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On the completeness of sets of complex exponentials

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Abstract

The completeness of sets of complex exponentials $\{e^{i\lambda f(n)\cdot} : f \in \mathbb{Z}\}$ in $L^p(-\pi, \pi)$, $1 < p \leqslant 2$, is considered under Levinson's sufficient condition in the non-trivial case $\lambda \geqslant 1 - (1/p)$. All such sets are determined explicitly.

1. *Introduction*

In 1934, Paley and Wiener [2] proved that a set of complex exponentials $\{e^{i\lambda_n \cdot}\}$, $\{\lambda_n\} \subset \mathbb{R}$, is complete on $L^2(-\pi, \pi)$ if there exists a constant $D < \pi^{-2}$ such that

$$|\lambda_n - n| \leqslant D \tag{1.1}$$

for every $n \in \mathbb{Z}$. Shortly after, Levinson [1] considered the $L^p(-\pi, \pi)$ case with $1 < p \leqslant 2$ and $1/p + 1/q = 1$. He proved that completeness holds on $L^p(-\pi, \pi)$ if

$$|\lambda_n - n| \leqslant D < 1/2q, \tag{1.2}$$

and that this result is optimal in the sense that completeness may fail if $D = 1/2q$.

In this short paper we consider the existence of solutions to Levinson's inequality (1.2) which are subsequences of the scaled integers $\lambda\mathbb{Z}$, $\lambda \in \mathbb{R}$, in other words, we consider *commensurable* solutions. In this case, the problem reduces to finding $f : \mathbb{Z} \to \mathbb{Z}$ such that $\lambda_n = \lambda f(n)$. The structure of the solutions and the action of f on the additive group $\mathbb{Z}/(2r+1)\mathbb{Z}$ are discussed once the reader has been introduced to the relevant concepts via a proof of the main theorem.

For $0 < \lambda < 1/q$, the existence of f is trivial because for every $n \in \mathbb{Z}$ there exists at least one $m \in \mathbb{Z}$ such that $|\lambda m - n| < \lambda/2$. On the other hand, for $\lambda \geqslant 1/q$, f is injective if it exists.

2. *Main theorem*

The special case $p = q = 2$ was solved in [3] via a complicated and laborious algorithm. We shall now prove the general theorem for $1 < p \leqslant 2$ using elementary principles from group theory, which demonstrate more clearly why the structure of the solutions is as such. Specifically (a non-trivial case being one in which $\lambda f(n) \neq n$ for at least one n), we shall prove the following theorem.

Theorem 2.1. *If $\lambda \geqslant 1/q$, there exists a non-trivial $f : \mathbb{Z} \to \mathbb{Z}$ such that*

$$|\lambda f(n) - n| \leqslant D < 1/2q \tag{2.1}$$

if and only if $\lambda = (2r+1)/s$ is an irreducible fraction with $s = 2qr + t$ and $1 \leqslant t < q$.

Proof. We begin by excluding the possibility that λ is irrational or rational with an even numerator. By Kronecker's approximation theorem, the fractional parts of the set $\lambda\mathbb{Z}$ are dense in the unit interval $(0, 1)$. It follows that an irrational λ cannot solve (1.2), so $\lambda = r/s$, where $r, s \in \mathbb{N}$, and, since f is one-to-one, we must have $f(r) = s$, so $\lambda = r/f(r)$. To exclude the possibility that the numerator is even, suppose that $\lambda = 2r/s$ is irreducible,

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so that $f(2r) \neq 2f(r)$. Dividing (2.1) by $\lambda n = 2rn/f(2r)$, we obtain the weaker condition

$$\left|\frac{f(n)}{n} - \frac{f(2r)}{2r}\right| \leqslant C < \frac{1}{2n}. \tag{2.2}$$

Now, since $f(2r) \neq 2f(r)$ and $f : \mathbb{Z} \to \mathbb{Z}$, we have $|f(2r) - 2f(r)| \geqslant 1$, so $|f(2r)/2r - f(r)/r| \geqslant 1/2r$, a contradiction. We must conclude that the numerator is odd, so $\lambda = (2r+1)/s$, $r, s \in \mathbb{N}$.

The fact that λ must be rational with an odd numerator means that we may replace $|\cdot| \leqslant D < 1/2q$ by $|\cdot| < 1/2q$, and work modulo $2r+1$ because f is actually a function on the additive group $\mathbb{Z}/(2r+1)\mathbb{Z}$, that is, $f(n) = f(j(2r+1)+k) = jf(2r+1) + f(k) = f(k) \bmod f(2r+1)$. Dividing (2.1) by $\lambda \geqslant 1/q$, we obtain the weaker condition

$$\left|f(n) - \frac{n}{\lambda}\right| < \frac{1}{2}, \tag{2.3}$$

so that $f(n) = \lfloor n/\lambda + 1/2 \rfloor$ is the nearest integer to n/λ in any case.

Let $s = m(2r+1) + l$. Since we need only consider λ irreducible, we have $(2r+1, s) = (2r+1, l) = 1$ and

$$\begin{aligned} g(ks) &= ks - (2r+1)\left\lfloor \frac{ks}{2r+1} + \frac{1}{2} \right\rfloor \\ &= kl - (2r+1)\left\lfloor \frac{kl}{2r+1} + \frac{1}{2} \right\rfloor \\ &= kl \mod 2r+1. \end{aligned} \tag{2.4}$$

For each $k \in [-r, r]$, the map $k \to kl$, $l \in [-r, r]$, is incongruent mod $2r+1$, and, since g is the corresponding element in the least residue system $[-r, r]$, g defines $\phi(2r+1)+1$ automorphisms of the additive group $\mathbb{Z}/(2r+1)\mathbb{Z}$. Since $\lambda \geqslant 1/q$ is irreducible, we have $(2r+1, s) = 1$ and it follows that g takes the values $\pm r$ independently of s, so

$$\sup_{n \in \mathbb{Z}} |\lambda f(n) - n| = \sup_{n \in \mathbb{Z}} \left|\frac{g(ns)}{s}\right| = \frac{r}{s} \tag{2.5}$$

and (2.1) holds if and only if $s = 2qr + t$, $1 \leqslant t < q$, which completes the proof. □

3. *Remarks on the action of f on the additive group $\mathbb{Z}/(2r+1)\mathbb{Z}$*

In the course of proving Theorem 2.1 we found that, for each solution λ, we must have $f(k) = \lfloor k/\lambda + 1/2 \rfloor$. The assertion of the theorem may be expressed as $s = q(2r+1) + l$, $1 - q \leqslant l < 0$, which, on substitution into the expression for f, gives $f(k) = kq + \lfloor kl/(2r+1) + 1/2 \rfloor$. Since $k \in [-r, r]$, this implies that $f : k \to kq$ if and only if $l = -1$, that is, $\lfloor kl/(2r+1) + 1/2 \rfloor = 0$, which is exactly the case in which λ is the best possible rational approximation to $1/q$ with numerator $2r+1$. It is interesting to note that this is the only case in which f is an automorphism of the additive group $\mathbb{Z}/(2r+1)\mathbb{Z}$.

For finite q, the solutions corresponding to the remaining congruences $l = -2, \ldots, 1-q$ do not yield functions which preserve the structure of the group. Yet, it is interesting to note that the only possibility in the L^1 (that is, $q = \infty$) and L^2 cases are the functions $f : k \to k$ and $f : k \to 2k$, respectively, which are both automorphisms of $\mathbb{Z}/(2r+1)\mathbb{Z}$.

It also seems worth remarking that for sufficiently large r, specifically $2r+1 > q$, we find that there are exactly $q-1$ distinct non-trivial solutions, and hence $q-1$ distinct non-trivial functions f. Therefore, including the trivial (identity) case $f(n) = n$, one has $q = (1 - (1/p))^{-1}$ distinct solutions. This fact easily follows from the observation that, for every $1 \leqslant t < q$, $2rq + t$ is not divisible by $2r+1$ if $2r+1 > q$.

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