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ON INEQUALITIES COMPLEMENTARY TO JENSEN'S

PAUL R. BEESACK

In a paper published in 1975 [**1**, § 3], D. S. Mitrinovič and P. M. Vasič used the so-called "centroid method" to obtain two new inequalities which are complementary to (the discrete version of) Jensen's inequality for convex functions. In this paper we shall present a very general version of such inequalities using the same geometric ideas used in [**1**] but not using the centroid method itself. At the same time we shall extend the domain of the inequalities (even in the discrete case), and clarify the value of the constant (λ) appearing in the inequality. We give applications of the theorems to some general means and also to the classical means. Our first result is given as

THEOREM 1. *Let ν be a nonnegative measure on a σ-algebra of subsets of a set D and let q, f be real ν-measurable functions on D such that $q(x) > 0$, $-\infty < x_1 \leqq f(x) \leqq x_2 < \infty$ for all $x \in D$ and $\int_D q d\nu = 1$. Let ϕ be a convex function on $I = [x_1, x_2]$ such that $\phi''(x) \geqq 0$ with equality for at most isolated points of I (so ϕ is strictly convex on I). If either* (i) *$\phi(x) > 0$ for all $x \in I$, or* (i′) *$\phi(x) > 0$ for $x_1 < x < x_2$ with either $\phi(x_1) = 0$, $\phi'(x_1) \neq 0$ or $\phi(x_2) = 0$, $\phi'(x_2) \neq 0$, or* (ii) *$\phi(x) < 0$ for all $x \in I$, or* (ii′) *$\phi(x) < 0$ for $x_1 < x < x_2$ with precisely one of $\phi(x_1) = 0$, $\phi(x_2) = 0$, then*

$$(1) \qquad \int_D q\phi(f)d\nu \leqq \lambda\phi\left(\int_D qfd\nu\right)$$

holds for some $\lambda > 1$ in cases (i), (i′) *or $\lambda \in (0, 1)$ in cases* (ii), (ii′). *More precisely, a value of λ (depending on x_1, x_2, ϕ) for* (1) *may be determined as follows: set $\mu = [\phi(x_2) - \phi(x_1)]/(x_2 - x_1)$. If $\mu = 0$, let $x = \bar{x}$ be the unique solution of the equation $\phi'(x) = 0$ $(x_1 < \bar{x} < x_2)$; then $\lambda = \phi(x_1)/\phi(\bar{x})$ suffices for* (1). *In case $\mu \neq 0$, let $x = \bar{x}$ be the unique solution in $[x_1, x_2]$ of the equation*

$$(2) \qquad g(x) = \mu\phi(x) - \phi'(x)[\phi(x_1) + \mu(x - x_1)] = 0;$$

then $\lambda = \mu/\phi'(\bar{x})$ suffices for (1). *Moreover we have $x_1 < \bar{x} < x_2$ in the cases* (i), (ii). *Moreover equality holds in* (1) *if and only if $f(x) = x_i$ for $x \in D_i$ where D_1, D_2 are ν-measurable subsets of D such that $D = D_1 \cup D_2$ and*

$$\bar{x} = x_1 \int_{D_1} qd\nu + x_2 \int_{D_2} qd\nu.$$

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Proof. We note that both integrals in (1) exist since both f and $\phi(f)$ are bounded measurable functions. In all cases, $\phi'(x)$ is continuous and strictly increasing on I so that by the mean value theorem applied to μ we have

$$(3) \qquad \phi'(x_1) < \mu < \phi'(x_2).$$

As in [**1**] we consider the points $A(x_1, \phi(x_1))$, $B(x_2, \phi(x_2))$ on the convex curve $y = \phi(x)$. The equation of the chord AB is

$$y = \phi(x_1) + \mu(x - x_1) \equiv m(x).$$

We also consider the family of convex curves with equations $y = \lambda\phi(x)$ $(\lambda > 0)$, and show there is a unique $\lambda > 0$ such that the curve will be tangent to the line AB at a point $\bar{P}(\bar{x}, \lambda\phi(\bar{x}))$ with $\bar{x} \in I$ (in fact $x_1 < \bar{x} < x_2$ in cases (i), (ii)). This is the case if and only if the pair of equations

$$(4) \qquad \lambda\phi'(x) = \mu,$$

$$(5) \qquad \lambda\phi(x) = m(x)$$

have a unique solution $(\bar{x}, \lambda)$ with $\bar{x} \in I$, $\lambda > 0$. In case $\mu = 0$ equations (4), (5) reduce to $\lambda\phi'(x) = 0$, $\lambda\phi(x) = \phi(x_1)$. If $\phi(x_1) \neq 0$, these equations have the unique solution $(\bar{x}, \lambda)$ determined by $\phi'(\bar{x}) = 0$, $\lambda = \phi(x_1)/\phi(\bar{x})$ where we observe that $x_1 < \bar{x} < x_2$ by the mean value theorem applied to μ. The case $\phi(x_1) = 0$ is impossible when $\mu \neq 0$ since then $\phi(x_2) = 0$ also, which is not the case. Note that $\lambda > 0$ when $\phi(x_1) \neq 0$.

When $\mu \neq 0$ we shall first consider only the cases (i), (ii). By (4), $\lambda \neq 0$ and eliminating λ from the pair of equations (4), (5), we see that $x = \bar{x}$ must be a solution of equation (2). We now show (as in [**1**]) that this equation has a unique solution on (x_1, x_2). First note that

$$g(x_1) = \phi(x_1)(\mu - \phi'(x_1)),\ g(x_2) = \phi(x_2)(\mu - \phi'(x_2));$$

since $\phi(x_1)$, $\phi(x_2)$ have the same sign, it follows from (3) that $g(x_1)$, $g(x_2)$ have opposite sign. Thus g has at least one zero on (x_1, x_2). Moreover,

$$g'(x) = -m(x)\phi''(x)$$

does not change sign on $[x_1, x_2]$. For, the linear function $m(x)$ has $m(x_i) = \phi(x_i)$ for $i = 1, 2$, and hence is either always positive in case (i) or always negative in case (ii), on $[x_1, x_2]$. It follows that g is a strictly monotonic function on $[x_1, x_2]$, and thus equation (2) has a unique solution $x = \bar{x} \in (x_1, x_2)$. Moreover $\phi'(\bar{x}) \neq 0$ since if it were then setting $x = \bar{x}$ in (2) would imply $0 = \mu\phi(\bar{x})$ which is impossible when $\mu \neq 0$ because $\phi(x) \neq 0$ on (x_1, x_2). If we now take $\lambda = \mu/\phi'(\bar{x})$ then it is easy to see that the pair $(\bar{x}, \lambda)$ satisfies equations (4), (5) and is the only such pair with $x_1 < \bar{x} < x_2$.

As for the value of λ, by (5) and the strict convexity of ϕ we obtain

$$\lambda\phi(\bar{x}) = \phi(x_1) + \mu(\bar{x} - x_1) = \left(1 - \frac{\bar{x} - x_1}{x_2 - x_1}\right)\phi(x_1) + \frac{\bar{x} - x_1}{x_2 - x_1}\phi(x_2)$$

or

(6) $\lambda\phi(\bar{x}) > \phi(\bar{x})$.

From this it follows that $\lambda > 1$ in case (i) and $\lambda < 1$ in case (ii). It remains to show that $\lambda > 0$ in case (ii) when $\mu \neq 0$. This follows from (5) since $\lambda\phi(\bar{x}) = m(\bar{x})$ and, as noted above, ϕ and m have the same sign on I.

As for the cases (i′), (ii′), which are relaxed versions of (i), (ii) respectively, we omit details but note that in case (i′), if $\phi(x_1) = 0$, $\phi'(x_1) \neq 0$ then we necessarily have $\phi'(x_1) > 0$, $\mu > 0$ while if $\phi(x_2) = 0$, $\phi'(x_2) \neq 0$ we must have $\mu < \phi'(x_2) < 0$. For the first of these, we have

$$g(x) = \mu\phi(x) - \mu(x - x_1)\phi'(x) = \mu(x - x_1)[\phi'(X) - \phi'(x)]$$

for $x_1 < X < x \leqq x_2$ whence $g(x) < 0$ for $x_1 < x \leqq x_2$ so $\bar{x} = x_1$ is the unique solution of (2) on $[x_1, x_2]$ and equations (4), (5) clearly have the unique solution $x = x_1$ on $[x_1, x_2]$ with $\lambda = \mu/\phi'(x_1) > 1$. A similar analysis applies to the second of (i′), where we now find $\bar{x} = x_2$, and $\lambda = \mu/\phi'(x_2) > 1$. For the two cases of (ii′) we observe in the first that $\phi'(x_1) < 0$ must hold, that $\bar{x} = x_1$, $0 < \lambda = \mu/\phi'(x_1) < 1$, and in the second that $\phi'(x_2) > 0$ must hold and $\bar{x} = x_2$, $0 < \lambda < \mu/\phi'(x_2) < 1$.

It only remains to prove the inequality (1) with the value of λ determined as above. To prove this we note that, since the line AB is tangent to the graph of the strictly convex (since $\lambda > 0$) function $\lambda\phi(x)$ at the point $\bar{P}$, we have for all $x \in I$

$$\lambda\phi(x) \geqq m(x) = \phi(x_1) + \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}(x - x_1),$$

with equality only for $x = \bar{x}$. We may take $x = \int_D qf d\nu$ since this $x \in I$. This gives

$$\lambda\phi\left(\int_D qf d\nu\right) \geqq \phi(x_1) + \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}\left(\int_D qf d\nu - x_1\right)$$
$$= \int_D q\left\{\phi(x_1) + \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}(f - x_1)\right\}d\nu \geqq \int_D q\phi(f)d\nu$$

precisely as at (6). Equality holds at the last step if and only if $f(x) = x_1$ or x_2 on ν-measurable subsets D_1 or D_2 of D. Hence equality holds in (1) precisely for such f where, in addition,

$$\bar{x} = \int_D qf d\nu = x_1 \int_{D_1} q d\nu + x_2 \int_{D_2} q d\nu.$$

In case the measure $Q(A) \equiv \int_A q d\nu$ is atomless, we observe that given any $\bar{x} \in [x_1, x_2]$ such sets D_1, D_2 exist, but are not in general unique.

COROLLARY 1. *Let all the hypotheses of Theorem* 1 *hold except that now*

ϕ is concave on I with $\phi''(x) \leqq 0$ with equality for at most isolated points of I. Then

$$(1') \quad \int_D q\phi(f)d\nu \geqq \lambda\phi\left(\int_D qfd\nu\right),$$

where λ is determined precisely as before. Now, $\lambda > 1$ holds if $\phi(x) < 0$ on (x_1, x_2) and $0 < \lambda < 1$ if $\phi(x) > 0$ on (x_1, x_2). Equality holds in (1′) for precisely the same f (if any) as in Theorem 1.

This follows from the theorem applied to the convex function $\phi_1 = -\phi$.

Remark 1. When $\mu \neq 0$ (so $\lambda \neq 0$) the unique solution $x = \bar{x}$ of (2) is related to λ by $\bar{x} = (\phi')^{-1}(\mu/\lambda)$ by (4). Note that this is meaningful since ϕ' is continuous and strictly monotonic on I. Hence it follows from (2) or (5) that the value of λ in (1) or (1′), when $\mu \neq 0$, is a solution of the equation (cf. [**1**, (3.1.5.)])

$$(7) \quad \lambda\phi[(\phi')^{-1}(\mu/\lambda)] = \phi(x_1) + \mu[(\phi')^{-1}(\mu/\lambda) - x_1].$$

If this equation has a unique root $\lambda > 1$ (when $\phi > 0$, convex or $\phi < 0$, concave), or a unique root $\lambda \in (0, 1)$ (when $\phi < 0$, convex or $\phi > 0$, concave) then this is the value of λ for (1) or (1′). However, equation (7) need not have a unique such root, in which case λ should be determined by the method stated in the theorem. For example, if $D = [x_1, x_2] = [0, \frac{1}{2}]$, $\phi(x) = x^2 - 1 < 0$ we find $\mu = \frac{1}{2}$, $\phi'(x) = 2x$, $(\phi')^{-1}(x) = \frac{1}{2}x$ and (7) reduces to

$$16\lambda^2 - 16\lambda + 1 = 0$$

which has the two roots $\lambda = (2 \pm \sqrt{3})/4$ both of which are in $(0, 1)$. On the other hand, equation (2) reduces to

$$x^2 - 4x + 1 = 0$$

with the two roots $x = 2 \pm \sqrt{3}$. Of these only $\bar{x} = 2 - \sqrt{3}$ is in $[0, \frac{1}{2}]$ whence

$$\lambda = \mu/\phi'(\bar{x}) = (2 + \sqrt{3})/4.$$

Remark 2. In case ϕ is convex but changes sign on I, no result such as (1) can hold, in general. For example, take $D = [x_1, x_2] = [0, 2]$, $\phi(x) = x^2 - 1$, $\nu =$ Lebesgue measure, $f(x) \equiv x$, $q(x) \equiv \frac{1}{2}$. All of the hypotheses of the theorem are satisfied except that ϕ changes sign. In this case $\mu = 2$ and

$$\int_D q\phi(f)d\nu = \frac{1}{2}\int_0^2 (x^2 - 1)dx = \frac{1}{2}\left(\frac{8}{3} - 2\right) > 0,$$

but for any λ we have

$$\lambda\phi\left(\int_D qfd\nu\right) = \lambda\phi(1) = 0,$$

so (1) can not hold.

In case (i′), if we remove the condition $\phi'(x_1) \neq 0$, although it is still clear that there exists $\lambda > 1$ such that (1) holds, no such λ can be determined by our method. For example, if $\phi'(x_1) = 0 = \phi(x_1) < \phi(x)$ for $x_1 < x \leqq x_2$, then $\mu > 0$ and all the curves $y = \lambda\phi(x)$ $(\lambda > 0, \lambda \neq 1)$ meet the line AB only in the point $A(x_1, 0)$ where they are not tangent to the line AB. Similar remarks apply to the cases where $\phi(x) < 0$.

Remark 3. If ϕ satisfies all of the hypotheses of the theorem except that it changes sign on $[x_1, x_2]$ we may proceed as follows. Let $m = \min \phi(x)$ for $x \in I$, so $m < 0$ and set $\phi_1(x) = \phi(x) - 2m$ so $\phi_1(x) \geqq -m > 0$. By part (i) of the theorem, for an appropriate $\lambda_1 > 1$ we will have

$$\int_D q\phi_1(f)d\nu \leqq \lambda_1\phi_1\left(\int_D qfd\nu\right)$$

which reduces to

$$\text{(8)} \qquad \int q\phi(f)d\nu \leqq \lambda_1\phi\left(\int_D qfd\nu\right) + 2m(1 - \lambda_1).$$

Note that $2m(1 - \lambda_1) > 0$. In (8) λ_1 is determined as λ is in (1) but replacing ϕ by ϕ_1 throughout. Observe that $\mu_1 = \mu$ however so, in case $\mu = 0$ for example, one finds that

$$\text{(9)} \qquad \lambda_1 = (\phi(x_1) - 2m)/(\phi(\bar{x}) - 2m)$$

where $x = \bar{x}$ is the unique solution of $\phi'(x) = 0$ on (x_1, x_2). If $\mu \neq 0$ we have $\lambda_1 = \mu/\phi'(\bar{x})$ where $\bar{x}$ is the unique solution of the modified equation

$$\text{(10)} \qquad g(x) - 2m(\mu - \phi'(x)) = 0$$

having $\bar{x} \in (x_1, x_2)$.

THEOREM 2. [**1**, 3.2] *Let ν, D, q, f, x_1, x_2 be as in Theorem 1, and let $\phi(x)$ be any differentiable function on $I = [x_1, x_2]$ such that $\phi'(x)$ exists and is strictly increasing on I. Then we have*

$$\text{(11)} \qquad \int_D q\phi(f)d\nu \leqq \lambda + \phi\left(\int_D qfd\nu\right)$$

for some λ satisfying $0 < \lambda < (x_2 - x_1)[\mu - \phi'(x_1)]$ where

$$\mu = [\phi(x_2) - \phi(x_1)]/(x_2 - x_1).$$

More precisely, λ may be determined for (11) *as follows: let $x = \bar{x}$ be the* *unique solution of the equation* $\phi'(x) = \mu(x_1 < \bar{x} < x_2)$; *then*

$$\lambda = \phi(x_1) - \phi(\bar{x}) + \mu(\bar{x} - x_1)$$

suffices in (11). *Equality holds in* (11) *only for* $f(x) = x_i$ $(x \in D_i)$, *where* D_1, D_2 *are* ν-*measurable subsets of* D *such that* $D = D_1 \cup D_2$ *and*

$$\bar{x} = x_1 \int_{D_1} q d\nu + x_2 \int_{D_2} q d\nu,$$

when such sets exist.

Proof. As in [**1**], the proof is similar to that given for Theorem 1, but is much easier. Using the same notation as before, we again have (3) and now look for the convex curve with equation $y = \lambda + \phi(x)$ which is tangent to the chord AB at a point $(\bar{x}, \bar{y})$ with $x_1 < \bar{x} < x_2$. This will occur if and only if $\bar{x}$, λ now satisfy the pair of equations

$$(4') \qquad \phi'(x) = \mu$$

$$(5') \qquad \lambda + \phi(x) = m(x).$$

Since ϕ' is strictly increasing on I it follows from the mean value theorem that equation (4′) has a unique solution $\bar{x} \in (x_1, x_2)$, and then λ is uniquely determined from (5′) as

$$\begin{aligned}\lambda = m(\bar{x}) - \phi(\bar{x}) &= \phi(x_1) - \phi(\bar{x}) + \mu(\bar{x} - x_1) \\ &= (\bar{x} - x_1)[\mu - \phi'(X)] \quad \text{where } x_1 < X < \bar{x}.\end{aligned}$$

From this we also obtain

$$0 < \lambda < (x_2 - x_1)[\mu - \phi'(x_1)].$$

The proof of the inequality (11) is just as before since we now have, with this value of λ,

$$\lambda + \phi(x) \geqq m(x) = \phi(x_1) + \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}(x - x_1)$$

for all $x \in I$. Again we set $x = \int_D qf d\nu$ and use the strict convexity of ϕ on I to obtain (11). The equality conditions follow precisely as in Theorem 1.

COROLLARY 2. *Let all the hypotheses of Theorem* 2 *be satisfied except that* $\phi'(x)$ *is strictly decreasing on* I. *Then*

$$(11') \qquad \phi\left(\int qf d\nu\right) \leqq \lambda + \int_D q\phi(f) d\nu$$

where

$$0 < \lambda < (x_2 - x_1)[\phi'(x_1) - \mu]$$

with

$$\mu = [\phi(x_2) - \phi(x_1)]/(x_2 - x_1).$$

In fact we may take $\lambda = \phi(\bar{x}) - \phi(x_1) - \mu(\bar{x} - x_1)$ *where* $x = \bar{x}$ *is the unique solution of the equation* $\phi'(x) = \mu(x_1 < \bar{x} < x_2)$. *Equality holds in* (11′) *under precisely the same conditions as in* (11).

To prove this we need only apply Theorem 2 to the function $\phi_1 = -\phi$ for which $\mu_1 = -\mu$, etc.

Remark 4. In a recent paper, M. L. Slater [**2**, Theorem 2] gave a somewhat different companion inequality to Jensen's inequality, under different hypotheses. We state a version of this inequality which allows a comparison to be made with Theorems 1 and 2, using our notation, namely

$$(12)\quad \int_D q\phi(f)d\nu \leqq \phi\left(\int_D qf\phi'(f)d\nu \Big/ \int_D q\phi'(f)d\nu\right)$$

provided ϕ is convex and increasing (or convex and decreasing) on I, and the integrals exist with $\int_D q\phi'(f)d\nu \neq 0$. For a comparison with both Theorems 1 and 2 we may take $\phi(x) = ax^2 + \epsilon$ with $a > 0$, $\epsilon > 0$, $I = [0, 1]$, $q \equiv 1$, $f(x) \equiv x$, so $\mu = 1$. Using Lebesgue measure, the upper bound on the right side of (12) reduces to

$$\phi(2/3) = (4a/9) + \epsilon = B_1.$$

For Theorem 1 we find

$$\bar{x} = -\epsilon + \epsilon[1 + (\epsilon a)^{-1}]^{1/2},\ \lambda = \{1 + [1 + (\epsilon a)^{-1})^{1/2}\}/2$$

so the upper bound on the right side of (1) is

$$B_2 = \lambda\phi(\tfrac{1}{2}) = \frac{a + 4\epsilon}{4}\,\frac{1 + \sqrt{1 + (1/\epsilon a)}}{2}.$$

For Theorem 2 we obtain $\bar{x} = 1/(2a)$, $\lambda = 1/(4a)$, and the upper bound on the right side of (11) is

$$B_3 = \lambda + \phi(\tfrac{1}{2}) = \epsilon + \frac{a}{4} + \frac{1}{4a}.$$

From this it is easy to verify that

$$B_3 < B_1 \leftrightarrow a > 3/\sqrt{7}, \quad \text{and} \quad B_2 < B_3 \leftrightarrow 4\epsilon > a(a^2 + 4a\epsilon).$$

A comparison of B_1 and B_2 is not so easy. We leave it to the reader to verify that if $\epsilon a > 2$ and $a > \epsilon$ then $B_2 < B_1$, while if $\epsilon a < 2$ and $2a < \epsilon$ then $B_1 < B_2$. However, when $a = \epsilon = 1$ for example, the bound B_1 is best (= least) of the three, and if $a = 3$, $\epsilon = 1$, B_3 is best and B_2 is worst.

We now give some applications of our theorems to the comparison of different general integral mean values of a function f with respect to continuous strictly monotonic functions ψ, χ. In [**3**, p. 169] such mean values $M_\psi(f)$ are defined by

$$(13)\quad M_\psi(f) = \psi^{-1}\left\{\int_D q\psi(f)d\nu\right\}.$$

Here D, ν, q are as in Theorem 1 and f is a ν-measurable function on D with $a \leqq f(x) \leqq b$ for $x \in D$ while ψ: $I \rightarrow \mathbf{R}$ is continuous and strictly monotonic on $I = [a, b]$ with inverse function ψ^{-1}. The cases $a = -\infty$ or $b = +\infty$ are allowed; if $a = -\infty$ for example we interpret continuity at a as the (true) statement $\psi(x) \rightarrow \psi(-\infty)$ as $x \rightarrow -\infty$, where the value $\psi(-\infty)$ may be finite or infinite. In case $\chi : I \rightarrow \mathbf{R}$ satisfies the same hypotheses as ψ, then a necessary and sufficient condition for

$$(14)\quad M_\psi(f) \leqq M_\chi(f)$$

to hold for all such f and q is that either χ be increasing and $\phi = \chi(\psi^{-1})$ be a convex function or that χ be decreasing and ϕ be concave. (See also [**3**, p. 70].) The opposite inequality to (14) holds for all such f and q if and only if either χ is decreasing and ϕ is convex, or χ is increasing and ϕ is concave. These conditions for (14) or its opposite can, of course, also be stated in terms of the character of ψ and $\phi_1 = \psi(\chi^{-1})$: (14) holds precisely when either ψ is increasing and ϕ_1 is concave, or ψ is decreasing and ϕ_1 is convex; the opposite inequality to (14) holds when ψ is decreasing and ϕ_1 is concave, or ψ is increasing and ϕ_1 is convex. The direct equivalence of these conditions with the preceding can be shown by using the fact that $\phi_1 = \phi^{-1}$, and the fact that when ϕ is increasing ϕ^{-1} is convex (concave) if and only if ϕ is convex (concave), while when ϕ is decreasing ϕ^{-1} is convex (concave) if and only if ϕ is concave (convex). Because of this equivalence we shall state all of our results only in terms of the character of χ and ϕ.

To obtain complementary inequalities to (14) we shall require that $I \subset [0, \infty)$ so that it is meaningful to consider the sub- or super-multiplicity of χ^{-1} as we shall do. We shall also want to consider terms such as $\chi^{-1}(\lambda)$ or $\chi^{-1}(\lambda^{-1})$ for $\lambda > 0$, and hence to apply Theorem 1 to $\phi = \chi(\psi^{-1})$ we must have $\chi(x) > 0$ for $x \in I$.

THEOREM 3. *Let ψ, $\chi : \mathbf{R}^+ = (0, \infty) \rightarrow \mathbf{R}$ be continuous and strictly monotonic functions with inverse functions ψ^{-1}, χ^{-1}, and suppose $\chi(\mathbf{R}^+) = \mathbf{R}^+$. Let D, ν, q, be as in Theorem* 1 *and let $f : D \rightarrow \mathbf{R}$ be ν-measurable on D with*

$$-\infty < x_1 \leqq \psi(f(x)) \leqq x_2 \leqq \infty \quad \textit{for } x \in D,$$

so $\psi(f(D)) = [x_1, x_2]$. Let $\phi = \chi(\psi^{-1})$ and set

$$\mu = [\phi(x_2) - \phi(x_1)]/(x_2 - x_1).$$

(a) *Suppose* $\phi''(x) \geqq 0$ *with equality for at most isolated points of* $[x_1, x_2]$ *and let* $\lambda > 1$ *be determined as in Theorem* 1. *Then*

$$(15)\quad M_\chi(f) \leqq \chi^{-1}(\lambda)M_\psi(f)$$

if χ *is increasing and supermultiplicative on* $\mathbf{R}^+$, *while the opposite inequality holds if* χ *is decreasing and supermultiplicative. Moreover*

$$(16)\quad M_\chi(f) \leqq [\chi^{-1}(\lambda^{-1})]^{-1}M_\psi(f)$$

if χ *is increasing and submultiplicative, with the opposite inequality holding when* χ *is decreasing and submultiplicative.*

(b) *Suppose* $\phi''(x) \leqq 0$ *with equality for at most isolated points of* $[x_1, x_2]$, *and let* λ *be determined by Theorem* 1. *Then*

$$(15')\quad M_\chi(f) \geqq \chi^{-1}(\lambda)M_\psi(f)$$

if χ *is increasing and submultiplicative, with the opposite inequality holding when* χ *is decreasing and submultiplicative. Moreover,*

$$(16')\quad M_\chi(f) \geqq [\chi^{-1}(\lambda^{-1})]^{-1}M_\psi(f)$$

if χ *is increasing and supermultiplicative, while the opposite inequality holds if* χ *is decreasing and supermultiplicative.*

Proof. (a) We have $\phi(x) > 0$ for $x \in [x_1, x_2]$ since $\psi^{-1}([x_1, x_2]) \subset \mathbf{R}^+$. Thus case (i) of Theorem 1 applies to ϕ so that

$$(*)\quad \int_D q\phi[\psi(f)]d\nu \geqq \lambda\phi\left(\int_D q\psi(f)d\nu\right).$$

If now χ is increasing and supermultiplicative on $\mathbf{R}^+$, then χ^{-1} is increasing and submultiplicative on $\mathbf{R}^+$, so applying χ^{-1} to (*) we obtain

$$M_\chi(f) \leqq \chi^{-1}\left[\lambda\phi\left(\int_D q\psi(f)d\nu\right)\right] \leqq \chi^{-1}(\lambda)M_\psi(f).$$

If however χ is decreasing and supermultiplicative, then χ^{-1} is decreasing and also supermultiplicative, whence (*) implies

$$M_\chi(f) \geqq \chi^{-1}\left[\lambda\phi\left(\int_D q\psi(f)d\nu\right)\right] \geqq \chi^{-1}(\lambda)M_\psi(f).$$

This proves the parts of (a) concerning (15) and its opposite. The proof of (16) and its opposite follows in the same way by first writing (*) in the form

$$\lambda^{-1}\int_D q\chi(f)d\nu \leqq \phi\left(\int_D q\psi(f)d\nu\right)$$

before applying χ^{-1}.

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(b) To prove the parts of (b) we use Corollary 1 to obtain the opposite inequality to (*) and proceed from there.

Remark 5. When χ^{-1} is submultiplicative we have

$$0 < \chi^{-1}(1) \leqq [\chi^{-1}(1)]^2$$

so $\chi^{-1}(1) \geqq 1$, and also

$$\chi^{-1}(1) \leqq \chi^{-1}(\lambda)\chi^{-1}(\lambda^{-1}),$$

whence

$$\chi^{-1}(\lambda) \geqq [\chi^{-1}(\lambda^{-1})]^{-1} \quad \text{for } \lambda > 0.$$

Similarly,

$$\chi^{-1}(\lambda) \leqq [\chi^{-1}(\lambda^{-1})]^{-1} \quad \text{for } \lambda > 0$$

when χ^{-1} is supermultiplicative. The factors appearing on the right sides of (15) and (16) are therefore, in general, equal only when χ^{-1} (hence also χ) is actually multiplicative. Unfortunately, the better (= smaller) factor is the one which does not appear on the right hand side of (15), (16) for the case of χ in question.

Remark 6. We have stated Theorem 3 in a form to which case (i) of Theorem 1 applies, for simplicity. One could also state a form to which case (i′) of Theorem 1 applies: this would require ψ, χ to be continuous and strictly increasing on $[0, \infty)$ with $\chi(0) = 0$, $\psi(0) = x_1$ and $\psi(A) = x_2$ for some $A > 0$. We omit any details since we will never require such a result.

As an application of Theorem 3 we consider the classical means $M_r(f)$ defined for $r \in \mathbf{R}$ by

$$M_r(f) = \left\{ \int_D q(x) f^r(x) d\nu \right\}^{1/r}, \quad r \neq 0,$$

$$M_0(f) = \exp \left\{ \int_D q(x) \log f(x) d\nu \right\}.$$

See [**3**, Chapter 6]. Note that $M_0 = M_\psi$ for $\psi(x) = \log x$ while for $r \neq 0$, $M_r = M_\psi$ for $\psi(x) = x^r$. In [**3**, 6.11] it is shown that $M_r < M_s$ for $r < s$. In all cases it is assumed that $f(x)$ is positive and ν-measurable on D. For inequalities which are complementary to the inequality $M_r < M_s$ $(r < s)$, we suppose that

$$0 < X_1 \leqq f(x) \leqq X_2 \leqq \infty \quad \text{for } x \in D,$$

and for $r, s \neq 0$ let

$$\mu = (X_2^s - X_1^s)(X_2^r - X_1^r)^{-1},$$
$$B_{r,s} = \left(\frac{\mu r}{s}\right)^{1/r} \cdot \left\{\frac{X_1^s X_2^r - X_1^r X_2^s}{\left(1 - \frac{s}{r}\right)(X_2^r - X_1^r)}\right\}^{(1/s)-(1/r)}.$$

Then

(17) $M_s(f) \leqq B_{r,s} M_r(f),\ r < s (r, s \neq 0).$

For cases involving $r = 0$ or $s = 0$, let

$$\left.\begin{aligned} \mu_t &= (X_2^t - X_1^t)/\log (X_2/X_1), \\ B_t &= \left(\frac{\mu_t}{et}\right)^{1/t} X_1^{-1} \exp (X_1^t/\mu_t). \end{aligned}\right\} \quad (t \neq 0)$$

Then

(18) $M_s(f) \leqq B_s M_0(f)$ if $0 < s$,

(19) $M_0(f) \leqq B_r^{-1} M_r(f)$ if $r < 0$.

The inequalities (17)–(19), using a somewhat different notation, were first given by W. Specht [**4**] for the case that $d\nu$ denotes either one-dimensional Lebesgue measure or a counting measure associated with a finite sum. The same results were later obtained by G. T. Cargo and O. Shisha [**5**] who also dealt with the cases of equality. The techniques used in both [**4**] and [**5**] are quite different from that used here, and in particular do not deduce the inequalities as special cases of a general result.

To prove (17) we take $\psi(x) = x^r$, $\chi(x) = x^s$, so $\phi(x) = \chi(\psi^{-1}(x)) = x^{s/r}$. If $0 < r < s$ or $r < 0 < s$, ϕ is convex and χ is increasing so (15) applies, while if $r < s < 0$, ϕ is concave and χ is decreasing so the opposite inequality to (15′) applies. In all cases we obtain an inequality of the form (17) with $B_{r,s} = \chi^{-1}(\lambda) = \lambda^{1/s}$. To compute λ we observe that the required hypothesis $x_1 \leqq \psi(f(x)) = f^r(x) \leqq x_2$ reduces to

$$0 < X_1 = x_1^{1/r} \leqq f(x) \leqq x_2^{1/r} = X_2 \quad \text{when } r > 0,$$

or to

$$0 < X_1 = x_2^{1/r} \leqq f(x) \leqq x_1^{1/r} = X_2 \quad \text{when } r < 0.$$

We find that μ as given in Theorem 3 reduces to the formula preceding (17) in all cases, and note that $\mu > 0$ when $s/r > 0$ while $\mu < 0$ when $s/r < 0$. If one solves equation (2) one obtains the unique solution

$$\bar{x} = (s/r\mu)(x_1^{s/r} - \mu x_1)[1 - (s/r)]^{-1},$$

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whence

$$\lambda = \mu/\phi'(\bar{x}) = \left(\frac{\mu r}{s}\right)^{s/r}\{(x_1^{s/r} - \mu x_1)[1 - (s/r)]^{-1}\}^{1-(s/r)}.$$

Using the fact that $x_1 = X_1^r$, $x_2 = X_2^r$ for $r > 0$, while $x_1 = X_2^r$, $x_2 = X_1^r$ for $r < 0$, the factor $B_{r,s} = \lambda^{1/s}$ reduces to that given preceding (17) in all cases.

To prove (18) we take $\chi(x) = x^s$, $\psi(x) = \log x$, so $\phi(x) = \chi(e^x) = e^{sx}$ is convex and χ is increasing ($s > 0$), and (15) applies to yield an inequality of the form (18) with $B_s = \chi^{-1}(\lambda) = \lambda^{1/s}$. The hypothesis

$$x_1 \leqq \psi(f(x)) = \log f(x) \leqq x_2$$

reduces to

$$0 < X_1 = e^{x_1} \leqq f(x) \leqq e^{x_2} = X_2$$

and we find that $\mu > 0$ in Theorem 3 is here given by $\mu = \mu_s$. We leave it to the reader to verify that $\chi^{-1}(\lambda)$ reduces to the constant B_s defined just preceding (18).

As for (19), observe that we may not take $\chi(x) = \log x$, $\psi(x) = x^r$ and try to use (15) or (16), because χ changes sign. To overcome this difficulty we rewrite the inequality which is opposite to (15) in the form

$$M_\psi(f) \leqq [\chi^{-1}(\lambda)]^{-1}M_\chi(f)$$

and take $\psi(x) = \log x$, $\chi(x) = x^r$. Then $\phi(x) = e^{rx}$ is convex and χ is decreasing ($r < 0$), so the last-displayed inequality applies to produce (19) with $B_r^{-1} = [\chi^{-1}(\lambda)]^{-1}$. The details are precisely as for (18) but with s replaced by r.

As an example, the case $r = 1$, $s = 2$ reduces to the inequality

$$M_2(f) \leqq \frac{X_1 + X_2}{2\sqrt{X_1X_2}}M_1(f), \quad (0 < X_1 \leqq f(x) \leqq X_2),$$

which is a reverse Cauchy-Schwarz inequality.

We may also apply Theorem 2 and its corollary to obtain an analogue of Theorem 3 involving general means, namely

THEOREM 4. *Let $I = (a, b)$, $-\infty \leqq a < b < +\infty$, and let $\psi, \chi : I \rightarrow \mathbf{R}$ be continuous, strictly monotonic functions with $(0, \infty) \subset \chi(I)$. Let D, ν, q, be as in Theorem* 1 *and let $f : D \rightarrow I$ be ν-measurable on D with $-\infty < x_1 \leqq \psi(f(x)) \leqq x_2 < \infty$ for all $x \in D$, so that $\psi(f(D)) = [x_1, x_2]$. Let $\phi = \chi(\psi^{-1})$ and set*

$$\mu = [\phi(x_2) - \phi(x_1)]/(x_2 - x_1)$$

and suppose $\lambda > 0$ is determined as in Theorem 2 *in case* (a), *or as in Corollary* 2 *in case* (b).

(a) *If $\phi'(x)$ exists and is strictly increasing on $[x_1, x_2]$ and χ is superadditive on I, then*

$$(20) \quad M_\chi(f) \leqq \chi^{-1}(\lambda) + M_\psi(f)$$

if χ is increasing, while the opposite inequality holds if χ is decreasing.

(b) *If $\phi'(x)$ exists and is strictly decreasing on $[x_1, x_2]$ and χ is superadditive, then*

$$(21) \quad M_\psi(f) \geqq \chi^{-1}(\lambda) + M_\chi(f)$$

if χ is increasing, with the opposite inequality holding if χ is decreasing.

The proof of these results is so similar to that of Theorem 3 that we may omit it. We remark that if the range of χ (= domain of χ^{-1}) is large enough, in particular if $\chi(I) = \mathbf{R}$, and if χ is subadditive one may also obtain inequalities analogous to (16) and (16′) and their opposites. For example if χ is increasing and subadditive, and ϕ' is strictly increasing,

$$M_\chi(f) \leqq -\chi^{-1}(-\lambda) + M_\psi(f).$$

We also note that $\chi(x) = x^t$ $(0 < x < \infty)$ is superadditive if $t \geqq 1$, but is subadditive if either $t < 0$ or $0 < t \leqq 1$. This restricts the range of applicability of Theorem 4 to the classical means M_t. For $r, s \neq 0$ the only case we obtain is

$$(22) \quad M_s(f) \leqq C_{r,s} + M_r(f) \quad \text{for } 0 \neq r < s,\ s \geqq 1,$$

provided $0 < X_1 \leqq f(x) \leqq X_2$, where $C_{r,s}$ is defined by

$$\mu = (X_2^s - X_1^s)(X_2^r - X_1^r)^{-1},$$

$$C_{r,s} = \left\{\frac{X_1^sX_2^r - X_1^rX_2^s}{X_2^r - X_1^r} + \frac{s-r}{r}\left(\frac{\mu r}{s}\right)^{s/(s-r)}\right\}^{1/s}.$$

The proof of this follows from (20) using $\chi(x) = x^s$, $\psi(x) = x^r$; we omit the details. Similarly using $\chi(x) = x^s$ and $\psi(x) = \log x$ and (20) we can obtain

$$(23) \quad M_s(f) \leqq C_s + M_0(f) \quad \text{for } s \geqq 1,$$

where C_s is defined by

$$\mu = (X_2^s - X_1^s)/\log (X_2/X_1),$$

$$C_s = \left\{\frac{X_1^s \log X_2 - X_2^s \log X_1}{\log (X_2/X_1)} + \frac{\mu}{s}\left[\log\left(\frac{\mu}{s}\right) - 1\right]\right\}^{1/s}.$$

The inequalities (22), (23) are not best possible, and indeed any inequalities obtained from (20), (21), can only be best possible when χ is additive. Best values of the constants $K_{r,s}$ appearing in the inequalities $M_s(f) \leqq K_{r,s} + M_r(f)$ for all r, s with $r < s$ were indicated by B. Mond and O. Shisha [**6**, **7**] for the case of finite sums. (The constants $K_{r,s}$ can not be given explicitly but involve terms which are solutions of transcendental equations.)

We conclude by using Theorem 2 to give inequalities of type (18), (19) involving $M_0(f)$ and a general mean $M_\psi(f)$.

THEOREM 5. *Let* $\psi : \mathbf{R}^+ \to \mathbf{R}$ *be continuous and strictly monotonic with inverse function* ψ^{-1}, *and set* $\phi(x) = \log \psi^{-1}(x)$. *Let* D, ν, q *be as in Theorem* 1 *and let* $f : D \to \mathbf{R}^+$ *be* ν*-measurable with* $0 < X_1 \leqq f(x) \leqq X_2$ *for all* $x \in D$. *Suppose that* $\phi'(x)$ *exists and is either strictly increasing or strictly decreasing on an interval* $[x_1, x_2] = \psi(f(D))$ *where* $x_1 = \psi(X_1)$, $x_2 = \psi(X_2)$ *if* ψ *is increasing, or* $x_1 = \psi(X_2)$, $x_2 = \psi(X_1)$ *if* ψ *is decreasing. Let*

$$\mu = \log (X_2/X_1)[\psi(X_2) - \psi(X_1)]^{-1}$$

and let $x = \bar{x}$ *be the unique solution on* $[x_1, x_2]$ *of the equation* $\phi'(x) = \mu$. *Then*

$$M_0(f) \leqq [X_1/\psi^{-1}(\bar{x})] \exp [\mu(\bar{x} - \psi(X_1))]M_\psi(f) \tag{24}$$

if ϕ' *is strictly increasing on* $[x_1, x_2]$, *while*

$$M_\psi(f) \leqq [X_1/\psi^{-1}(\bar{x})]^{-1} \exp [-\mu(\bar{x} - \psi(X_1))]M_0(f) \tag{24$'$}$$

if ϕ' *is strictly decreasing on* $[x_1, x_2]$.

The proof follows (with some calculation) by applying Theorem 2 or its Corollary (with f replaced by $\psi(f)$) to give

$$\int_D q \log f d\nu \leqq \lambda + \log M_\psi(f)$$

when ϕ' is increasing, or

$$\log M_\psi(f) \leqq \lambda + \int_D q \log f d\nu$$

when ϕ' is decreasing. One then shows that, with λ as in Theorem 2 or Corollary 2, e^λ reduces to the factors appearing on the right side of (24) and (24′).

A second proof of (18) and (19) can be obtained from (24′) with $\psi(x) = x^s$, and from (24) with $\psi(x) = x^r$ respectively. (Recall that the μ in (18), (19) is the reciprocal of the μ in Theorem 5.)

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Carleton University,
Ottawa, Ontario