Cambridge University Press

Consider the angular integrations

$$\int_0^{2\pi} \mu_2 \, d\varphi = 2\pi\mu_1\mu \tag{56}$$

$$\int_0^{2\pi} \mu_2^2 \, d\varphi = 2\pi\mu_1^2\mu^2 + \pi(1-\mu_1^2)(1-\mu^2) \tag{57}$$

$$\int_{-1}^{1} \mu_1 \, d\mu_1 = 0. \tag{58}$$

Because of (56) and (58), the s term of (53) is zero.

Integrating all the angles, except μ, in the s^2 term gives

$$\langle \Delta F_{12} \rangle = 16\sqrt{\pi}\, G^2 m^2 n \omega^3 s^2 \times \times \int_\varepsilon^\lambda \int_\varepsilon^\lambda \int_{-1}^{1} \frac{3\mu^2 - 1}{x\varrho} \exp[-\omega^2 (\mathbf{x} - \boldsymbol{\varrho})^2] \, d\mathbf{x} \, d\boldsymbol{\varrho} \, d\mu. \tag{59}$$

Performing the μ integration we obtain

$$\langle \Delta F_{12} \rangle = -16\sqrt{\pi}\, G^2 m^2 n \omega^3 s^2 \times \times \left\{ \int_\varepsilon^\lambda \int_\varepsilon^\lambda G(x, \varrho) \, dx \, d\varrho + \int_\varepsilon^\lambda \int_{-\varepsilon}^{-\lambda} G(x, \varrho) \, dx \, d\varrho \right\}, \tag{60}$$

where

$$G(x, \varrho) = \frac{1}{x\varrho} \left(\frac{1}{\omega^2 x\varrho} + \tfrac{3}{2} \frac{1}{\omega^4 x^2 \varrho^2} + \tfrac{3}{4} \frac{1}{\omega^6 x^3 \varrho^3} \right) \exp[-\omega^2 (x + \varrho)^2]. \tag{61}$$

The indefinite integral of $G(x, \varrho)$ can be obtained by integration by parts,

$$\int\int G(x, \varrho) \, d\varrho \, dx = \left\{ \frac{1}{12\omega^6 x^3 \varrho^3} - \frac{1}{6\omega^4 \varrho^3 x} - \frac{1}{6\omega^4 \varrho x^3} + \right. \left. + \frac{1}{6\omega^4 \varrho^2 x^2} \right\} \exp[-\omega^2 (x + \varrho)^2] - - \frac{1}{3\omega^3} \left(\frac{1}{\varrho^3} + \frac{1}{x^3} \right) \frac{\sqrt{\pi}}{2} [\omega(x + \varrho)]. \tag{62}$$

We shall consider here the case of $\lambda = \infty$, in which case

$$\langle \Delta F_{12} \rangle = 6\sqrt{\pi} G^2 m^2 n \omega^3 s^2 \left\{ \left(\frac{1}{12\omega^6 \varepsilon^6} \quad \frac{1}{6\omega^4 \varepsilon^4} \right) \{1 - \exp[-(2\omega\varepsilon)^2]\} - \right. \left. - \frac{1}{3\omega^4 \varepsilon^4} + \frac{\sqrt{\pi}}{3\omega^3 \varepsilon^3} \phi(2\varepsilon\omega) \right\} \tag{63}$$

and inserting (63) into (49)

$$\langle(\Delta v_{12})^2\rangle = 16\sqrt{\pi}\, G^2 m^2 n j^3 a^2 \left\{ \tfrac{4}{15} \frac{T^5}{(2\varepsilon j)^6} \{1 - \exp[-(2\varepsilon j/T)^2]\} + \right.$$
$$+ \tfrac{4}{3} \frac{T^3}{(2\varepsilon j)^4} \{\tfrac{4}{5} \exp[-(2\varepsilon j/T)^2] - 1\} +$$
$$+ \tfrac{8}{15} \frac{T}{(2\varepsilon j)^2} \exp[-(2\varepsilon j/T)^2] - \frac{8\sqrt{\pi}}{15(2\varepsilon j)} \left[1 - \varphi\left(\frac{2\varepsilon j}{T}\right)\right] +$$
$$\left. + \tfrac{4}{3} \frac{\sqrt{\pi}}{(2\varepsilon j)^3} T^2 \varphi(2\varepsilon j/T) \right\}. \tag{64}$$

For long times (64) asymptotically approaches

$$\langle(\Delta \mathbf{v}_{12})^2\rangle = 16\sqrt{\pi}\, G^2 m^2 n j^3 a^2 \frac{2T}{(2\varepsilon j)^2}, \qquad T \to \infty. \tag{65}$$

Equating (65) to the square of (46), and using

$$j^2 = \frac{3}{2\langle v^2\rangle}$$
$$\varepsilon \sim \frac{2Gm}{\langle v^2\rangle}$$

we have for the time of dissolution (for equal masses)

$$\tau = \sqrt{\frac{2}{3\pi}} \frac{Gm}{a^3 v^3 n}. \tag{66}$$

This agrees, in functional dependence, with the expression of Ambartsumian for the case of $a \ll 2Gm/\langle v^2\rangle$.

Acknowledgement

The author gratefully acknowledges a number of criticisms and corrections made by Drs S. Aarseth and D. Heggie and for bringing to the authors attention the paper by M. Hénon.

Note added in proof. Dr Heggie has made an extensive study of the evolution of binary stars ('The Dynamical Evolution of Binary Stars', Thesis, Cambridge University Press, 1971). His result for hard binaries is that their disruption rate is exponentially small. Dr Heggie has pointed out (private communication) that the assumption of $a \ll \varepsilon$ implies (if the result of Equation (65) is to be applied to the binary problem) that the relative motion of the binary components is much faster than that of the field stars and hence the assumption of keeping **s** constant may be a poor one. The calculation can be modified by taking **s** to be time dependent and a function of the relative velocity of the binary components. This may change the final result significantly. But nonetheless, it is clear that the statistical theory of Chandrasekhar and Von Neumann will give a field star velocity dependence for the dissolution time if the autocorrelation function is modified as described in Section 1.

Regarding the numerical experiments of Cruz-Gonzalez and Poveda, M. Hénon (*Astron. Astrophys.* **19** (1972), 488), has shown that the method of simulating the field stars was incorrect and that when proper account is taken of this fact the numerical results yield a better agreement with Oort's formula for the dissolution of binaries.

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DISCUSSION

Severne: Is it consistent here to use simultaneously the approximation of straight line trajectories and finite system size?

Cohen: One could take other than linear orbits depending on the problem of hand. But for most situation linear orbits are a good approximation – and simple to work with.

Lynden-Bell: I would just like to get clear exactly what stars you consider the force from. Your calculation essentially considers only forces from the stars that move with the point considered.

Cohen: Yes. Also, stars are counted in the averaging only if they are within the system at *both* times.