ON HOMOTOPIC HARMONIC MAPS

PHILIP HARTMAN

1. Introduction. Let M, M' be C^∞ Riemann manifolds such that

(1.0) M is compact;

(1.1) M' is complete and its sectional curvatures are non-positive.

In terms of local coordinates $x = (x^1, \ldots, x^n)$ on M and $y = (y^1, \ldots, y^m)$ on M', let the respective Riemann elements of arc-length be

$$ds^2 = g_{ij}\,dx^i\,dx^j, \qquad ds'^2 = g'_{\alpha\beta}\,dy^\alpha\,dy^\beta$$

and $\Gamma^i{}_{jk}$, $\Gamma'^\alpha{}_{\beta\gamma}$ be the corresponding Christoffel symbols. When there is no danger of confusion, x (or y) will represent a point of M (or M') or its coordinates in some local coordinate system.

A map $f\colon M \to M'$ of class C^∞ is said to be harmonic if, in local coordinates,

$$\Delta f^\alpha(x) + g^{ij}(x)\Gamma'^\alpha_{\beta\gamma}(f)f^\beta_i(x)f^\gamma_j(x) = 0, \qquad \text{for } \alpha = 1, \ldots, m, \tag{1.2}$$

where Δ is the Laplace–Beltrami operator on M,

$$\Delta u = (\det g_{hk})^{-\frac{1}{2}} \frac{\partial}{\partial x^j}\left[g^{ij}\frac{\partial u}{\partial x^i}(\det g_{hk})^{\frac{1}{2}}\right], \tag{1.3}$$

and $f^\beta_i = \partial f^\beta/\partial x^i$; cf. **(2, 7, 1)** and historical remarks in **(1**, p. 110**)**. In order to discuss the existence of harmonic maps, Eells and Sampson **(1)** consider initial value problems associated with the corresponding parabolic equations

$$f^\alpha_t = \Delta f^\alpha + g^{ij}\Gamma'^\alpha_{\beta\gamma}f^\beta_i f^\gamma_j \qquad \text{for } \alpha = 1, \ldots, m, \tag{1.4}$$

$$f(\cdot, 0) = f_0, \tag{1.4$_0$}$$

where $f_t^\alpha = \partial f^\alpha/\partial t$. It will always be assumed that $f_0\colon M \to M'$ is of class $C^1(M)$. By a solution $f(x, t)$ of (1.4)–(1.4$_0$) on $[0, t_1)$, or, equivalently, by a solution of (1.4) on $[0, t_1)$ reducing to f_0 at $t = 0$, will be meant a function $f(x, t)\colon M \times [0, t_1) \to M'$ of class $C^1(M \times [0, t_1)) \cap C^\infty(M \times (0, t_1))$ satisfying (1.4) on $M \times (0, t_1)$ and (1.4$_0$) at $t = 0$. Eells and Sampson **(1)** prove:

(0) (i) Let M'' be a compact subset of M' and $C > 0$. Then there exists an $\epsilon = \epsilon(M'', C) > 0$ with the property that if $f_0\colon M \to M'$, $f_0(M) \subset M''$, and

$$e_0[f_0] \equiv \sup g^{ij} g'_{\alpha\beta} f^\alpha_{0i} f^\beta_{0j} \leqslant C,$$

Received May 4, 1966. This research was supported by the Air Force Office of Scientific Research.

then (1.4)–(1.4_0) has a unique solution on $M \times [0, \epsilon]$; Theorem 10B, p. 154. (ii) Also, there exists a $C_0 > 0$ independent of t_1 and f_0 such that if f is a solution of (1.4)–(1.4_0) for $(x, t) \in M \times [0, t_1)$, then $e_0[f(\cdot, t)] \leqslant C_0\, e_0[f_0]$ for $0 \leqslant t < t_1$; Theorem 9B, pp. 147–148.

(α) Under an assumption (p. 144, (12)) involving an embedding $w: M' \to R^q$ of M' into a Euclidean space (which is always fulfilled if M' is compact), the solution $f(x, t)$ of (1.4)–(1.4_0) exists for all $t \geqslant 0$; Theorem 10C, pp. 154–155.

(β) If, in addition, the range of $f(x, t)$: $M \times [0, \infty) \to M'$ is in a compact subset of M', then there exists an unbounded sequence $0 < t_1 < t_2 < \ldots$ such that $f_\infty(x) = \lim f(x, t_n)$ exists uniformly on M (in fact, in $C^k(M)$ for all k) and is a harmonic map of M into M' (which is of course homotopic to $f_0(x)$).

The object of this paper is to prove the following assertions:

(A) *If f_0: $M \to M'$ is a C^1 map, then the initial value problem* (1.4)–(1.4_0) *has a solution $f(x, t)$ for all $t \geqslant 0$*; i.e., the condition on the embedding $w: M' \to R^q$ in (α) above can be omitted.

(B) *If $f(x, t)$ is a solution of* (1.4)–(1.4_0) *for all $t \geqslant 0$ and its range is in a compact subset of M', then*

$$f_\infty(x) = \lim_{t\to\infty} f(x, t) \tag{1.5}$$

exists in $C^k(M)$ for every k and is a harmonic map f_∞: $M \to M'$ homotopic to f_0; i.e., the selection of a sequence $t_1 < t_2 < \ldots$ in (β) above is unnecessary.

(C) *If $f(x, t)$ is a solution of* (1.4)–(1.4_0) *for all $t \geqslant 0$, then its range is in a compact subset of M' if and only if there exists a harmonic map homotopic to $f_0 = f(\cdot, 0)$.*

(D) *A harmonic map f_∞ is a stable stationary point of the differential equation* (1.4). *It is an asymptotically stable stationary point if and only if there exists no other harmonic map homotopic to it*; *in this case, it is globally asymptotically stable* (in the sense that if f_0: $M \to M'$ is any C^1 map homotopic to f_∞, then the solution $f(x, t)$ of (1.4)–(1.4_0) exists for $t \geqslant 0$ and $f(\cdot, t) \to f_\infty$ as in (B)).

(E) *If f_0: $M \to M'$ is a C^1 map, the set of harmonic maps homotopic to $f_0(x)$ is connected*; i.e., if $f_{\infty 0}, f_{\infty 1}$ are two homotopic harmonic maps, then there exists a continuous map $f_\infty(x, u)$: $M \times [0, 1] \to M'$ such that $f_\infty(\cdot, 0) = f_{\infty 0}$, $f_\infty(\cdot, 1) = f_{\infty 1}$, and $f_\infty(\cdot, u)$ is harmonic for fixed u, $0 \leqslant u \leqslant 1$.

In order to state the next assertion, introduce the following notation: if y, z are points of M', let $r'(y, z)$ denote the distance between them in the metric of M'; if f, g: $M \to M'$ are two maps, let $|f, g|_\infty = \sup r'(f(x), g(x))$ for $x \in M$; if M'' is a compact subset of M', let $\delta = \delta(M'') > 0$ denote a number with the property that if $y \in M''$ and $r'(y, z) < \delta$, then there is a unique minimizing geodesic joining y and z.

(F) *Let M'' be a compact subset of M' and $f_{\infty 0}, f_{\infty 1}$ be harmonic maps such that $f_{\infty 0}(M) \subset M''$ and $|f_{\infty 0}, f_{\infty 1}|_\infty < \delta(M'')$. For x fixed, let $F(x, u)$, where $0 \leqslant u \leqslant 1$, be the unique minimizing arc (or point) joining $F(x, 0) = f_{\infty 0}(x)$ and $F(x, 1) = f_{\infty 1}(x)$ with u proportional to the arc-length. Then, for fixed u, $F(\cdot, u)\colon M \to M'$ is a harmonic map and the length of the geodesic arc $F(x, u)$, $0 \leqslant u \leqslant 1$, is independent of x.*

Equations (1.4) are the Euler–Lagrange equations for the variation of the "energy,"

$$E(f) = \int_M g^{ij}(x)\, g'_{\alpha\beta}(f)\, f^\alpha_i f^\beta_j\, (\det g_{hk})^{\frac{1}{2}}\, dx, \tag{1.6}$$

of a C^1 map $f\colon M \to M'$. The assertions (E) and (F) imply that if $f_{\infty 0}, f_{\infty 1}$ are homotopic harmonic maps, then there exists a piecewise smooth homotopy through harmonic maps. Hence, we have

COROLLARY. *The energy $E(f)$ is constant on any set of homotopic harmonic maps.*

This corollary and (F) will be used to prove

(G) *If $f_{\infty 0}, f_{\infty 1}$ are homotopic harmonic maps, then there exists a C^∞ homotopy $f_\infty(x, u)\colon M \times [0, 1] \to M'$ from $f_\infty(\cdot, 0) = f_{\infty 0}$ to $f_\infty(\cdot, 1) = f_{\infty 1}$ with the following properties*: (i) *for a fixed u, $f_\infty(\cdot, u)\colon M \to M'$ is a harmonic map and* (ii) *for fixed x, the arc $f_\infty(x, u)$ for $0 \leqslant u \leqslant 1$ is a geodesic arc with length independent of x, and u proportional to the arc-length.*

(H) *Let $f_\infty\colon M \to M'$ be a harmonic map and let there exist a point $x_0 \in M$ such that all sectional curvatures of M' through $f_\infty(x_0)$ are negative. Then a necessary condition that there exist a harmonic map $\neq f_\infty$ homotopic to f_∞ is that there exist a geodesic γ_0 on M' and a neighbourhood U_0 of x_0 such that $f_\infty(U_0) \subset \gamma_0$.*

COROLLARY. *In* (H), *if the rank of the Jacobean matrix $(\partial f^\alpha_\infty / \partial x^i)$ at $x = x_0$ is larger than* 1, *then there is no harmonic map $\neq f_\infty$ homotopic to f_∞.*

If M and M' are analytic Riemann manifolds, then harmonic maps are analytic; **(1**, p. 117**)**. In this case, (H) has the following corollary.

COROLLARY. In (H), *let M and M'' be analytic Riemann manifolds. Then a necessary and sufficient condition that there exist a harmonic map $\neq f_\infty$ homotopic to f_∞ is that $f_\infty(M)$ be a point or a closed geodesic.*

When M and M' are only C^∞, then the assumption of this corollary involving one point $f_\infty(x_0)$ has to be made at every point of $f_\infty(M)$:

(I) *Let $f_\infty\colon M \to M'$ be a harmonic map and suppose that all sectional curvatures of M' are negative at every point of $f_\infty(M)$. Then a necessary and sufficient condition that there exist a harmonic map $\neq f_\infty$ homotopic to f_∞ is that either* (i) *$f_\infty(M)$ is a point or* (ii) *$f_\infty(M)$ is a closed geodesic γ. In the latter case, all har-*

monic maps homotopic to f_∞ are obtained by a "rotation of f_∞" (i.e., by moving each point $f_\infty(x)$ a fixed oriented distance u along γ) *and, conversely, every "rotation of f_∞" is a harmonic map homotopic to f_∞.*

It should be noted that in (H) and (I) there are no curvature assumptions on M. This contrasts with the corollary of **(1**, p. 124**)**.

The proof of assertions (A)–(I) will depend on a priori estimates of Eells and Sampson **(1)** and on a device of Lewis **(5)**, as exploited by Hartman **(3, 4)**.

Assertion (C) shows that the example **(1**, p. 155**)** of a solution

$$f(x, t)\colon M \times [0, \infty) \to M'$$

which does not have a precompact range is superfluous. It can be mentioned, however, that a modification of this example shows that the part of (C) which is converse to (B) is false if one drops the assumption that the sectional curvatures of M' are non-positive:

Let $v = v(u) > 0$ be an even real-valued function of class C^∞ for all real u such that $dv/du < 0$ for $u > 0$, so that v has a maximum at $u = 0$. Let M' be the surface of revolution $y^1 = v(u) \cos \phi$, $\quad y^2 = v(u) \sin \phi$, $\quad y^3 = u$ in R^3 and $M = S^1$, parametrized by the central angle θ. The corresponding equation (1.4) is

$$u_t = u_{\theta\theta} + v_u v_{uu}(1 + v_u^2)^{-1} u_\theta^2 - v v_u (1 + v_u^2)^{-1} \phi_\theta^2,$$
$$\phi_t = \phi_{\theta\theta} + 2(v_u/v) u_\theta \phi_\theta.$$

If $f\colon S^1 \to M'$ satisfies the initial condition $u_\theta = 0$, $\phi = \theta$ at $t = 0$, then the same is true for the solution for $t > 0$. Thus the initial value problem reduces to

$$u_t = -v v_u/(1 + v_u^2), \qquad u(0) = u_0.$$

If $u_0 = 0$, then the solution $u = 0$, $\phi = \theta$ for all $t \geqslant 0$, is independent of t, and is a harmonic map. If $u_0 > 0$, then it is easy to see that the solution exists for all $t \geqslant 0$ and $u \to \infty$ as $t \to \infty$. Of course, the harmonic map $\theta \to (u, \phi) = (0, \theta)$ is homotopic to the initial map $\theta \to (u, \phi) = (u_0, \phi)$.

In this example, the harmonic map $\theta \to (u, \phi) = (0, \theta)$ is an unstable stationary point of (1.4). Thus (D) is also false without the assumption on the sectional curvatures of M' in (1.1).

An obvious modification of this example shows that (E) is also false without this assumption.

2. Main lemma. Let $r'(y, z)$ denote the distance between two points $y, z \in M'$ in the metric on M'. If $f, g\colon M \to M'$ are two continuous maps, introduce the distance functions

$$|f, g|_\infty = \sup r'(f(x), g(x)) \qquad \text{for } x \in M, \tag{2.1$_\infty$}$$

$$|f, g|_p = \left(\frac{1}{V}\int_M |r'(f(x), g(x))|^p \, dV\right)^{1/p}, \tag{2.1$_p$}$$

https://doi.org/10.4153/CJM-1967-062-6 Published online by Cambridge University Press

where $1 \leqslant p < \infty$, dV is the element of volume on M (so that in local coordinates $dV = (\det g_{ik})^{\frac{1}{2}}\, dx$), and $V = \int_M dV$ is the volume of M.

LEMMA 2.1. *Let $F(x, u)$: $M \times [0, a] \to M'$ be of class C^1 and, for a fixed u, let $f(x, u, t)$ be the solution of (1.4) on $0 \leqslant t < t_1$ $(\leqslant \infty)$ reducing to $F(x, u)$ at $t = 0$. Then*

$$(2.2_\infty) \qquad D(t, \infty) = a \sup(g'_{\alpha\beta} f_u^\alpha f_u^\beta)^{\frac{1}{2}} \qquad \text{for } x \in M,\ 0 \leqslant u \leqslant a,$$

$$(2.2_p) \qquad D(t, p) = a\left\{\frac{1}{aV}\int_M \int_0^a (g'_{\alpha\beta} f_u^\alpha f_u^\beta)^{p/2}\, du\, dV\right\}^{1/p},$$

where $1 \leqslant p < \infty$, are non-increasing functions of t on $[0, t_1)$.

Only the case $p = \infty$ of this lemma will be needed below for the proofs of (A)–(I), but the proof for $1 \leqslant p < \infty$ will also be given. (Of course, the case $p = \infty$ can also be obtained as a consequence of the result for large p.)

Proof. For fixed u, $f(x, u, t)$ and $f_u = \partial f/\partial u$ are of class

$$C^1(M \times [0, t_1)) \cap C^\infty(M \times (0, t_1))$$

with the corresponding partial derivatives continuous functions of (x, u, t). This can be proved by a re-examination (and differentiation with respect to u) of the successive approximations used in the proof of the local existence theorem (0) for (1.4) in (**1**, Section 10C).

Let $f = f(x, u, t)$ and subscripts $t, u, i, j, \ldots$ denote partial differentiation with respect to $t, u, x^i, x^j, \ldots$. Define the function

$$(2.3) \qquad Q(x, u, t) = g'_{\alpha\beta}(f)\, f_u^\alpha f_u^\beta.$$

In view of the relations

$$\partial g'_{\alpha\beta}/\partial y^\sigma = g'_{\alpha\tau}\, \Gamma'^\tau_{\beta\sigma} + g'_{\beta\tau}\, \Gamma'^\tau_{\alpha\sigma},$$

differentiation of (2.3) with respect to t and x^i gives

$$(2.4) \qquad \tfrac{1}{2} Q_t = g'_{\alpha\beta} f_{ut}^\alpha f_u^\beta + g'_{\alpha\tau}\, \Gamma'^\tau_{\beta\sigma} f_t^\sigma f_u^\alpha f_u^\beta,$$

$$(2.5) \qquad \tfrac{1}{2} Q_i = g'_{\alpha\beta} f_{ui}^\alpha f_u^\beta + g'_{\alpha\tau}\, \Gamma'^\tau_{\beta\sigma} f_i^\sigma f_u^\alpha f_u^\beta.$$

From the last formula and (1.3) it follows that

$$\Delta(\tfrac{1}{2}Q) = g'_{\alpha\beta}(\Delta f_u^\alpha) f_u^\beta + g'_{\alpha\tau}\, \Gamma'^\tau_{\beta\sigma} f_u^\alpha f_u^\beta (\Delta f^\sigma)$$
$$+ g^{ij} f_{ui}^\alpha \frac{\partial}{\partial x^j}(g'_{\alpha\beta} f_\mu^\beta) + g^{ij} f_i^\sigma \frac{\partial}{\partial x^j}(g'_{\alpha\tau}\, \Gamma'^\tau_{\beta\sigma} f_u^\alpha f_u^\beta).$$

Carrying out the indicated differentiations and using (2.4), (1.4), and the differentiated form of (1.4) given by

$$f_{ut}^\alpha = \Delta f_u^\alpha + g^{ji} \frac{\partial \Gamma'^\alpha_{\lambda\mu}}{\partial y^\sigma} f_u^\sigma f_i^\lambda f_j^\mu + 2 g^{ij}\, \Gamma'^\alpha_{\lambda\mu} f_{iu}^\lambda f_j^\mu,$$

we obtain

$$(2.6)\qquad (\tfrac{1}{2}Q_t) = \Delta(\tfrac{1}{2}Q) + g^{ij} R'_{\lambda\alpha,\mu\beta} f^\alpha_u f^\beta_u f^\lambda_i f^\mu_j - g^{ij} g'_{\alpha\beta} \frac{D^\alpha f_u}{\partial x^i} \frac{D^\beta f_u}{\partial x^j},$$

where $R'_{\lambda\alpha,\mu\beta}$ is the curvature tensor

$$R'_{\lambda\alpha,\mu\beta} = g'_{\alpha\gamma}\left\{\frac{\partial \Gamma'^{\gamma}_{\lambda\mu}}{\partial y^\beta} - \frac{\partial \Gamma'^{\alpha}_{\beta\lambda}}{\partial y^\mu} + \Gamma'^{\gamma}_{\beta\tau}\Gamma'^{\tau}_{\lambda\mu} - \Gamma'^{\gamma}_{\mu\tau}\Gamma'^{\tau}_{\lambda\beta}\right\}$$

and

$$\frac{D^\alpha f_u}{\partial x^i} = f^\alpha_{ui} + \Gamma'^{\alpha}_{\lambda\mu} f^\lambda_u f^\mu_i.$$

On the case $p = \infty$. Thus, for a fixed u, $Q(x, u, t)$ satisfies the parabolic differential inequality $Q_t - \Delta Q \leqslant 0$. Hence, the maximum principle implies that if $0 \leqslant s \leqslant t$, then $\max_x Q(x, u, t) \leqslant \max_x Q(x, u, s)$ for every fixed $u \in [0, a]$; cf., e.g., (**8**, Theorem 4, p. 171). Consequently

$$\max_{x,u} Q(x, u, t) \leqslant \max_{x,u} Q(x, u, s).$$

This shows that $D(t, \infty)$ is non-increasing.

On the case $1 \leqslant p < \infty$. Define the functions

$$R(u, t) = \int_M Q^{\frac{1}{2}p}\, dV,$$

$$H(t) = V a^{1-p} D^p(t, p) = \int_0^a R(u, t)\, du.$$

In view of the relation

$$H_t(t) = \int_0^a R_t(u, t)\, du,$$

it suffices to show that $R_t(u, t) \leqslant 0$. From

$$\frac{R_t}{p} = \int_M Q^{\frac{1}{2}p-1} (\tfrac{1}{2}Q)_t\, dV$$

and (2.6) it is seen that $R_t/p = \mathrm{I} + \mathrm{II} + \mathrm{III}$, where

$$\mathrm{I} = \int_M Q^{\frac{1}{2}p-1} g^{ij} R'_{\lambda\alpha,\mu\beta} f^\alpha_u f^\beta_u f^\lambda_i f^\mu_j\, dV \leqslant 0,$$

$$\mathrm{II} = -\int_M Q^{\frac{1}{2}p-2} g^{ij} (g'_{\lambda\mu} f^\lambda_u f^\mu_u)\left(g'_{\alpha\beta} \frac{D^\alpha f_u}{\partial x^i} \frac{D^\beta f_u}{\partial x^j}\right) dV \leqslant 0,$$

$$\mathrm{III} = \int_M Q^{\frac{1}{2}p-1} \Delta(\tfrac{1}{2}Q) dV = -(p-2) \int_M Q^{\frac{1}{2}p-2} g^{ij} (\tfrac{1}{2}Q_i)(\tfrac{1}{2}Q_i)\, dV.$$

If $p \geqslant 2$, then $\mathrm{III} \leqslant 0$. If $1 \leqslant p \leqslant 2$, so that $|p - 2| \leqslant 1$, it follows from Schwarz's inequality that $\mathrm{II} + \mathrm{III} \leqslant 0$ for, by (2.5),

$$\tfrac{1}{2}Q_i = g'_{\alpha\beta} \frac{D^\alpha f_u}{\partial x^i} f^\beta_u.$$

Thus, in all cases, $1 \leqslant p < \infty$, we have that $R_t \leqslant 0$. This completes the proof of Lemma 2.1.

The usefulness of Lemma 2.1 arises from the following two remarks.

REMARK 1. *For* $1 \leqslant p \leqslant \infty$ *and* $0 \leqslant t < t_1$,

$$|f(\cdot, 0, t), f(\cdot, a, t)|_p \leqslant D(t, p). \tag{2.7}$$

This is trivial if $p = \infty$. If $0 \leqslant p < \infty$, the expression on the left of (2.7) does not exceed

$$\left\{\frac{1}{V}\int_M\left[\int_0^a (g'_{\alpha\beta} f_u^\alpha f_u^\beta)^{\frac{1}{2}} du\right]^p dV\right\}^{1/p}.$$

If $p = 1$, this is $D(t, 1)$ and (2.7) follows. If $1 < p < \infty$, an obvious application of Hölder's inequality to the inner integral gives (2.7).

REMARK 2. *For each fixed* x, *let* $F(x, u)$: $[0, a] \to M'$ *be a geodesic of minimal length for* $0 \leqslant u \leqslant a$ *and let* u *be proportional to the arc-length* $r'[F(x, 0), F(x, u)]$. *Then, for* $1 \leqslant p \leqslant \infty$,

$$D(0, p) = |F(\cdot, 0), F(\cdot, a)|_p. \tag{2.8}$$

For if $L(x, u) = r'(F(x, 0)), F(x, u))$, so that $u = aL(x, u)/L(x, a)$, then

$$g'_{\alpha\beta}(F)\, F_u^\alpha\, F_u^\beta = L^2(x, a)/a^2 = |r'(F(x, 0)), F(x, a))|^2/a^2$$

for $0 \leqslant u \leqslant a$. Since $f(x, u, 0) = F(x, u)$, the inner integral of (2.2_p), at $t = 0$, is $|r'(F(x, 0), F(x, a))|^p a^{1-p}$. Hence (2.8) follows for $1 \leqslant p < \infty$. The case $p = \infty$ follows by the same argument.

Lemma 2.1 and (2.7) imply the following

COROLLARY 2.1. *Under the conditions of Lemma* 2.1,

$$|f(\cdot, 0, t), f(\cdot, a, t)|_p \leqslant D(0, p), \qquad 1 \leqslant p \leqslant \infty, \tag{2.9}$$

for $0 \leqslant t < t_1$.

From Lemma 2.1 and Remarks 1 and 2, we obtain the following corollary. Recall that if M'' is a compact subset of M', then $\delta = \delta(M'') > 0$ is a number with the property that if $y \in M''$, $z \in M'$, and $r'(y, z) < \delta$, then there is a unique geodesic of minimal length joining y and z.

COROLLARY 2.2. *Let* M'' *be a compact subset of* M'. *Let* $f_0(x, t)$ *be a solution of* (1.4) *for* $x \in M$, $0 \leqslant t < t_1$ $(\leqslant \infty)$ *with its range in* M''. *Let* $f_1(x)$: $M \to M'$ *be a* C^1 *map satisfying*

$$|f_0(\cdot, 0), f_1|_\infty < \delta = \delta(M''). \tag{2.10}$$

Then the solution $f_1(x, t)$ *of* (1.4) *reducing to* f_1 *at* $t = 0$ *exists for* $0 \leqslant t < t_1$ *and*

$$|f_0(\cdot, t), f_1(\cdot, t)|_p \tag{2.11}$$

is non-increasing on $[0, t_1)$ *for* $1 \leqslant p \leqslant \infty$.

Proof. For a fixed x, let $F(x, u)$: $[0, 1] \to M'$ be the unique geodesic joining $f_0(x, 0)$ and $f_1(x)$ with u proportional to the arc-length parameter. Then $F(x, u) \in C^1(M \times [0, 1])$ and Remarks 1 and 2 show that if $f_1(x, t)$ exists on an interval $[0, T]$, $0 \leqslant T < t_1$, then

$$|f_0(\cdot, T), f_1(\cdot, T)|_p \leqslant |f(\cdot, 0), f_1(\cdot)|_p$$

for $1 \leqslant p \leqslant \infty$. In particular,

$$|f_0(\cdot, T), f_1(\cdot, T)|_\infty < \delta,$$

Hence (2.11) holds on any common interval $[0, t_0) \subset [0, t_1)$ of existence of $f_0(x, t)$ and $f_1(x, t)$.

But if $0 < t_0 < t_1$, then the range of $f_1(x, t)$: $M \times [0, t_0) \to M'$ is in a compact set and the local existence theorem for (1.4) mentioned before (α) above implies that $f_1(x, t)$ can be defined beyond t_0. This completes the proof.

3. Proof of (A). Let the initial value problem (1.4)–(1.4_0) have a solution $f(x, t)$ on $M \times [0, t_1)$, where $t_1 < \infty$. In order to prove (A), it suffices to show that $f(x, t)$ can be continued as a solution of (1.4) beyond $t = t_1$. In view of the local existence theorem (**1**, Theorem 10B, p. 154) quoted as (0) above, it is enough to prove that the range of $f(x, t)$: $M \times [0, t_1) \to M'$ is contained in a compact subset of M'.

To this end, apply Corollary 2.1 to $F(x, u) = f(x, u)$ for $0 \leqslant u \leqslant a$ with $a = t_1/2$, where $f(x, u, t) = f(x, u + t)$ for $0 \leqslant t < t_1/2$. By (2.9),

$$|f(\cdot, t), f(\cdot, t + t_1/2)|_\infty \leqslant D(0, \infty) \qquad \text{for } 0 \leqslant t < t_1/2.$$

Since the range of $f(x, t)$: $M \times [0, t_1/2] \to M'$ is compact, it follows that the range of $f(x, t)$ is also in a compact set of M' for $x \in M$, $t_1/2 \leqslant t < t_1$. This proves (A).

4. Proof of (B). Let $f(x, t)$ be a solution of (1.4)–(1.4_0) for all $t \geqslant 0$ with its range in a compact subset of M'. By (β), there exists an unbounded t-sequence $0 < t_1 < t_2 < \ldots$ such that $f_\infty(x) = \lim f(x, t_n)$ exists uniformly on M and is a harmonic map of M into M'. In particular,

$$\liminf_{t \to \infty} |f(\cdot, t), f_\infty|_\infty = 0.$$

Hence Corollary 2.2 implies that $|f(\cdot, t), f_\infty|_\infty$ is non-increasing for large t. Thus

$$(4.1) \qquad |f(\cdot, t), f_\infty|_\infty \to 0 \qquad \text{as } t \to \infty,$$

i.e., $f(\cdot, t) \to f_\infty$ in $C^0(M)$ as $t \to \infty$. The a priori estimates of (**1**) (as used in the proof of the theorem in Section 11A, pp. 156–157) show that $f(\cdot, t) \to f_\infty$ in $C^k(M)$ for every k.

5. Proof of (C). Only the following partial converse of (B) has to be proved: If $f(x, t)$: $M \times [0, \infty) \to M'$ is a solution of (1.4) and there exists a harmonic map f_1: $M \to M'$ homotopic to $f(\cdot, 0)$, then the range of $f(x, t)$ is in a compact subset of M'.

Let $F(x, u)$: $M \times [0, 1] \to M'$ be a homotopy from $F(x, 0) = f(x, 0)$ to $F(x, 1) = f_1(x)$. It can be supposed that $F(x, u) \in C^1 (M \times [0, 1])$. (For if $F(x, u)$ is only continuous, suppose that F is independent of u for u near 0 and 1. Considering F as a map of the manifold $M \times (0, 1)$ without boundary into M', it follows that there is a C^1 map of $M \times (0, 1) \to M'$ coinciding with F for u near 0 and 1; cf., e.g. (**6**, Exercise (c), top of p. 39).)

Hence, Corollary 2.1 implies that

$$|f(\cdot, t), f_1|_\infty \leqslant D(0, \infty) \qquad \text{for } 0 \leqslant t < \infty.$$

This proves (C).

6. Proof of (D). Let f_∞: $M \to M'$ be a harmonic map. Then, by Corollary 2.2 and (A), there is a $\delta > 0$ with the property that if f_0: $M \to M'$ is a C^1 map and $|f_\infty, f_0|_\infty < \delta$, then the solution $f(x, t)$ of (1.4) reducing to f_0 at $t = 0$ exists for $t \geqslant 0$ and satisfies $|f_\infty, f(\cdot, t)|_\infty < \delta$. This is the first part of (D). The last part follows from (B).

7. Proof of (E). Let $F(x, u)$: $M \times [0, 1] \to M'$ be a homotopy from $F(\cdot, 0) = f_{\infty 0}$ to $F(\cdot, 1) = f_{\infty 1}$. As above, it can be supposed that $F(x, u)$ is of class C^1. For a fixed u, let $f(x, u, t)$ be the solution of (1.4) for $t \geqslant 0$ reducing to $F(x, u)$ at $t = 0$. Thus, for each u, (B) and (C) imply that there exists a harmonic map $f_\infty(\cdot, u)$ such that

$$|f(\cdot, u, t), f_\infty(\cdot, u)|_\infty \to 0 \qquad \text{as } t \to \infty.$$

It has to be shown that $f_\infty(x, u)$ is continuous in (x, u). But this is a consequence of Corollary 2.2, which implies that if $0 \leqslant u, v \leqslant 1$, and $|v - u|$ is sufficiently small, then

$$|f(\cdot, u, t), f(\cdot, v, t)|_\infty \leqslant |F(\cdot, u), F(\cdot, v)|_\infty$$

for $0 \leqslant t < \infty$. Letting $t \to \infty$ gives

$$|f_\infty(\cdot, u), f_\infty(\cdot, v)|_\infty \leqslant |F(\cdot, u), F(\cdot, v)|_\infty. \tag{7.1}$$

This completes the proof.

8. Proof of (F). Let $f_{\infty 0}, f_{\infty 1}$, $F(x, u)$ be as in assertion (F). Let $f(x, u, t)$ and $f_\infty(x, u) = \lim f(x, u, t)$, $t \to \infty$, as in the last section. Thus $f_\infty(x, u)$ is continuous for $(x, u) \in M \times [0, 1]$, $f_\infty(\cdot, u)$ is a harmonic map for fixed u, and $f_\infty(\cdot, u) = f_{\infty 0}, f_{\infty 1}$ for $u = 0, 1$.

Let

$$Q(x, u, t) = g'_{\alpha\beta}(f) f_u^\alpha f_u^\beta,$$

https://doi.org/10.4153/CJM-1967-062-6 Published online by Cambridge University Press

where $f = f(x, u, t)$. Since $f_{\infty 0}(x) = f(x, 0, t)$ and $f_{\infty 1}(x) = f(x, 1, t)$ for $t \geqslant 0$, we have, by Lemma 2.1,

$$r'(f_{\infty 0}(x), f_{\infty 1}(x)) \leqslant \int_0^1 Q^{\frac{1}{2}}(x, u, t)\, du \leqslant D(t, \infty) \leqslant D(0, \infty), \tag{8.1}$$

where

$$D(t, \infty) = \sup_{x,v} Q^{\frac{1}{2}}(x, v, t). \tag{8.2}$$

The maximum of the function on the left of (8.1) is $|f_{\infty 0}, f_{\infty 1}|_\infty$, which equals $D(0, \infty)$ by Remark 2 following Lemma 2.1. Let this maximum be assumed at $x = x_0$, so that

$$D(0, \infty) \leqslant \int_0^1 Q^{\frac{1}{2}}(x_0, u, t)\, du \leqslant D(t, \infty) \leqslant D(0, \infty).$$

In particular,

$$Q^{\frac{1}{2}}(x_0, u, t) = \sup_{x,v} Q^{\frac{1}{2}}(x, v, t) = D(0, \infty). \tag{8.3}$$

Thus, for fixed u and t, $Q^{\frac{1}{2}}(x, u, t)$ takes its maximum at $x = x_0$ and this maximum is independent of t.

The proof of Lemma 2.1 shows that Q satisfies the parabolic differential inequality $Q_t - \Delta Q \leqslant 0$ for fixed u. Hence the strong maximum principle implies that $Q^{\frac{1}{2}}(x, u, t) = c_0(u)$ does not depend on (x, t); cf., e.g., (**8**, Theorem 4, p. 171). In view of (8.3), $c_0(u) = D(0, \infty)$ is independent of u.

Since $F(x, u) = f(x, u, 0)$, the definition of F gives

$$r'(f_{\infty 0}(x), f_{\infty 1}(x)) = \int_0^1 Q^{\frac{1}{2}}(x, u, 0)\, du.$$

Consequently,

$$r'(f_{\infty 0}(x), f_{\infty 1}(x)) = \int_0^1 Q^{\frac{1}{2}}(x, u, t)\, du = D(0, \infty)$$

is independent of x, and $f(x, u, t)$, for $0 \leqslant u \leqslant 1$, is the unique minimizing geodesic arc joining $f_{\infty 0}(x), f_{\infty 1}(x)$ with u proportional to the arc-length. Hence $f(x, u, t) \equiv F(x, u)$ for $t \geqslant 0$. In particular, $F(x, u) \equiv f_\infty(x, u)$ and the assertion (F) is proved.

9. Proof of (G). Let $f_{\infty 0}, f_{\infty 1}$ be given distinct homotopic, harmonic maps. Consider the set Ω of homotopies $f(x, u)\colon M \times [0, 1] \to M'$ from $f(\cdot, 0) = f_{\infty 0}$ to $f(\cdot, 1) = f_{\infty 1}$ such that, for fixed u, $f(\cdot, u)$ is a harmonic map, and that $f(x, u)$ is uniformly Lipschitz continuous with respect to u (i.e., there is a constant C such that

$$r'(f(x, u_2), f(x, u_1)) \leqslant C\, |u_1 - u_2|$$

for $x \in M$, $0 \leqslant u_1, u_2 \leqslant 1$). That the set Ω is not empty follows, for example, from the proof of (E) in § 7; cf. (7.1).

For $f(x, u) \in \Omega$, define

$$D(f) = \text{ess sup } (g'_{\alpha\beta}(f) f_u^\alpha f_u^\beta)^{\frac{1}{2}} \qquad \text{for } x \in M,\ 0 \leqslant u \leqslant 1,$$

and let

$$\omega = \inf D(f) \qquad \text{for } f \in \Omega.$$

Note that

$$r'(f(x, u_2), f(x, u_1)) \leqslant D(f)\, |u_2 - u_1|. \tag{9.1}$$

Let $f_{(1)}(x, u), f_{(2)}(x, u), \ldots$ be elements of Ω such that $D(f_{(k)}) \to \omega$ as $k \to \infty$. It is clear that there exists a compact set M'' of M' such that

$$f_{(k)}(x, u) \in M'' \qquad \text{for } x \in M,\ 0 \leqslant u \leqslant 1;\ k = 1, 2, \ldots . \tag{9.2}$$

According to the theorem in **(1**, Section 8B**)**, there exists a constant c_0 such that if $f\colon M \to M'$ is a harmonic map, then

$$g^{ij}(x)\, g'_{\alpha\beta}(f)\, f_i^\alpha f_j^\beta \leqslant c_0 E(f),$$

where $E(f)$ is the energy integral (1.6). By the corollary of assertion (F), $E(f(\cdot, u)) = E(f_{\infty 0})$ for all $f(x, u) \in \Omega$. Consequently, (9.2) and the last formula line show that the sequence of functions $f_{(k)}(x, u)\colon M \times [0, 1] \to M'$ is equicontinuous. Hence there exists a subsequence $f_{n(1)}, f_{n(2)}, \ldots$ such that

$$f_\infty(x, u) = \lim_{k\to\infty} f_{n(k)}(x, u)$$

exists uniformly in $M \times [0, 1]$. Furthermore, for a fixed u, $f_\infty(\cdot, u)$ is a harmonic map; cf. **(1**, Section 8 and the arguments in Section 11**)**. Also, $f_\infty(x, u)$ is uniformly Lipschitz continuous with respect to u; in fact,

$$r'(f_\infty(x, u_2)\, f_\infty(x, u_1)) \leqslant \omega\, |u_2 - u_1|.$$

Finally, $f_\infty(\cdot, 0) = f_{\infty 0}$ and $f_\infty(\cdot, 1) = f_{\infty 1}$. Thus $f_\infty(x, u) \in \Omega$.

Since the last formula line implies that $D(f_\infty) \leqslant \omega$, it follows from the definition of ω that $f_\infty(x, u) \in \Omega$ is a "minimizing" element, $D(f_\infty) = \omega$. Note that $f_{\infty 0} \not\equiv f_{\infty 1}$ implies that $\omega > 0$.

In order to complete the proof, it suffices to show that if $\delta = \delta(M'')$, $0 \leqslant u_1 < u_2 \leqslant 1$, $\omega(u_2 - u_1) < \delta$, then, for a fixed x, the arc $f_\infty(x, u)$, where $u_1 \leqslant u \leqslant u_2$, is a geodesic arc of length independent of x with $u - u_1$ proportional to arc-length. Suppose, if possible, that this is not the case. By (F), it follows that

$$r'(f_\infty(x, u_2), f_\infty(x, u_1)) \leqslant \omega(u_2 - u_1) < \delta$$

is independent of x. Let

$$d = r'(f_\infty(x, u_2), f_\infty(x, u_1)) = |f_\infty(\cdot, u_2), f_\infty(\cdot, u_1)|_\infty.$$

The assumption that $f_\infty(x, u)$ does not have the desired property for $u_1 \leqslant u \leqslant u_2$ implies that

$$d/(u_2 - u_1) < \sup(g'_{\alpha\beta}(f)\, f_{\infty u}^\alpha f_{\infty u}^\beta)^{\frac{1}{2}} \qquad \text{for } x \in M,\ u_1 \leqslant u \leqslant u_2.$$

https://doi.org/10.4153/CJM-1967-062-6 Published online by Cambridge University Press

Hence $d/(u_2 - u_1) < \omega$.

Without affecting these inequalities or the minimizing property $D(f_\infty) = \omega$ of f_∞, the function $f_\infty(x, u)$ on $u_1 \leqslant u_0 \leqslant u_2$ can be replaced by the homotopy supplied by (F) from $f_\infty(x, u_1)$ to $f_\infty(x, u_2)$ along geodesics.

Let $\theta < 1$ be near 1. In terms of θ, define a continuous, piecewise linear, increasing function $u = u(v)$ from $0 \leqslant v \leqslant 1$ onto $0 \leqslant u \leqslant 1$ with slope θ on I_1: $0 \leqslant v \leqslant u_1/\theta$ and on I_3: $1 - (1 - u_2)/\theta \leqslant v \leqslant 1$ and an appropriate slope $1/\theta_0 > 1$ on the complementary interval I_2. Obviously, $\theta_0 = \theta_0(\theta) \to 1$ as $\theta \to 1$. Define $F(x, v) = f_\infty(x, u(v))$. It is clear that $F \in \Omega$. On I_1 and I_3,

$$(g'_{\alpha\beta}(F)\, F_v^\alpha\, F_v^\beta)^{\frac{1}{2}} = \theta(g'_{\alpha\beta}(f)\, f_{\infty u}^\alpha\, f_{\infty u}^\beta)^{\frac{1}{2}} \leqslant \omega\theta$$

and, on I_2,

$$(g'_{\alpha\beta}(F)\, F_v^\alpha\, F_v^\beta)^{\frac{1}{2}} = d/\theta_0(u_2 - u_1) < \omega/\theta_0.$$

If θ, hence θ_0, is sufficiently near 1, then $D(F) < \omega$. But this contradicts $F \in \Omega$ and the definition of ω. Hence (G) is proved.

10. Proof of (H)**.** Let f_∞: $M \to M'$ be a harmonic map, x_0 a point of M such that

(10.1) the sectional curvatures of M' at $f_\infty(x_0)$ are negative.

Suppose that there exists a harmonic map $f_{\infty d} \not\equiv f_\infty$ homotopic to f_∞. By (E), it can be supposed that

$$d = |f_{\infty d}, f_\infty|_\infty > 0 \tag{10.2}$$

is fixed so small that, by (F),

$$r'(f_{\infty d}(x), f_\infty(x)) = d \qquad \text{for all } x \in M$$

and there exists a homotopy $f(x, u)$: $M \times [0, d] \to M'$ such that $f(x, u) \in C^\infty$, $f(\cdot, 0) = f_\infty$, $f(\cdot, d) = f_{\infty d}$, $f(\cdot, u)$: $M \to M'$ is a harmonic map for fixed u, $f(x, \cdot)$ for $0 \leqslant u \leqslant d$ is a geodesic arc with u as arc-length. In particular,

$$Q(x, u) = g'_{\alpha\beta}(f)\, f_u^\alpha f_u^\beta \equiv 1, \tag{10.3}$$

$$f_{uu}^\alpha + \Gamma'^\alpha_{\lambda\mu}(f)\, f_u^\lambda f_u^\mu = 0. \tag{10.4}$$

Let U be a neighbourhood of x_0 on M and $d_0 > 0$ a small positive number with the property that

(10.5) the sectional curvatures of M' are negative at $f(x, u)$ for $x \in U$, $0 \leqslant u \leqslant 4d_0$ $(\leqslant d)$.

From (10.3) and equation (2.6) in the proof of Lemma 2.1, we have, on $M \times [0, d]$,

$$R'_{\lambda\alpha,\mu\beta}\, f_u^\alpha f_u^\beta f_i^\lambda f_j^\mu = 0, \tag{10.6}$$

$$D^\alpha f_u/\partial x^i \equiv f_{ui}^\alpha + \Gamma'^\alpha_{\lambda\mu} f_u^\lambda f_i^\mu = 0. \tag{10.7}$$

https://doi.org/10.4153/CJM-1967-062-6 Published online by Cambridge University Press

(It may be of interest to remark that if (10.7) is considered a system of linear total differential equations for the functions f_u, then (10.6) is a consequence of the integrability conditions.)

In view of (10.5) and (10.6), there exist functions $c^i(x, u)$ on $U \times [0, 4d_0]$ satisfying

$$(10.8) \qquad f_i^\alpha(x, u) = c^i(x, u) f_u^\alpha(x, u) \qquad \text{for } \alpha = 1, \ldots, m \text{ and } i = 1, \ldots, n.$$

Since the vector $f_u \not\equiv 0$, by (10.3), it follows that the functions $c^i(x, u)$ are of class C^1 in (x, u) and of class C^∞ in x (for fixed u).

Differentiating (10.8) with respect to u,

$$f_{iu}^\alpha = c^i f_{uu}^\alpha + c_u^i f_u^\alpha,$$

and using (10.4) and (10.8),

$$(10.9) \qquad f_{iu}^\alpha + \Gamma'^\alpha_{\lambda\mu} f_u^\lambda f_i^\mu = c_u^i f_u^\alpha.$$

By (10.7), the left side is 0, and, by (10.3), $f_u \not\equiv 0$, so that $c_u^i \equiv 0$. Thus $c^i(x) = c^i(x, u)$ does not depend on u.

Differentiating (10.8) with respect to x^j and using (10.7),

$$(10.10) \qquad f_{ij}^\alpha + \Gamma'^\alpha_{\lambda\mu} f_i^\lambda f_j^\mu = c_j^i f_u^\alpha.$$

Since the left side of (10.10) is symmetric with respect to i and j,

$$\partial c^i / \partial x^j = \partial c^j / \partial x^i.$$

Thus on any open simply connected neighbourhood U_0 of x_0, $x_0 \in U_0 \subset U$, there exists a unique, real-valued function $\phi(x) \in C^\infty$ such that

$$\phi_i = \partial\phi / \partial x^i = -c^i$$

and $\phi(x_0) = 2d_0$. It will be supposed that U_0 is so small that $d_0 \leqslant \phi(x) \leqslant 3d_0$ for $x \in U_0$.

Let $|s| \leqslant d_0$ and replace u by $s + \phi(x)$. Then (10.8) becomes

$$0 = f_i^\alpha(x, s + \phi(x)) + \phi_i(x) f_u^\alpha(x, s + \phi(x)) = \partial f^\alpha(x, s + \phi(x)) / \partial x^i.$$

Hence, if $y_0(s) = f(x_0, s + 2d_0)$, then $f(x, s + \phi(x)) \equiv y_0(s)$ is independent of $x \in U_0$ for $|s| \leqslant d_0$.

Let $\gamma(x)$ denote the complete geodesic on M' containing the geodesic arc $f(x, u)$, $0 \leqslant u \leqslant d$ (and x fixed). In particular, $y_0(s) = f(x_0, s + 2d_0) \in \gamma(x_0)$ for $|s| \leqslant d_0$. On the other hand, $y_0(s) = f(x, s + \phi(x)) \in \gamma(x)$ for $|s| \leqslant d$. Hence $\gamma(x) \equiv \gamma(x_0)$ does not depend on $x \in U_0$.

Consequently, $f(x, 0) \in \gamma(x) = \gamma(x_0)$ for $x \in U_0$, so that

$$f_\infty(U_0) = f(U_0, 0) \subset \gamma(x_0).$$

This proves (H).

Remark 1. Let $y = y(s)$, $-\infty < s < \infty$, be an arc-length parametrization of the complete geodesic $\gamma(x_0)$ on M', such that $y(u) = f(x_0, u)$ for $0 \leqslant u \leqslant d$.

Since $f_\infty(U_0) \subset \gamma(x_0)$ and U_0 is simply connected, there exists a unique continuous function $s = s(x)$ defined on U_0 satisfying $s(x_0) = 0$ and

$$(10.11) \qquad f_\infty(x) = y(s(x)).$$

It is easy to see that $s(x) \in C^\infty$ and that

$$(10.12) \qquad f^\alpha_{\infty i}(x) = y^\alpha_s s_i, \qquad \text{where } y^\alpha_s = dy^\alpha/ds.$$

(If this is compared with (10.8), one can show that $s(x) = 2d_0 - \phi(x)$.) From (10.12) and (1.3), it follows that

$$(10.13) \qquad \Delta f^\alpha_\infty + g^{ij}\,\Gamma'^\alpha_{\lambda\mu}(f_\infty) f^\lambda_{\infty 1} f^\mu_{\infty j} = y^\alpha_s(s(x))\Delta s(x),$$

so that $f = f_\infty$ satisfies the partial differential equation (1.2) if and only if $\Delta s = 0$.

In particular, $s(x) \equiv 0$ or $s(x)$ has no local maximum or minimum on U_0. In the first case, $f_\infty(U_0) = f_\infty(x_0)$. In the second case, the map f_∞: $U_0 \to \gamma(x_0)$ is open; i.e., open subsets of U_0 are mapped onto open arcs on $\gamma(x_0)$.

Remark 2. Let $x^0 \in U_0$ and

$$\sum_\alpha \sum_i |f^\alpha_{\infty i}(x^0)| \neq 0; \qquad \text{e.g., } f^1_{\infty 1}(x^0) \neq 0,$$

so that $s_1(x^0) \neq 0$. Then, on a neighbourhood $U^0 \subset U_0$ of x^0, it is possible to make the change of variables

$$x \to x^* = (s(x) - s(x^0), x^2 - x^{02}, \ldots, x^n - x^{0n}),$$

which has the Jacobian determinant $s_1(x^0) \neq 0$ at $x = x^0$, and the point x^0 acquires the new coordinates $x^* = 0$. Then (10.11) goes over into the normal form $f_\infty(x^*) = y(x^{*1} + s(x^0))$ for $x^* = (x^{*1}, \ldots, x^{*n}) \in U^0$.

11. Proof of (I).

Necessity of (i) *or* (ii). It follows from (H) that if there exists a harmonic map $\neq f_\infty$ homotopic to f_∞, then $f_\infty(M)$ is contained in some complete geodesic γ of M'.

Suppose that $f_\infty(M)$ is not a point. It has to be shown that $f_\infty(M)$ is a closed geodesic curve. Suppose that this is not the case; then $f_\infty(M)$ is a closed subarc of γ. This is impossible since the map f_∞: $M \to \gamma$ is open, by the Remark 1 at the end of § 10. (Another contradiction can be obtained by the use of assertion (G) since $f_\infty(M)$ would be homotopic on γ to a point.)

Sufficiency of (i) *or* (ii). Since the case where $f_\infty(M)$ is a point is trivial, suppose that $f_\infty(M)$ is a closed geodesic curve γ on M'. Let γ: $y = y(s)$, $-\infty < s < \infty$, be an arc-length parametrization of γ, so that $y(s)$ is a periodic function of s.

Let $x_0 \in M$ and s_0 satisfy $f_\infty(x_0) = y(s_0)$. Then in a simply connected neighbourhood of x_0 there is a unique continuous function $s(x)$ of x satisfying (10.11) and $s(x_0) = s_0$. As in § 10, $s(x) \in C^\infty$ and (1.2) is equivalent to $\Delta s(x) = 0$.

https://doi.org/10.4153/CJM-1967-062-6 Published online by Cambridge University Press

Since f_∞: $M \to M'$ is a harmonic map, the same is true of $f(x, u)$ defined, for fixed u, by $f(x, u) = y(s(x) + u)$.

Completion of proof. It has to be shown that all harmonic maps homotopic to f_∞ are "rotations of f_∞". But this is clear from (G) and the proof of (H) in § 10.

References

1. J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., *86* (1964), 109–160.

2. F. B. Fuller, *Harmonic mappings*, Proc. Nat. Acad. Sci., U.S.A., *40* (1954), 987–991.

3. P. Hartman, *On stability in the large for systems of ordinary differential equations*, Can. J. Math., *13* (1961), 480–492.

4. ——— *On the existence and stability of stationary points*, Duke Math. J., *33* (1966), 281–290.

5. D. C. Lewis, *Metric properties of differential equations*, Amer. J. Math., *71* (1949), 294–312.

7. J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math., *63* (1956), 20–64.

6. J. R. Munkres, *Elementary differential topology*, Ann. of Math. Studies, No. 54 (Princeton, 1963).

8. L. Nirenberg, *A strong maximal principle for parabolic equations*, Comm. Pure Appl. Math., *6* (1953), 153–177.

The Johns Hopkins University