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OPTIMAL WEIGHTED ESTIMATES FOR THE CAUCHY-RIEMANN EQUATION ON ANALYTIC POLYHEDRA

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Andersson obtained weighted L^p estimates for $\overline{\partial}$ on analytic polyhedra. We provide an example to show why they cannot be improved.

1. INTRODUCTION

A bounded domain $\Omega \subset \mathbb{C}^n$ is an analytic polyhedron with defining functions ϕ_j if

$$\Omega = \left\{z \in \mathbb{C}^n \ ; \ \left|\phi_j(z)\right| < 1, \quad j = 1, \dots, N\right\},$$

where the defining functions ϕ_j are holomorphic in some neighbourhood of $\overline{\Omega}$. For a multi-index $I \subset \{1, \dots, N\}$ we let $\sigma_I = \left\{z \in \overline{\Omega} \ ; \ \left|\phi_j(z)\right| = 1, \ j \in I\right\}$.

For $\alpha = (\alpha_1, \dots, \alpha_N)$ let $L^p_\alpha(\Omega)$ denote the L^p-space with respect to the weight

$$dV_\alpha = \prod_{j=1}^{N} \left(1 - \left|\phi_j(z)\right|^2\right)^{\alpha_j} dV.$$

In [**2**], Andersson obtained the following L^p_α-estimates for $\overline{\partial}$ on analytic polyhedra. For the estimates he used a generalisation of the Henkin-Ramirez formulas which permits weight factors (see [**4**]).

THEOREM. *Let $\Omega \Subset \mathbb{C}^n$ be an analytic polyhedron and let f be a $\overline{\partial}$-closed (s,q)-form in $L^p_\alpha(\Omega)$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_j \geq 0$, $1 \leq p < \infty$. If $\wedge_{j \in I} \partial\phi_j \neq 0$ on σ_I for all $|I| < n - q$, then there is a solution u to $\overline{\partial} u = f$ in $L^p_\alpha(\Omega)$.*

The main result of the paper is the sharpness of the L^p_α-estimates for $\overline{\partial}$ in Andersson's theorem. More precisely, we provide a closed $(0,1)$-form f on a polydisc Δ^2 in $\mathbb{C}^2$ such that if $f \in L^p_\alpha(\Delta^2)$ $(2 \leq p < \infty, \ \alpha \geq 0)$ and $v \in L^r_\beta(\Delta^2)$ is a solution to $\overline{\partial} v = f$, then $\alpha \leq \beta$ and $r \leq p$.

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2. Weighted Bergman projections on the unit polydisc

In this section, we restrict ourselves to the unit polydisc Δ^n in $\mathbb{C}^n$.

Let $A^p_\alpha(\Delta^n)$ denote the weighted Bergman space of holomorphic functions f in $L^p_\alpha(\Delta^n)$. Then $A^p_\alpha(\Delta^n)$ is a closed subspace of $L^p_\alpha(\Delta^n)$. $L^2_\alpha(\Delta^n)$ is a Hilbert space with the inner product $(\cdot,\cdot)_\alpha$ defined by

$$(f,g)_\alpha = \int_{\Delta^n} f(\zeta)\overline{g(\zeta)} \prod_{j=1}^n \left(1-|\zeta_j|^2\right)^{\alpha_j} dV(\zeta).$$

Since $A^2_\alpha(\Delta^n)$ is a closed subspace of $L^2_\alpha(\Delta^n)$, there is an orthogonal projection $P_\alpha : L^2_\alpha(\Delta^n) \to A^2_\alpha(\Delta^n)$. It can be shown that

$$P_\alpha f(z) = C_\alpha \int_{\Delta^n} f(\zeta) \prod_{j=1}^n \frac{\left(1-|\zeta_j|^2\right)^{\alpha_j}}{\left(1-\overline{\zeta_j} z_j\right)^{\alpha_j+2}} dV(\zeta), \quad z \in \Delta^n,$$

where $C_\alpha = \prod_{j=1}^n (\alpha_j + 1)$ [**1**, **2**, **3**, **6**]. We call the projection P_α by the weighted Bergman projection.

Using [**5**, Proposition 1.4.10] repeatly, we can obtain the following result.

Lemma 2.1. *For $\alpha \geq 0$, let*

$$K_\alpha(\zeta, z) = C_\alpha \prod_{j=1}^n \frac{\left(1-|\zeta_j|^2\right)^{\alpha_j}}{\left(1-\overline{\zeta_j} z_j\right)^{\alpha_j+2}}$$

and

$$h(z) = \prod_{j=1}^n \left(1-|z_j|^2\right)^{-c_j}.$$

(i) *If $0 < c_j < 1 + \alpha_j$, $j = 1, \dots, n$, then*

$$\int_{\zeta\in\Delta^n} \left|K_\alpha(\zeta, z)\right| h(\zeta)\, dV(\zeta) \lesssim h(z), \quad z \in \Delta^n.$$

(ii) *If $0 < c_j + \alpha_j < 1 + \alpha_j$, $j = 1, \dots, n$, then*

$$\int_{z\in\Delta^n} \left|K_\alpha(\zeta, z)\right| h(z)\, dV(z) \lesssim h(\zeta), \quad \zeta \in \Delta^n.$$

By the mean value property, it follows that

$$\int_0^{2\pi} \rho e^{i\theta} h(\zeta_1, \rho e^{i\theta}) d\theta = 2\pi \cdot 0 \cdot h(\zeta_1, 0) = 0.$$

Thus we have $(h, u)_\beta = 0$, that is, u is orthogonal to all L^2_β holomorphic functions on Δ^2. If $v \in L^p_\beta(\Delta^2)$ is a solution to $\overline{\partial} v = f$ in Δ^2, then by Theorem 2.2, $u = v - P_\beta(v)$ would be in $L^p_\beta(\Delta^2)$. This is impossible. Hence there is no solution v in $L^p_\beta(\Delta^2)$ to the equation $\overline{\partial} v = f$ for $\beta < \alpha$.

If we take $d = (2 + \alpha_1)/r$, then it also follows by a similar process to the above that every solution v to $\overline{\partial} v = f$ is not in $L^r_\alpha(\Delta^2)$ for $p < r$.

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