THE MINIMUM OF A DEFINITE TERNARY QUADRATIC FORM

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A purely algebraic solution for the problem of the extreme definite ternary quadratic form has been given by Mordell **(2)**. The equivalent geometric problem of the critical lattice of a sphere was solved by Mrs. Ollerenshaw **(3)** under a preliminary assumption that was said to be justified by "simple considerations" (p. 297). The present treatment eliminates this weakness, and avoids the use of trigonometric functions, without adding undue complications. It arose from a remark of Hilbert and Cohn-Vossen **(1**, p. 45**)** which states that their method for the analogous two-dimensional problem **(1**, pp. 35–37**)** can be generalized to any number of dimensions. However, the generalization is not trivial, even in three dimensions, and the four-dimensional treatment of Mrs. Ollerenshaw **(4)** involves a preliminary assumption analogous to that mentioned above.

We begin by briefly recapitulating Gauss's statement of the connection between the algebraic and geometric problems. Three independent vectors $\mathbf{t}_r$ $(r = 1, 2, 3)$ generate a system of vectors $u_r\mathbf{t}_r$ (u_r integers) which lead from the origin to the points of a lattice. We may regard (u_1, u_2, u_3) as affine coordinates for the lattice points. The square of the distance from $(0, 0, 0)$ to (u_1, u_2, u_3) is

$$(\sum u_r\mathbf{t}_r)^2 = \sum\sum a_{rs}\, u_r\, u_s,$$

where $a_{rs} = \mathbf{t}_r\cdot\mathbf{t}_s$. Thus the lattice determines a positive definite ternary quadratic form whose minimum m is the square of the minimum distance c between lattice points. If the vector $\mathbf{t}_r$ has Cartesian components (x_r, y_r, z_r), the determinant of the form is

$$D = \det\,(\mathbf{t}_r\cdot\mathbf{t}_s) = \det\,(x_r\,x_s + y_r\,y_s + z_r\,z_s)$$
$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = V^2,$$

where V is the volume of the elementary cell (or primitive parallelepiped) of the lattice. Since

$$\frac{D}{m^3} = \left(\frac{V}{c^3}\right)^2,$$

the form of smallest determinant for its minimum value corresponds to the lattice of smallest cell for its minimum distance. Such a lattice we now seek.

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Consider any point A of a given lattice whose elementary cell has volume V. Choose a lattice point B at the minimum distance c from A, and a lattice point C outside the line AB, at the shortest distance b ($\geqslant c$) from A. These points can always be chosen so that $\angle CAB \leqslant \frac{1}{2}\pi$, and the sides $a = BC$, $b = CA$, $c = AB$ of the triangle ABC satisfy

$$a \geqslant b \geqslant c, \quad a^2 \leqslant b^2 + c^2.$$

Let Δ and R denote the area and circumradius of this triangle, so that

$$16\Delta^2 = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2, \qquad 4R\Delta = abc.$$

In a parallel lattice plane nearest to the plane ABC, there is a lattice point D whose orthogonal projection D_1 on the plane ABC lies inside or on a side of the parallelogram $ABA'C$, and so, by choice of notation, inside or on a side of the triangle ABC. Denote by d the distance DD_1 from D to the plane ABC, so that

$$V = 2\Delta d.$$

Since none of AD, BD, CD is parallel to AB, all of them are greater than or equal to b. Since the triangle ABC has no obtuse angle, circles of radius R with centres at the vertices overlap in such a way that every point of the triangle except the circumcentre is inside at least one of them (**3**, p. 298, footnote). Therefore the distance of D_1 from at least one vertex is less than R, except that it is equal to R when D_1 is the circumcentre. Thus at least one of AD, BD, CD is less than or equal to $(R^2 + d^2)^{\frac{1}{2}}$, and consequently

$$b^2 \leqslant R^2 + d^2,$$

with equality only when D_1 is the circumcentre. Hence

$$\begin{aligned} V^2 &= (2\Delta d)^2 \geqslant 4\Delta^2(b^2 - R^2) \\ &= \tfrac{1}{4} b^2(-a^4 - b^4 - c^4 + 2b^2c^2 + c^2a^2 + 2a^2b^2) \\ &= \tfrac{1}{2} c^6 + \tfrac{1}{4} c^2(b^2 - c^2)(3b^2 + 2c^2) + \tfrac{1}{4} b^2(a^2 - b^2)(b^2 + c^2 - a^2) \\ &\geqslant \tfrac{1}{2} c^6, \end{aligned}$$

with equality only when

$$d^2 = b^2 - R^2, \quad b = c,$$

and either

$$\text{(i) } a = b \quad \text{or} \quad \text{(ii) } b^2 + c^2 = a^2.$$

These conditions (i) and (ii) are sufficient to determine two "critical" lattices Λ_1 and Λ_2 for either of which D/m^3 or $(V/c^3)^2$ attains its minimum value $\frac{1}{2}$.

In case (i), the tetrahedron $ABCD$ is regular, and we may choose Cartesian coordinates

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0)$$

for its vertices. Thus

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$$\mathbf{t}_1 = AB = (0, 1, 1), \quad \mathbf{t}_2 = AC = (1, 0, 1), \quad \mathbf{t}_3 = AD = (1, 1, 0)$$

and

$$\begin{aligned}(\textstyle\sum u_r \mathbf{t}_r)^2 &= (u_2 + u_3, u_3 + u_1, u_1 + u_2)^2 \\ &= (u_2 + u_3)^2 + (u_3 + u_1)^2 + (u_1 + u_2)^2 \\ &= 2(u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1 + u_1 u_2).\end{aligned}$$

In case (ii), the triangle ABC is right-angled isosceles. Choosing A, B and D as before, we now have C at $(0, 1, -1)$. Thus

$$\mathbf{t}_1 = AB = (0, 1, 1), \quad \mathbf{t}_2 = AC = (0, 1, -1), \quad \mathbf{t}_3 = AD = (1, 1, 0)$$

and

$$\begin{aligned}(\textstyle\sum u_r \, \mathbf{t}_r)^2 &= (u_3, u_1 + u_2 + u_3, u_1 - u_2)^2 \\ &= u_3^2 + (u_1 + u_2 + u_3)^2 + (u_1 - u_2)^2 \\ &= 2(u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1).\end{aligned}$$

In either case, the lattice generated by the $\mathbf{t}$'s consists of all points with integral coordinates whose sum is even. Thus the lattices Λ_1 and Λ_2 are the same, viz., the face-centred cubic lattice (**1**, pp. 46, 56). (The points whose coordinates are all even form an ordinary cubic lattice, while those which have two odd coordinates are the centres of the square faces.) Since the two sets of $\mathbf{t}$'s are different bases for the same lattice, the two forms are equivalent: there is only one class of extreme ternary forms.

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