ON CENTRAL PRIMITIVE IDEMPOTENT MEASURES

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Let S be a compact semitopological semigroup and let $P(S)$ be the convolution semigroup of probability measures on S. An idempotent measure μ in $P(S)$ is defined to be primitive if and only if the only idempotent measures in $\mu P(S)\mu$ are μ and the zero element m of $P(S)$. In a previous paper [2] we give some characterization of primitive idempotent measures on S. Let $\Pi(P(S))$ be the set of primitive idempotents in $P(S)$ and let Π_c be the set of central primitive idempotents in $P(S)$. It is shown in [1] that $\Pi(P(S))$ is neither an ideal nor even a subsemigroup of $P(S)$ in general. The purpose of this paper is to investigate the structure of Π_c.

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In the following theorem we use Pym's decomposition of an idempotent measure. Notations and terminology used may be found in the references numbered ([2] and [5]).

THEOREM 1. *If the minimal* (*two-sided*) *ideal* $K(S)$ *of* S *is not a group, then* $\Pi_c = \phi$.

PROOF. Suppose there is a measure μ in Π_c. Then, since $K(S)$ is not a group, $\mu \in K(P(S))$. Hence $\operatorname{supp} \mu \cap K(S) \neq \phi$ ([2], Theorem 1). Let e be an idempotent in $\operatorname{supp} \mu \cap K(S)$. We denote the set of idempotents in $K(S)e$ and $eK(S)$ by E and F respectively and write $G = eK(S)e$.

It is easy to see that $e \in E \cap F$. Moreover, since $K(S$ is not a group, either E or F contains an idempotent other than e. Suppose $f \in F$ and $f \neq e$.

Let $\mu = \mu_E m_G \mu_F$ be the Pym's decomposition of μ with respect to the idempotent e, where m_G is the Haar measure of G ([2], Theorem 1). Let $\nu = \delta_e m_G \delta_f$ where δ_x is the unit mass point mass at x in S. Then, since $\mu\nu = \nu\mu$,

$$\mu_E m_G \mu_F \delta_e m_G \delta_f = \delta_e m_G \delta_f \mu_E m_G \mu_F.$$

Now since $\operatorname{supp}(\mu_F \delta_e) \subseteq G$ and $\operatorname{supp}(\delta_f \mu_E) \subseteq G$, we see

$$\mu_E m_G \delta_f = \delta_e m_G \mu_F$$

Therefore $\mu_E = \delta_e, \mu_F = \delta_f$.

On the other hand, let $\tau = \delta_f m_G \delta_e$. Similar arguments show that $\mu_E = \delta_f$. Hence $e = f$, a contradiction. We conclude that $\Pi_c = \phi$.

It thus remains for us to discuss the case in which $K(S)$ is a group. It is known that $K(S)$ is a group if and only if $K(S)$ is a compact group and that $P(S)$ has a zero element m if and only if $K(S)$ is a group (see, for example, [2]).

THEOREM 2. *Let $K(S)$ be a group. Then Π_c is a semigroup with the mulitplication*

$$\mu\nu = \begin{cases} \mu & \text{if } \mu = \nu \\ m & \text{if } \mu \neq \nu \end{cases}$$

for μ, $\nu \in \Pi_c$.

PROOF. Let μ, ν be in Π_c. Then, since μ, ν are central, $\mu\nu$ is an idempotent measure in both $\mu P(S)\mu$ and $\nu P(S)\nu$. Now since μ, ν are primitive idempotents, we have $\mu = \nu$ or $\mu\nu = m$.

COROLLARY. *Let $K(S)$ be a group and let μ, ν be in Π_c. Then the supports of μ, ν are either disjoint or both contained in $K(S)$.*

We omit the proof of the corollary, all we need is to point out that the support of a central idempotent measure is a compact subgroup in S ([2] Theorem 2).

Although in general the limit of a net of idempotents may not be an idempotent in a compact semitopological semigroup, we still have the next theorem.

THEOREM 3. *Let $K(S)$ be a group and let μ, ν be cluster points of Π_c in $P(S)$. Then $\mu\nu = m$. In particular $\mu^2 = m$ for each cluster point μ of Π_c in $P(S)$.*

PROOF. Since $P(S)$ is weak* compact Hausdorff space, there exist two nets of distinct measures $\{\mu_\alpha : \alpha \in D_1\}$ and $\{\nu_\beta : \beta \in D_2\}$ in Π_c such that $\{\mu_\alpha\}$ and $\{\nu_\beta\}$ converge to μ and ν respectively; where D_1 and D_2 are two directed sets. By the separately continuity of multiplication, the net $\{\mu_\alpha \nu_\beta : \beta \in D_2\}$ converges to $\mu_\alpha \nu$ for each $\alpha \in D_1$. But $\mu_\alpha \nu_\beta = m$ if $\mu_\alpha \neq \nu_\beta$ by the above theorem, so $\mu_\alpha \nu = m$ for each $\alpha \in D_1$. It follows that $\mu\nu = m$.

References

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