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THE UNIVERSAL THEORY OF ORDERED EQUIDECOMPOSABILITY TYPES SEMIGROUPS

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ABSTRACT. We prove that a commutative preordered semigroup embeds into the space of all equidecomposability types of subsets of some set equipped with a group action (in short, a *full type space*) if and only if it satisfies the following axioms: (i) $(\forall x, y)$ $(x \leq x + y)$; (ii) $(\forall x, y)((x \leq y \text{ and } y \leq x) \Rightarrow x = y)$; (iii) $(\forall x, y, u, v)((x + u \leq y + u$ and $u \leq v) \Rightarrow x + v \leq y + v)$; (iv) $(\forall x, u, v)((x + u = u$ and $u \leq v) \Rightarrow x + v = v)$; (v) $(\forall x, y)(mx \leq my \Rightarrow x \leq y)$ (all $m \in \mathbb{N} \setminus \{0\}$). Furthermore, such a structure can always be embedded into a reduced power of the space $\mathbf{T}$ of nonempty initial segments of $\mathbf{Q}_+$ with rational (possibly infinite) endpoints, equipped with the addition defined by $\mathfrak{a}+\mathfrak{b} = \{x+y : x \in \mathfrak{a} \text{ and } y \in \mathfrak{b}\}$ and the ordering defined by $\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow (\exists \mathfrak{c})(\mathfrak{a}+\mathfrak{c} = \mathfrak{b})$. As a corollary, the set of all universal formulas of $(+, \leq)$ satisfied by all full type spaces is decidable.

0. **Introduction.** Let a group G act on a set Ω. Say that two subsets X and Y of Ω are *G-equidecomposable* when there are finite partitions $X = \bigcup_{i<n} X_i$ and $Y = \bigcup_{i<n} Y_i$ and g_i $(i < n)$ in G such that $(\forall i < n)(g_i X_i = Y_i)$. The quotient space of $\mathcal{P}(\Omega)$ by this equivalence embeds naturally into a commutative monoid, denoted throughout this paper by $\mathrm{S}(\Omega)/G$. This monoid can be equipped with the *minimal* preordering [21], *i.e.* defined by $x \leq y \Leftrightarrow (\exists z)(x + z = y)$. These preordered monoids will be called *full type spaces*. There are relatively few things known about full type spaces, although they satisfy interesting non-trivial first-order statements [1, 18, 19, 20], which may depend on properties of G such as amenability. Among these, the only two that are known which can be expressed by *universal* formulas (*i.e.* of the form $(\forall \vec{x})\varphi(\vec{x})$ where φ is quantifier-free) are the *Cantor-Bernstein property*

$$\text{(CB)} \qquad (\forall x, y)\big((x \leq y \text{ and } y \leq x) \Rightarrow x = y\big),$$

and the *multiplicative cancellation property*

$$\text{(MC)} \qquad (\forall x, y)(mx \leq my \Rightarrow x \leq y) \quad (\text{all } m \in \mathbb{N} \setminus \{0\}),$$

the latter being called in this paper "unperforation" by reference to [5], with non-trivial proofs (especially for (MC)). On the other hand, properties of *specific* full type spaces can be found in [11, 12, 19].

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Thus, a bold (and false) conjecture which could be formulated about full type spaces is that they can all be embedded into direct powers of $\overline{\mathbb{P}} = ([0,+\infty],+,0,\leq)$, as it is the case for their natural idealizations, Tarski's *cardinal algebras*, and connected objects [15, 22]. A simple counterexample may be found in [12], where it is also proved that the underlying *monoid* of a full type space can be embedded into a power of $\mathbb{R} \cup \{\infty\}$ (nothing is said about the ordered structure).

A better looking conjecture would be that every full type space can be embedded into some *reduced* power of $\overline{\mathbb{P}}$. This turns out to be true if G is exponentially bounded [23] (but $\overline{\mathbb{P}}$ does not embed into any "exponentially bounded full type space"). But later, M. Laczkovich found an example of full type space having two elements a and b such that $a+b = 2b$ and $a \not\leq b$ [14], thus showing that not all full type spaces can be embedded into reduced powers of $\overline{\mathbb{P}}$ and answering a question of [23].

Thus a natural third conjecture is the following: *are (CB) and (MC) in fact enough to characterize the universal theory of full type spaces?* And at long last, the answer turns out to be "essentially yes", an additional axiom, called here "preminimality", having to be added, yielding *subrational* P. O. M.*'s* (Definition 1.5). Furthermore, every subrational P. O. M. (in particular, every full type space) can be embedded into a reduced power of a certain simple structure, denoted here by $\mathbb{T}$. By definition, $\mathbb{T}$ is the set of all intervals $\mathfrak{a}$ of $\mathbf{Q}_+$ such that $0 \in \mathfrak{a}$ with rational (possibly infinite) endpoints, equipped with the addition $\mathfrak{a} + \mathfrak{b} = \{a+b : a \in \mathfrak{a} \text{ and } b \in \mathfrak{b}\}$ and the minimal ordering (this reminds the construction of the reals with Dedekind cuts of $\mathbf{Q}$ [2], but note that this time, $[0,1]$ and $[0,1)$ are distinct [and incomparable] elements of $\mathbb{T}$). The aim of this paper is to prove these two statements. As a consequence, image spaces of "abstract measures" are in most cases not worse than $\mathbb{T}$ itself.

Section 1 recalls the basic properties of full type spaces (including (CB) and (MC)), plus the useful Lemma 1.9.

Section 2 presents another class of subrational P. O. M.'s, which are spaces of initial segments of positive cones of linearly ordered real vector spaces. Its main results are Propositions 2.11 and 2.24, giving a hint of the fundamental character of $\mathbb{T}$.

In Section 3, we prove that every member of a special class of subrational P. O. M.'s called *rational* P. O. M.*'s* can be embedded into a reduced power of $\mathbb{T}$.

In Section 4, we conclude that every subrational P. O. M. can be embedded into a rational P. O. M., thus into a reduced power of $\mathbb{T}$.

In Section 5, M. Laczkovich's construction comes up in a crucial way to show that $\mathbb{T}$ embeds into a full type space (Corollary 5.2); by using Lemma 1.9, one gets the aim of this paper, Theorem 5.3, which can be stated, "those P. O. M.'s that can be embedded into a reduced power of $\mathbb{T}$ (resp. a full type space) are exactly the subrational P. O. M.'s". Using again $\mathbb{T}$, one concludes (Corollary 5.5) that the set of all universal formulas holding in all full type spaces is decidable.

The main topic of this paper (*subrational* P. O. M.*'s*) concerns only *antisymmetric* P. O. M.'s, but, in view of further generalizations, we will leave sometimes open the possibility to apply some theorems to the non antisymmetric case (*e.g.* with "type spaces" where pieces live in some non-σ-complete Boolean algebra)—this may also emphasize the non-trivial character of (CB). Such is *e.g.* the case with Proposition 4.3. The corresponding increase in the global length is less than half a page.

For any two sets X and Y, ${}^XY$ will denote the set of all maps from X to Y. Let $\mathcal{F}$ be a [proper] filter on a set I. For every family $(S_i)_{i\in I}$ of sets, one defines [3] the *reduced product* of $(S_i)_{i\in I}$ modulo $\mathcal{F}$, which we will note $\prod_{\mathcal{F}}(S_i)_i$ or simply $\prod_{\mathcal{F}} S_i$, by taking the quotient of $\prod_{i\in I} S_i$ by the equivalence relation defined by $(x_i)_i \equiv_{\mathcal{F}} (y_i)_i$ if and only if $\{i \in I : x_i = y_i\} \in \mathcal{F}$ (and we will denote by $\langle x_i : i \in I\rangle_{\mathcal{F}}$ the equivalence class of $(x_i)_{i\in I}$ modulo $\equiv_{\mathcal{F}}$). This operation extends naturally to arbitrary first-order structures. We refer to [3] for details. We denote by ω the first limit ordinal, and by ON the (proper) class of all ordinals. For every set Ω, let $\mathfrak{S}_\Omega$ denote the set of all permutations of Ω. If X and Y are two subsets of a given preordered set P, then we will write $X \leq Y$ for $\big(\forall(x,y) \in X\times Y\big)(x \leq y)$. If $X = \{a\}$ (resp. $Y = \{a\}$), then we will write $a \leq Y$ (resp. $X \leq a$). If $X = \{a_1, \ldots, a_m\}$ and $Y = \{b_1, \ldots, b_n\}$, then we will write $a_1, \ldots, a_m \leq b_1, \ldots, b_n$. Similarly for $\geq$, $<$, *etc.*, instead of $\leq$. For every subset X of P, we will write $\downarrow X = \{y \in P : (\exists x \in X)(y \leq x)\}$ and $\uparrow X = \{y \in P : (\exists x \in X)(x \leq y)\}$; we will write $\downarrow a$ (resp. $\uparrow a$) instead of $\downarrow\{a\}$ (resp. $\uparrow\{a\}$). A subset X of P is an *initial* (resp. *final*) segment of P when $X = \downarrow X$ (resp. $X = \uparrow X$); X is *directed* when $(\forall x, y \in X)(\exists z \in X)(x \leq z$ and $y \leq z)$. A *semigroup* is a set equipped with an associative operation; a monoid is a semigroup with unit.

Without the result of [14], Section 5 would not have existed. Thus the author would like to thank deeply Miklós Laczkovich for having allowed him to include the results of [13] and especially the crucial [14], bringing a contribution that would have entitled him to be a co-author of this paper.

1. **Subrational** P. O. M.**'s; full type spaces.**

We shall first recall some definitions; we will mainly follow the terminology of [21, 22, 23], but also sometimes of [5].

DEFINITION 1.1. A P. O. M. *(positively ordered monoid)* is a structure $(A, +, 0, \leq)$ such that $(A, +, 0)$ is a commutative monoid and $\leq$ is a preordering on A satisfying both following axioms:

(i) $(\forall x)(0 \leq x)$;

(ii) $(\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z)$.

A P. O. M. A is *minimal* [21] when it satisfies $(\forall x, y)\big(x \leq y \Rightarrow (\exists z)(x + z = y)\big)$, *antisymmetric* when it satisfies $(\forall x, y)\big((x \leq y$ and $y \leq x) \Rightarrow x = y\big)$. It is *preminimal* [23] when it satisfies both following axioms:

$$(\forall x, y, u, v)\big((x + u \leq y + u \text{ and } u \leq v) \Rightarrow x + v \leq y + v\big)$$
$$(\forall x, y, u, v)\big((x + u = y + u \text{ and } u \leq v) \Rightarrow x + v = y + v\big).$$

Of course, if A is antisymmetric, then it suffices to verify the first condition above. Every minimal P. O. M. is preminimal, and every sub-P. O. M. of a preminimal P. O. M. is preminimal. Note (§4) that there are preminimal P. O. M.'s which do not embed into any minimal P. O. M. The P. O. M. A is *separative* [23] when it satisfies both following axioms:

$$(\forall x, y, z)\big((x+z \leq y+z \text{ and } z \leq y) \Rightarrow x \leq y\big)$$
$$(\forall x, y, z)\big((x+z = y+z \text{ and } z \leq x, y) \Rightarrow x = y\big).$$

Of course, if A is antisymmetric, then it suffices to verify the first condition above. Thus separativeness if a weak form of cancellativeness, while it implies preminimality. Finally, if $m \in \mathbb{N} \setminus \{0\}$, say that A is *m-unperforated* (see [5] where this terminology is used for abelian ordered groups) when it satisfies both following axioms:

$$(\forall x, y)(mx \leq my \Rightarrow x \leq y)$$
$$(\forall x, y)(mx = my \Rightarrow x = y).$$

Of course, if A is antisymmetric, then it suffices to verify the first condition above. Say that A is unperforated when it is m-unperforated for all $m \in \mathbb{N} \setminus \{0\}$.

Now, let Ω be a set. Let $\mathrm{S}(\Omega)$ (resp. $\mathrm{S}_c(\Omega)$) denote the space of all bounded $\mathbb{N}$-valued (resp. $\mathbb{R}_+$-valued) functions defined on Ω; for all $X \subseteq \Omega$, identify X with its characteristic function $\mathbf{1}_X$. If a group G acts on Ω, then it acts on $\mathrm{S}(\Omega)$ and on $\mathrm{S}_c(\Omega)$ by translations [20]. Then, as in [20], one can define the space $\mathrm{S}(\Omega)/G$ of all *equidecomposability types* of subsets of Ω modulo G, by taking the quotient of $\mathrm{S}(\Omega)$ by the congruence $\equiv_G$ defined by

$$\varphi \equiv_G \psi \Longleftrightarrow$$
$$(\exists n \in \omega \setminus \{0\})(\exists_{i<n} g_i \in G)\big(\exists_{i<n}\varphi_i \in \mathrm{S}(\Omega)\big)\Big(\varphi = \sum_{i<n} \varphi_i \text{ and } \psi = \sum_{i<n} g_i\varphi_i\Big).$$

Thus $\mathrm{S}(\Omega)/G$ is a commutative monoid. We equip it with the minimal preordering, so that it becomes a minimal P. O. M. We will call such a P. O. M. a *full type space*.

One defines similarly $\mathrm{S}_c(\Omega)/G$ by replacing $\mathrm{S}(\Omega)$ by $\mathrm{S}_c(\Omega)$ in the definition above. If φ belongs to $\mathrm{S}(\Omega)$ (resp. $\mathrm{S}_c(\Omega)$), then we will denote by $[\varphi]$ (resp. $[\varphi]_c$) its equidecomposability type in $\mathrm{S}(\Omega)/G$ (resp. $\mathrm{S}_c(\Omega)/G$); we will add an index G when the context does not make it clear, as in $[\varphi]_G$ (resp. $[\varphi]_{c,G}$).

The proof of the following classical result is mainly the Cantor-Bernstein argument (without choice), and it is well-known [1, 18, 19]:

PROPOSITION 1.2. *The* P. O. M.*'s* $\mathrm{S}(\Omega)/G$ *and* $\mathrm{S}_c(\Omega)/G$ *are antisymmetric.* ∎

Note that the proof of this result does not depend on any choice assumption. Indeed, it generalizes to the case where the pieces used in decompositions live in some σ-algebra of subsets of Ω [18]. On the contrary, the following Corollary 1.4 (whose proof can be easily obtained from the classical proof of the "cancellation property"—see [19, Section 9]) is not known in weaker contexts than the Boolean prime ideal theorem (to prove the infinite marriage theorem). Lemma 1.3 is a seemingly much stronger form of Corollary 1.4, due to M. Laczkovich [13]; we reproduce it here, with the authorization of the author:

LEMMA 1.3 (M. LACZKOVICH). *Let φ, ψ in* $\mathrm{S}(\Omega)$. *Then* $[\varphi] \leq [\psi]$ *if and only if* $[\varphi]_c \leq [\psi]_c$.

PROOF. We prove the non trivial direction. Using the embedding procedure of [19, Section 9], one can assume without loss of generality that $\varphi = \mathbf{1}_X$ and $\psi = \mathbf{1}_Y$ for some subsets X and Y of Ω. By assumption, there are $n \in \omega \setminus \{0\}$ and $g_i \in G$, $\varphi_i \colon \Omega \longrightarrow [0,1]$ $(i < n)$ such that $\mathbf{1}_X \leq \sum_{i<n} \varphi_i$ and $\sum_{i<n} g_i\varphi_i \leq \mathbf{1}_Y$. Put $\Gamma = \{(x,y) \in X \times Y : (\exists i < n)(y = g_i x)\}$. It suffices to prove that Γ has a *matching*, *i.e.* a one-to-one map $f \colon X \longrightarrow Y$ such that $(\forall x \in X)\Big(\big(x, f(x)\big) \in \Gamma\Big)$. For all $U \subseteq \Omega$, put $\Gamma[U] = \Big\{y \in \Omega : (\exists x \in U)\big((x,y) \in \Gamma\big)\Big\}$. If $U \subseteq X$ is finite, then $|\Gamma[U]| \leq n|U|$ so that $\Gamma[U]$ is finite. Thus, by the (infinite) marriage Theorem [7], it suffices to prove that for every finite $U \subseteq X$, we have $|\Gamma[U]| \geq |U|$. We have $\Gamma[U] = W \cap Y$ where $W = \bigcup_{i<n} g_i U$, thus

$$
\begin{aligned}
|\Gamma[U]| &= \sum_{y\in W} \mathbf{1}_Y(y) \geq \sum_{y\in W}\sum_{i<n} \varphi_i(g_i^{-1}y) = \sum_{i<n}\sum_{y\in W} \varphi_i(g_i^{-1}y) \\
&\geq \sum_{i<n}\sum_{y\in g_i U} \varphi_i(g_i^{-1}y) \\
&= \sum_{i<n}\sum_{x\in U} \varphi_i(x) \\
&= \sum_{x\in U}\sum_{i<n} \varphi_i(x) \\
&\geq \sum_{x\in U} \mathbf{1}_X(x) \\
&= |U|,
\end{aligned}
$$

and we are done. ∎

Since $\mathrm{S}_c(\Omega)/G$ is trivially unperforated, we deduce immediately the following

COROLLARY 1.4. $\mathrm{S}(\Omega)/G$ *is unperforated.* ∎

DEFINITION 1.5. A *subrational* P. O. M. is an antisymmetric, preminimal, unperforated P. O. M. A *full measure* P. O. M. is a P. O. M. which can be embedded into a full type space.

Thus, by Proposition 1.2 and Corollary 1.4 (and minimality of full type spaces), *every full measure* P. O. M. *is a subrational* P. O. M.. The principal aim of this paper is to prove the converse.

Now, we recall the construction presented in [19, Section 9]. If a group G acts on a set Ω, say that a partial function $f \colon \Omega \longrightarrow \Omega$ is *piecewise in* G when there exist $n \in \omega \setminus \{0\}$, g_i $(i < n)$ in G and mutually disjoint subsets X_i $(i < n)$ of Ω such that $\mathrm{dom}(f) = \bigcup_{i<n} X_i$ and for all $i < n$ and all $x \in X_i$, $f(x) = g_i \cdot x$.

DEFINITION 1.6. Let G be a group acting on a set Ω. Define an enlarged action as follows. Let $\hat{\Omega} = \Omega \times \omega$, let $G^* = G \times \mathfrak{S}_\omega$ act on $\hat{\Omega}$ componentwise; let $\hat{G}$ denote the group of all permutations of $\hat{\Omega}$ which are piecewise in G^*. Say that a subset X of $\hat{\Omega}$ is *bounded* when for large enough $n \in \omega$, we have $X \cap (\Omega \times \{n\}) = \emptyset$.

Then to every $\varphi \in \mathrm{S}(\Omega)$, one can associate naturally a bounded subset X_φ of $\hat{\Omega}$ the following way: if $\varphi = \sum_{i<n} \mathbf{1}_{X_i}$ ($X_i \subseteq \Omega$ "components" of φ—choose any such representation), take $X_\varphi = \bigcup_{i<n}(X_i \times \{i\})$. Then the proof of the following lemma is routine:

LEMMA 1.7. *One can define a* P. O. M.*-embedding the following way:*

$$\mathrm{S}(\Omega)/G \longrightarrow \mathrm{S}(\hat{\Omega})/\hat{G}, \quad [\varphi] \longmapsto [X_\varphi],$$

which is of course independent of the choice of the "components" above. ∎

The embedding above is of course not an isomorphism (there are unbounded subsets of $\hat{\Omega}$). We will identify $\mathrm{S}(\Omega)/G$ with its image in $\mathrm{S}(\hat{\Omega})/\hat{G}$. The sole purpose of its introduction in this paper is the following

LEMMA 1.8. *Let X, Y be bounded subsets of $\hat{\Omega}$. Then $[X] \leq [Y]$ (resp. $[X] = [Y]$) if and only if there exists $g \in \hat{G}$ such that $g \cdot X \subseteq Y$ (resp. $g \cdot X = Y$).*

PROOF. Suppose $[X] \leq [Y]$. Thus by definition, there exists a (partial) one-to-one function $g_0\colon X \longrightarrow Y$, piecewise in G^*. To conclude, it suffices to extend g_0 to an element of $\hat{G}$. Let $n \in \omega$ such that $X \cup Y \subseteq \Omega \times (\omega \setminus n)$; since ω and $\omega \setminus n$ are equidecomposable using bijections $\omega \longrightarrow \omega$, $\Omega \times \omega$ and $\Omega \times (\omega \setminus n)$ are $\hat{G}$-equidecomposable, whence, by Proposition 1.2, $\hat{\Omega} \setminus X \equiv_{\hat{G}} \hat{\Omega} \equiv_{\hat{G}} \hat{\Omega} \setminus g_0 X$ and we are done (every bijection piecewise in $\hat{G}$ is in $\hat{G}$). The proof is similar for the case where $[X] = [Y]$. ∎

Note that the lemma does not apply for unbounded X (*e.g.* $X = \hat{\Omega}$).

Now, we are ready to prove the following

LEMMA 1.9. *Every reduced product of full measure* P. O. M.*'s is a full measure* P. O. M.

PROOF. Let $(A_i)_{i \in I}$ ($I \neq \emptyset$) be a family of full measure P. O. M.'s, let $\mathcal{F}$ be a (proper) filter on I. For all $i \in I$, A_i embeds by assumption into some full type space, say $\mathrm{S}(X_i)/G_i$; let $\hat{X}_i$, $\hat{G}_i$ be as in Definition 1.6. One can assume without loss of generality that the X_i's are mutually disjoint. Then, put $U_i = \hat{X}_i \times \omega$, $Y_i = \hat{X}_i \times \{\omega\}$. Then for every $J \subseteq I$, put $U_J = \bigcup_{i \in J} U_i$, $Y_J = \bigcup_{i \in J} Y_i$, $X_J = U_J \cup Y_J$, $U = U_I$, $Y = Y_I$ and $X = X_I$. For all $i \in I$, $\hat{G}_i$ acts on Y_i *via* the bijection $\hat{X}_i \longrightarrow Y_i$, $x \longmapsto (x, \omega)$. Let G be the group of permutations of X such that for some J in $\mathcal{F}$, $gU_J = U_J$, $gY_J = Y_J$ and for all $i \in J$, $g|_{Y_i} \in \hat{G}_i$. We first try to define a map $e\colon \prod_{\mathcal{F}} \mathrm{S}(X_i)/G_i \longrightarrow \mathrm{S}(X)/G$ by $e(\langle [A_i]_{\hat{G}_i} : i \in I \rangle_{\mathcal{F}}) = \big[U \cup \bigcup_{i \in I} A'_i\big]_G$ where $A_i \subseteq \hat{X}_i$ are bounded and $A'_i = A_i \times \{\omega\}$. Suppose that for some $J \in \mathcal{F}$, we have $(\forall i \in J)([A_i]_{\hat{G}_i} = [B_i]_{\hat{G}_i})$. By Lemma 1.8, for all $i \in J$, there exists $g_i \in \hat{G}_i$ such that $g_i A_i = B_i$. Define $g \in G$ by $g|_{U \cup Y_{I \setminus J}} = \mathrm{id}$, and $g|_{Y_i} = g_i$ for all $i \in J$. It is clear that $g \in G$ and that $U \cup \bigcup_{i \in J} B'_i = g \cdot [U \cup \bigcup_{i \in J} A'_i]$. Furthermore, by using the fact that ω is equidecomposable to $\omega + 1$ using permutations of $\omega + 1$, it is easy to construct some $h \in G$ such that $h|_{X_J} = \mathrm{id}$ and $h \cdot U = U \cup Y_{I \setminus J}$, whence $[U]_G = [U \cup Y_{I \setminus J}]_G$. Finally, we get $[U \cup \bigcup_{i \in I} A'_i]_G = [U \cup \bigcup_{i \in I} B'_i]_G$, whence e is well-defined. Using the fact that ω is paradoxical [19] using permutations of ω, is is easy to prove that U is G-paradoxical (one uses $g \in G$ such that $g|_Y = \mathrm{id}$). Thus, it follows that *e is a semigroup homomorphism.*

Finally, we prove that e is an embedding. Let $\alpha = \langle [A_i] : i \in I \rangle_{\mathcal{F}}$, $\beta = \langle [B_i] : i \in I \rangle_{\mathcal{F}}$ and suppose that $e(\alpha) \leq e(\beta)$. This means that $[U \cup \bigcup_{i \in I} A_i']_G \leq [U \cup \bigcup_{i \in I} B_i']_G$, using group-elements g_l $(l < k)$ of G. There is $J \in \mathcal{F}$ such that for all $l < k$, $g_l U_J = U_J$ and $g_l Y_J = Y_J$ and for all $i \in J$, $g_l|_{Y_i} \in \hat{G}_i$. The first two conditions imply that $[\bigcup_{i \in J} A_i']_G \leq [\bigcup_{i \in J} B_i']_G$ using the group-elements $h_l = g_l|_{Y_J}$ (all $l < k$). Thus for all $i \in J$, $[A_i'] \leq [B_i']$ using the group-elements $h_l|_{Y_i}$ $(l < k)$. Thus $[A_i] \leq [B_i]$ for all $i \in J$, whence $\alpha \leq \beta$. Thus *e is an ordered semigroup embedding.*

A last problem to solve is that $e(0) = [U] \neq 0$, but this is easy to fix: just define $f\colon \prod_{\mathcal{F}} \mathrm{S}(X_i)/G_i \longrightarrow \mathrm{S}(X)/G$ by $f(0) = 0$ and $f(\alpha) = e(\alpha)$ if $\alpha \neq 0$. Then f is a P. O. M.-embedding. Since $\prod_{\mathcal{F}} A_i$ embeds into $\prod_{\mathcal{F}} \mathrm{S}(X_i)/G_i$, we are done. ∎

COROLLARY 1.10. *Every direct limit of full measure* P. O. M.*'s is a full measure* P. O. M.

PROOF. It is easy to prove that the direct limit of a family of P. O. M.'s (or much more general first-order structures) embeds into some reduced power of these structures. We conclude by Lemma 1.9. ∎

2. **Initial segments of linearly ordered vector spaces.**

We start with a classical definition.

DEFINITION 2.1. An *ordered vector space* is a $\mathbb{R}$-vector space E equipped with an ordering $\leq$ which is compatible with the structure of vector space, *i.e.* satisfies both following axioms:

(i) $(\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z)$;

(ii) $(\forall x)(0 \leq x \Rightarrow 0 \leq \lambda x)$ (all $\lambda \in \mathbb{R}_+$).

Its *positive* (resp. *negative*) *cone* E_+ (resp. E_-) is the set of all positive (resp. negative) elements of E (0 included). Define a *vector line* to be a linearly ordered vector space. A *positive cone* (resp. *linear cone*) is the positive cone of some vector space ordering (resp. vector line ordering). Thus a positive cone is a nonempty convex, positively homogeneous subset P of some $\mathbb{R}$-vector space E such that $P \cap (-P) = \{0\}$.

We start with the following folklore lemma.

LEMMA 2.2. *Let P be a positive cone of a vector space E, let C be a convex subset of E such that $P \cap C = \emptyset$. Then there is a linear cone Q such that $P \subseteq Q$ and $Q \cap C = \emptyset$.*

PROOF. By Zorn's lemma, there exists a maximal positive cone Q such that $P \subseteq Q$ and $Q \cap C = \emptyset$. We show that Q is a linear cone. Otherwise, there exists $a \in E \setminus \big(Q \cup (-Q)\big)$. Since $a \notin -Q$, $Q + \mathbb{R}_+ a$ is a positive cone; it contains strictly Q, thus it meets C. Similarly, $Q + \mathbb{R}_+(-a)$ meets C. Thus, there are x, y in Q and α, β in $\mathbb{R}_+$ such that $u = x + \alpha a$ and $v = y - \beta a$ belong to C. Since $Q \cap C = \emptyset$, $\alpha > 0$ and $\beta > 0$. Thus $\frac{\beta}{\alpha+\beta} x + \frac{\alpha}{\alpha+\beta} y = \frac{\beta}{\alpha+\beta} u + \frac{\alpha}{\alpha+\beta} v$ belongs to $Q \cap C$, a contradiction. ∎

By taking C to be an open half-line, we get immediately the not less classical

COROLLARY 2.3. *Let E be an ordered vector space. Then the ordering of E is the intersection of all vector line orderings of E containing it.* ∎

COROLLARY 2.4. *Let E be a vector line, let $\mathfrak{a}$ be an initial segment of E. Then there are a vector line F containing E and $a \in F$ such that $\mathfrak{a} = \downarrow a \cap E$.*

PROOF. Let $F = E \times \mathbb{R}$ (the ordering is not defined yet), and define

$$P = \{(x,0) : x \geq 0\} \cup \big\{(x,\lambda) \in E \times (\mathbb{R}_+ \setminus \{0\}) : (-1/\lambda)x \in \mathfrak{a}\big\}$$
$$C = \{(x,0) : x < 0\} \cup \big\{(x,\lambda) \in E \times (\mathbb{R}_+ \setminus \{0\}) : (-1/\lambda)x \notin \mathfrak{a}\big\}.$$

It is easy to verify that P is a positive cone of F, that C is a convex subset of F and that $P \cap C = \emptyset$. By Lemma 2.2, there exists a linear cone Q containing P such that $Q \cap C = \emptyset$. Equip F with the linear ordering with positive cone Q. Embed E into F *via* $x \longmapsto (x,0)$. Using the definitions of P and C, it is easy to verify that this map is also an order-embedding, and that for all $x \in E$, $x \in \mathfrak{a}$ if and only if $(0,1) - (x,0) \in Q$. Thus, identifying E and its image in F, F and $a = (0,1)$ satisfy the required conditions. ∎

COROLLARY 2.5 ("AMALGAMATION PROPERTY OF VECTOR LINES"). *Any diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\big\downarrow & & \\ B & & \end{array}$$

of vector line embeddings can be completed into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle \bar{e}} \\ B & \xrightarrow{\bar{f}} & D \end{array}$$

of vector line embeddings. In addition, if $\dim(B/A) \leq 1$, *then one can take* $\dim(D/C) \leq 1$.

PROOF. Put $I = \big\{\big(e(a), -f(a)\big) : a \in A\big\}$, and put $D = B \times C/I$. For all $b \in B$ and $c \in C$, write $[b,c] = (b,c) + I \in D$. There are natural vector space homomorphisms $\bar{e}\colon c \longmapsto [0,c]$ and $\bar{f}\colon b \longmapsto [b,0]$. Let $P = \{\xi \in D : \xi \cap (B_+ \times C_+) \neq \emptyset\}$. It is clear that $0 \in P$, $\mathbb{R}_+ P \subseteq P$ and $P + P \subseteq P$. Let $\xi \in P \cap (-P)$. Then $\xi = [x,y]$ for some $x \in B_+$, $y \in C_+$. Since $-\xi \in P$, there exists $a \in A$ such that $-x + e(a) \geq 0$ and $-y - f(a) \geq 0$. Since e and f are embeddings, we have $a = 0$, whence $(x,y) = (0,0)$, so that $\xi = 0$. Thus P is a positive cone on D.

Now, let $N = \big\{\xi \in D : \xi \cap \big(B_- \times C_- \setminus \{(0,0)\}\big) \neq \emptyset\big\}$. Clearly, N is a convex subset of D. Let $\xi \in P \cap N$. Since $\xi \in N$, there are $x \in B_+$ and $y \in C_+$ such that $(x,y) \neq (0,0)$ and $\xi = -[x,y]$. Since $\xi \in P$, there exists $a \in A$ such that $-x + e(a) \geq 0$ and $-y - f(a) \geq 0$. We conclude as before that $x = y = 0$, a contradiction. Thus $P \cap N = \emptyset$. Thus, by Lemma 2.2, there exists a linear cone Q containing P such that $Q \cap N = \emptyset$. Equip D with the linear ordering with positive cone Q. Since $P \subseteq Q$, $\bar{e}$ and $\bar{f}$ are ordered vector space homomorphisms; in fact, since $Q \cap N = \emptyset$, it is easy to verify that they are embeddings; this proves the first part of the lemma. The second part is trivial, by just replacing D by $\bar{e}C + \mathbb{R}\bar{f}(b)$. ■

We shall now construct a special type of vector line. Let A be a linearly ordered set. Denote by $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ the set of all maps $x: A \longrightarrow \mathbb{R}$ such that $\operatorname{supp}(x) = \{\alpha \in A : x(\alpha) \neq 0\}$ is well-ordered. Then put $\operatorname{val}(x) =$ least α such that $x(\alpha) \neq 0$ for $x \neq 0$, and $\operatorname{val}(0) = +\infty$. Then $\operatorname{val}(x+y) \geq \min\bigl(\operatorname{val}(x), \operatorname{val}(y)\bigr)$ for all x, y in $\mathbb{R}\langle\!\langle A\rangle\!\rangle$. Define $\mathbb{P}\langle\!\langle A\rangle\!\rangle = \{0\} \cup \bigl\{x \in \mathbb{R}\langle\!\langle A\rangle\!\rangle \setminus \{0\} : x\bigl(\operatorname{val}(x)\bigr) > 0\bigr\}$. Then $\mathbb{P}\langle\!\langle A\rangle\!\rangle$ is a linear cone on $\mathbb{R}\langle\!\langle A\rangle\!\rangle$. Equip $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ with the linear ordering with positive cone $\mathbb{P}\langle\!\langle A\rangle\!\rangle$. For each $\alpha \in A$, identify α with the 'vector' $e_\alpha = (\delta_{\alpha\beta})_{\beta\in A}$ (where δ is the Kronecker symbol), so that elements of $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ can be written $x = \sum_\alpha x_\alpha \alpha$, $x_\alpha \in \mathbb{R}$.

In any ordered vector space, write $x \preceq y \Leftrightarrow (\exists n \in \mathbb{N})(|x| \leq n|y|)$, $x \asymp y \Leftrightarrow (x \preceq y$ and $y \preceq x)$, $x \lll y \Leftrightarrow (\forall n \in \mathbb{N})(n|x| \leq |y|)$. In $\mathbb{R}\langle\!\langle A\rangle\!\rangle$, we clearly have

(i) $(\forall x \neq 0)(x \asymp e_{\operatorname{val}(x)})$ (*i.e.* $x \asymp \operatorname{val}(x)$ with the previous identification), and

(ii) $(\forall \alpha, \beta \in A)(\alpha <_A \beta \Leftrightarrow \beta \lll \alpha)$.

Clearly, if A is a subset of a linearly ordered set B, then $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ embeds naturally into $\mathbb{R}\langle\!\langle B\rangle\!\rangle$, in a *functorial* way; thus we will identify $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ with its natural image in $\mathbb{R}\langle\!\langle B\rangle\!\rangle$. Similarly, if A is fixed and $C \subseteq A$, then, for all $x \in \mathbb{R}\langle\!\langle A\rangle\!\rangle$, we will denote by $x|_C$ the element y of $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ defined by $y(\alpha) = x(\alpha)$ for $\alpha \in C$ and $y(\alpha) = 0$ if $\alpha \notin C$.

For all x, y in a vector line E such that $x > 0$ and $y \preceq x$, put $(y : x) = \sup\{r \in \mathbb{R} : rx \leq y\}$.

LEMMA 2.6. *We have $y - (y : x)x \lll x$. Furthermore, if $y \asymp x$, then $(y : x) \neq 0$.*

PROOF. An easy verification. ■

Now, for every linearly ordered set A, let $\tilde{A}$ be the linearly ordered set of all initial segments of the lexicographical product $A \times \{0, 1\}$, equipped with the inclusion relation. Identify A with its natural image (*via* $a \longmapsto \downarrow(a, 0)$) in $\tilde{A}$.

LEMMA 2.7. *Let U, V be subsets of A such that $U \cup V = A$ and $U < V$. Then there exists $\gamma \in \tilde{A}$ such that $U = \downarrow\gamma \cap A$ and $V = \uparrow\gamma \cap A$.*

PROOF. It is immediate that U is an initial segment of A and V is a final segment of A. Put $\gamma = U \times \{0, 1\}$. It is easy to verify that γ satisfies the required condition. ■

LEMMA 2.8. *Let A be a linearly ordered set, let E be a vector line containing $\mathbb{R}\langle\!\langle A\rangle\!\rangle$. Put $H = \bigl\{x \in E : (\forall y \in \mathbb{R}\langle\!\langle A\rangle\!\rangle \setminus \{0\})(x \not\asymp y)\bigr\}$. Then every element of E can be written under the form $x + h$, where $x \in \mathbb{R}\langle\!\langle A\rangle\!\rangle$, $h \in H$ and $(\forall \alpha \in \operatorname{supp}(x))(h \lll \alpha)$.*

PROOF. Suppose otherwise. So there exists $a \in E$ without any decomposition as above. Construct inductively $\alpha_i \in A$ and $\lambda_i \in \mathbb{R} \setminus \{0\}$, $i \in ON$, as follows.

Let $i \in ON$, suppose that α_j, λ_j have been constructed for all $j < i$, with $(\alpha_j)_{j<i}$ strictly increasing in A, $\lambda_j \in \mathbb{R} \setminus \{0\}$ for all $j < i$, and $(\forall j < i)(a - \sum_{k\leq j} \lambda_k \alpha_k \lll \alpha_j)$. Put $a' = a - \sum_{j<i} \lambda_j \alpha_j$. Thus $(\forall j < i)(a' \lll \alpha_j)$. Thus, by assumption, $a' \notin H$, thus there exists a [unique] $\alpha_i \in A$ such that $a' \asymp \alpha_i$. Necessarily, $\alpha_j < \alpha_i$ (in A) for all $j < i$. Put $\lambda_i = (a' : \alpha_i)$, so that $\lambda_i \in \mathbb{R} \setminus \{0\}$. By Lemma 2.6, $a' - (a' : \alpha_i)\alpha_i \lll \alpha_i$, *i.e.* $a - \sum_{j \leq i} \lambda_j \alpha_j \lll \alpha_i$, and this is nothing but the induction step.

Thus the ON-sequence $(\alpha_i)_{i \in ON}$ is strictly increasing (with all the α_i in A), a contradiction. ∎

LEMMA 2.9. *Let A be a linearly ordered set, let $E \subseteq F$ be vector lines with $\dim(F/E) \leq 1$, let $f\colon E \longrightarrow \mathbb{R}\langle\!\langle A \rangle\!\rangle$ be an embedding. Then one can form a commutative diagram of embeddings*

$$\begin{array}{ccc} F & \xrightarrow{g} & \mathbb{R}\langle\!\langle \tilde{A} \rangle\!\rangle \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & \mathbb{R}\langle\!\langle A \rangle\!\rangle \end{array}$$

where unlabeled arrows are the natural ones.

PROOF. By Corollary 2.5, there is a commutative diagram of vector line embeddings

$$\begin{array}{ccc} F & \xrightarrow{g_0} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & \mathbb{R}\langle\!\langle A \rangle\!\rangle \end{array}$$

where $\dim(F'/\mathbb{R}\langle\!\langle A \rangle\!\rangle) \leq 1$. Thus $F' = \mathbb{R}\langle\!\langle A \rangle\!\rangle + \mathbb{R}a$ for some $a \in F'$. By Lemma 2.8, $a \in \mathbb{R}\langle\!\langle A \rangle\!\rangle + H$ where $H = \{x \in F' : (\forall y \in \mathbb{R}\langle\!\langle A \rangle\!\rangle \setminus \{0\})(x \not\asymp y)\}$; thus one can suppose without loss of generality that $a \in H$. Furthermore, without loss of generality, $a > 0$. Let $U = \{\xi \in A : \xi \lll a\}$ and $V = \{\xi \in A : a \lll \xi\}$. Thus $U < V$ and, since $a \in H$, $U \cup V = A$. By Lemma 2.7, there exists $\gamma \in \tilde{A}$ such that $U = \downarrow\gamma \cap A$ and $V = \uparrow\gamma \cap A$. Let φ be the unique linear map from F' to $\mathbb{R}\langle\!\langle \tilde{A} \rangle\!\rangle$ defined by $\varphi|_{\mathbb{R}\langle\!\langle A \rangle\!\rangle} = \mathrm{id}$ and $\varphi(a) = \gamma$. It remains to show that φ is an order-embedding, *i.e.* that for all $x \in \mathbb{R}\langle\!\langle A \rangle\!\rangle$, we have

$$\begin{cases} x \leq a \Leftrightarrow x \leq \gamma & \text{(it suffices to verify it for } x > 0) \\ x \geq a \Leftrightarrow x \geq \gamma \end{cases}.$$

However, this is obvious by definition of γ and since $a \in H$. Thus, the conclusion follows with $g = \varphi \circ g_0$. ∎

Now, an easy induction argument (taking at limit stages the union of the corresponding linearly ordered sets) yields the following

LEMMA 2.10. *Let A be a linearly ordered set, let $E \subseteq F$ be vector lines, let $f\colon E \longrightarrow \mathbb{R}\langle\!\langle A \rangle\!\rangle$ be an embedding. Then there exists a linearly ordered set B containing A such that one can form the following commutative diagram of embeddings*

$$\begin{array}{ccc} F & \xrightarrow{g} & \mathbb{R}\langle\!\langle B \rangle\!\rangle \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & \mathbb{R}\langle\!\langle A \rangle\!\rangle \end{array}$$

where unlabeled arrows are the natural ones. ∎

We deduce immediately the

PROPOSITION 2.11. *Every vector line can be embedded into some $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ where A is a linearly ordered set.* ■

Now, let E be a vector line. Denote by $\mathrm{In}(E)$ the set of all *nonempty* initial segments of E, equipped with the addition defined by

$$\mathfrak{a}+\mathfrak{b}=\{a+b : a\in\mathfrak{a} \text{ and } b\in\mathfrak{b}\}.$$

Put $\mathbf{0}=E_-$. An element $\mathfrak{a}$ of $\mathrm{In}(E)$ is *positive* when $\mathbf{0}\subseteq\mathfrak{a}$. If $\mathfrak{a},\mathfrak{b}\in\mathrm{In}(E)$, say that $\mathfrak{a}\leq\mathfrak{b}$ when there exists some *positive* $\mathfrak{c}\in\mathrm{In}(E)$ such that $\mathfrak{a}+\mathfrak{c}=\mathfrak{b}$. Clearly, $\mathfrak{a}$ is positive if and only if $\mathbf{0}\leq\mathfrak{a}$, and $\mathfrak{a}\leq\mathfrak{b}$ if and only if $\mathfrak{b}-\mathfrak{a}$ is positive and $\mathfrak{a}+(\mathfrak{b}-\mathfrak{a})=\mathfrak{b}$, where $\mathfrak{b}-\mathfrak{a}=\{x\in E : x+\mathfrak{a}\subseteq\mathfrak{b}\}$ (note that $\mathfrak{b}-\mathfrak{a}$ is always an element of $\mathrm{In}(E)\cup\{\emptyset\}$). Furthermore, $\mathfrak{a}\leq\mathfrak{b}$ implies $\mathfrak{a}\subseteq\mathfrak{b}$, but the converse is false: for example, take $E=\mathbb{R}$, $\mathfrak{a}=(-\infty,0)$, $\mathfrak{b}=(-\infty,0]$. It is obvious that $(\mathrm{In}(E),+,\mathbf{0},\leq)$ is an ordered monoid; it is certainly not a P. O. M., except for $E=\{0\}$. This structure is rather to be compared with the structure of *commutative inverse semigroups* [8], which it seems (except for the unperforation) to generalize.

A subset $\mathfrak{h}$ of E_+ is an *ideal* of E_+ when $\mathfrak{h}$ is a nonempty initial segment of E_+ and $\mathfrak{h}+\mathfrak{h}\subseteq\mathfrak{h}$. Then, put $\mathfrak{h}^+=\mathfrak{h}\cup E_-$ and $\mathfrak{h}^-=-(E_+\setminus\mathfrak{h})=\{-x : x\in E_+\setminus\mathfrak{h}\}$. Note that $\mathfrak{h}^+\in\mathrm{In}(E)$ and $\mathfrak{h}^+\geq\mathbf{0}$, and that if $\mathfrak{h}\neq E_+$, then $\mathfrak{h}^-\in\mathrm{In}(E)$ and $\mathfrak{h}^-<\mathbf{0}$. Denote by $\mathrm{Idl}(E)$ the set of all ideals of E_+.

LEMMA 2.12. *Let A be a linearly ordered set, let $\mathfrak{a}\in\mathrm{In}\,\mathbb{R}\langle\!\langle A\rangle\!\rangle$. Then there are $a\in\mathbb{R}\langle\!\langle A\rangle\!\rangle$ and $\mathfrak{h}\in\mathrm{Idl}(\mathbb{R}\langle\!\langle A\rangle\!\rangle)$ such that* $\mathrm{supp}(a)\cap\mathfrak{h}=\emptyset$ *and either* $\mathfrak{a}=a+\mathfrak{h}^+$ *or* $\mathfrak{a}=a+\mathfrak{h}^-$.

PROOF. By Corollary 2.4, there are a vector line E containing $\mathbb{R}\langle\!\langle A\rangle\!\rangle$ and an element b of E such that $\mathfrak{a}=\downarrow b\cap\mathbb{R}\langle\!\langle A\rangle\!\rangle$. By Lemma 2.8, there are $a\in\mathbb{R}\langle\!\langle A\rangle\!\rangle$ and $h\in H$ (H defined as in Lemma 2.8) such that $b=a+h$ and $(\forall\alpha\in\mathrm{supp}(a))(h\lll\alpha)$. Then, let $\mathfrak{h}$ be the ideal of $\mathbb{P}\langle\!\langle A\rangle\!\rangle$ generated by $|h|$, *i.e.* $\mathfrak{h}=\downarrow|h|\cap\mathbb{P}\langle\!\langle A\rangle\!\rangle$ (it is an ideal since $|h|\in H$ and $|h|\geq 0$). Since $(\forall\alpha\in\mathrm{supp}(a))(h\lll\alpha)$, we have $\mathrm{supp}(a)\cap\mathfrak{h}=\emptyset$. Furthermore, it is easy to verify that if $h\geq 0$, then $\mathfrak{a}=a+\mathfrak{h}^+$ and that if $h<0$, then $\mathfrak{a}=a+\mathfrak{h}^-$. ■

From now on until Lemma 2.21, put $E=\mathbb{R}\langle\!\langle A\rangle\!\rangle$, A fixed linearly ordered set. For all $\mathfrak{a}$ in $\mathrm{In}(E)$, a in E and $\mathfrak{h}$ in $\mathrm{Idl}(E)$, say that $\mathfrak{a}=a+\mathfrak{h}^{\pm}$ is a *normal form* of $\mathfrak{a}$ when $\mathrm{supp}(a)\cap\mathfrak{h}=\emptyset$. Thus the lemma above asserts *existence* of a normal form for every $\mathfrak{a}$ in $\mathrm{In}(E)$. We shall now prove *uniqueness*.

LEMMA 2.13. *Let $a\in E$, $\mathfrak{h}\in\mathrm{Idl}(E)$ and $\varepsilon\in\{+,-\}$; put $\mathfrak{a}=a+\mathfrak{h}^{\varepsilon}$. Then $\mathfrak{h}=\{x\in E_+ : x+\mathfrak{a}\subseteq\mathfrak{a}\}$.*

PROOF. We distinguish two cases.

CASE 1. $\mathfrak{a}=a+\mathfrak{h}^+$.
Then for all $x\in E_+$, $x+\mathfrak{a}\subseteq\mathfrak{a}$ if and only if $x+\mathfrak{h}\subseteq\mathfrak{h}$, if and only if $x\in\mathfrak{h}$.

CASE 2. $\mathfrak{a}=a+\mathfrak{h}^-$.
Then for all $x\in E$, $x+\mathfrak{a}\subseteq\mathfrak{a}$ if and only if $x+\big(-(E_+\setminus\mathfrak{h})\big)\subseteq-(E_+\setminus\mathfrak{h})$. If $x\in\mathfrak{h}$, then for all $y\in E_+\setminus\mathfrak{h}$, $x-y\in-(E_+\setminus\mathfrak{h})$ (otherwise $y\in x+\mathfrak{h}\subseteq\mathfrak{h}$, contradiction),

thus $x + \big(-(E_+ \setminus \mathfrak{h})\big) \subseteq -(E_+ \setminus \mathfrak{h})$. If $x \in E_+ \setminus \mathfrak{h}$, then $x + (-x) = 0 \notin -(E_+ \setminus \mathfrak{h})$ with $-x \in -(E_+ \setminus \mathfrak{h})$, thus $x + \big(-(E_+ \setminus \mathfrak{h})\big) \not\subseteq -(E_+ \setminus \mathfrak{h})$. The conclusion follows. ∎

Thus for every $\mathfrak{a} \in \operatorname{In}(E)$, note $I(\mathfrak{a}) = \{x \in E_+ : x + \mathfrak{a} \subseteq \mathfrak{a}\}$.

LEMMA 2.14. *Let a, b in E, let $\mathfrak{h}$ be an ideal of E_+. Then $a + \mathfrak{h}^+ \neq b + \mathfrak{h}^-$.*

PROOF. Suppose that $a + \mathfrak{h}^+ = b + \mathfrak{h}^-$. Thus $a \in a + \mathfrak{h}^+ \subseteq b + \mathfrak{h}^-$, thus $b - a \in E_+ \setminus \mathfrak{h}$. (so that $a < b$). Thus $(b-a)/2 \in E_+ \setminus \mathfrak{h}$ (because $\mathfrak{h}$ is an ideal), whence $b + \big(-(b-a)/2\big) \in b + \mathfrak{h}^- = a + \mathfrak{h}^+$, whence $(a+b)/2 \in a + \mathfrak{h}^+$, whence $(b-a)/2 \in \mathfrak{h}$, a contradiction. ∎

LEMMA 2.15. *Every element of* $\operatorname{In}(E)$ *admits exactly one normal form.*

PROOF. The existence part has been proved in Lemma 2.12. Now, let $\mathfrak{a} = a_i + \mathfrak{h}_i^{\varepsilon_i}$ $(i = 0, 1)$ be two normal forms of $\mathfrak{a} \in \operatorname{In}(E)$. By Lemma 2.13, $\mathfrak{h}_0 = \mathfrak{h}_1$ (so denote it by $\mathfrak{h}$). By Lemma 2.14, $\varepsilon_0 = \varepsilon_1$ (so denote it by ε). Assume without loss of generality that $a_0 < a_1$. If $a = a_1 - a_0$, then $\operatorname{supp}(a) \cap \mathfrak{h} = \emptyset$, whence $a \notin \mathfrak{h}$. By Lemma 2.13, $a + \mathfrak{h}^\varepsilon \neq \mathfrak{h}^\varepsilon$, a contradiction. ∎

Let $\mathfrak{a} \in \operatorname{In}(E)$, with normal form $a + \mathfrak{h}^\varepsilon$. We will note $a = \pi(\mathfrak{a})$, $\mathfrak{h} = I(\mathfrak{a})$ and $\varepsilon = \epsilon(\mathfrak{a})$.

LEMMA 2.16. *Let $a \in E$, $\mathfrak{h} \in \operatorname{Idl}(E)$ and ε in $\{+, -\}$. Then the normal form of $a + \mathfrak{h}^\varepsilon$ is $b + \mathfrak{h}^\varepsilon$ where $b = a|_{A \setminus \mathfrak{h}}$.*

PROOF. Let $b = a|_{A\setminus\mathfrak{h}}$ and $c = a|_{\mathfrak{h}}$. It is clear that $|c| \in \mathfrak{h}$, whence, by Lemma 2.13, $c + \mathfrak{h}^\varepsilon = \mathfrak{h}^\varepsilon$. Since $\operatorname{supp}(b) \cap \mathfrak{h} = \emptyset$, the conclusion follows. ∎

LEMMA 2.17. *Let $\mathfrak{h}$, $\mathfrak{k}$ be ideals of E_+ such that $\mathfrak{h} \subseteq \mathfrak{k}$. Then the following holds:*

(i) $\mathfrak{h}^+ + \mathfrak{k}^+ = \mathfrak{k}^+$;

(ii) $\mathfrak{h}^+ + \mathfrak{k}^- = \mathfrak{k}^-$;

(iii) If $\mathfrak{h} \subsetneq \mathfrak{k}$, *then* $\mathfrak{h}^- + \mathfrak{k}^+ = \mathfrak{k}^+$;

(iv) $\mathfrak{h}^- + \mathfrak{k}^- = \mathfrak{k}^-$.

PROOF. (i) is immediate since $\mathfrak{h} + \mathfrak{k} = \mathfrak{k}$. (ii) $\mathfrak{k}^- \subseteq \mathfrak{h}^+ + \mathfrak{k}^-$ is trivial. Conversely, it suffices to prove $\mathfrak{h} + \mathfrak{k}^- \subseteq \mathfrak{k}^-$. Let $a \in \mathfrak{h}$, $b \in E_+ \setminus \mathfrak{k}$ (thus $a < b$). Then $b - a \notin \mathfrak{k}$, otherwise (since $a \in \mathfrak{h} \subseteq \mathfrak{k}$) $b \in \mathfrak{k}$, a contradiction. Thus $a - b \in \mathfrak{k}^-$. (iii) $\mathfrak{h}^- + \mathfrak{k}^+ \subseteq \mathfrak{k}^+$ is trivial. Conversely, fix c in $\mathfrak{k} \setminus \mathfrak{h}$. Let $b \in \mathfrak{k}^+$. Then $b + c \in \mathfrak{k}^+$; but $-c \in \mathfrak{h}^-$, whence $b = (-c) + (b + c) \in \mathfrak{h}^- + \mathfrak{k}^+$. (iv) $\mathfrak{h}^- + \mathfrak{k}^- \subseteq \mathfrak{k}^-$ is clear. Conversely, let $b \in \mathfrak{k}^-$. Then $b \in \mathfrak{h}^-$ (because $\mathfrak{h} \subseteq \mathfrak{k}$), whence $b/2 \in \mathfrak{h}^- \cap \mathfrak{k}^-$, thus $b = (b/2) + (b/2) \in \mathfrak{h}^- + \mathfrak{k}^-$. ∎

Now we can state the

LEMMA 2.18. *Let $\mathfrak{a}$, $\mathfrak{b}$ in* $\operatorname{In}(E)$. *Then the following holds:*

(i) $I(\mathfrak{a} + \mathfrak{b}) = I(\mathfrak{a}) \cup I(\mathfrak{b})$;

(ii) $\pi(\mathfrak{a} + \mathfrak{b}) = \big(\pi(\mathfrak{a}) + \pi(\mathfrak{b})\big)|_{A \setminus (I(\mathfrak{a}) \cup I(\mathfrak{b}))}$.

PROOF. Immediate from Lemmas 2.16 and 2.17 and from the fact that the inclusion relation is a linear ordering of $\operatorname{Idl}(E)$. ∎

LEMMA 2.19. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *in* $\mathrm{In}\,(E)$. *Then* $\mathfrak{a} \leq \mathfrak{b}$ *if and only if the following condition is satisfied:*

$$I(\mathfrak{a}) \subseteq I(\mathfrak{b}) \quad \textit{and} \quad \pi(\mathfrak{a})|_{A\setminus I(\mathfrak{b})} \leq \pi(\mathfrak{b}) \textit{ and}$$
$$\Big[\big(\epsilon(\mathfrak{a}) = + \textit{ and } \epsilon(\mathfrak{b}) = -\big) \Rightarrow \pi(\mathfrak{a})|_{A\setminus I(\mathfrak{b})} < \pi(\mathfrak{b})\Big] \textit{ and}$$
$$\Big[\big(I(\mathfrak{a}) \neq I(\mathfrak{b}) \textit{ and } \epsilon(\mathfrak{b}) = -\big) \Rightarrow \pi(\mathfrak{a})|_{A\setminus I(\mathfrak{b})} < \pi(\mathfrak{b})\Big] \textit{ and}$$
$$\Big[\big(I(\mathfrak{a}) = I(\mathfrak{b}) \textit{ and } \epsilon(\mathfrak{b}) = +\big) \Rightarrow \epsilon(\mathfrak{a}) = +\Big].$$

PROOF. Suppose first that $\mathfrak{a}+\mathfrak{c} = \mathfrak{b}$ for some positive $\mathfrak{c}$. By Lemma 2.18, $I(\mathfrak{a}) \subseteq I(\mathfrak{b})$. By possibly replacing $\mathfrak{c}$ by $\mathfrak{c} + I(\mathfrak{b})^+$ (note that $\mathfrak{b} + I(\mathfrak{b})^+ = \mathfrak{b}$), we may assume without loss of generality that $I(\mathfrak{c}) = I(\mathfrak{b})$. Let $a = \pi(\mathfrak{a})$, $b = \pi(\mathfrak{b})$, $c = \pi(\mathfrak{c})$. Since $\mathfrak{c}$ is positive, $c \geq 0$. We have $\mathfrak{b} = a + c + I(\mathfrak{b})^{\epsilon(\mathfrak{b})}$, whence, by Lemma 2.16, $b = (a + c)|_{A\setminus I(\mathfrak{b})} = a|_{A\setminus I(\mathfrak{b})} + c \geq a|_{A\setminus I(\mathfrak{b})}$. Now, we distinguish cases.

CASE 1. $\epsilon(\mathfrak{a}) = +$ and $\epsilon(\mathfrak{b}) = -$. By Lemma 2.17 (i), $\epsilon(\mathfrak{c}) = -$. But $\mathfrak{c} = c + I(\mathfrak{b})^-$ is positive, whence $c > 0$, *i.e.* $a|_{A\setminus I(\mathfrak{b})} < b$.

CASE 2. $I(\mathfrak{a}) \neq I(\mathfrak{b})$ and $\epsilon(\mathfrak{b}) = -$. By Lemma 2.17 (i,iii), $\epsilon(\mathfrak{c}) = -$, whence, as before, $c > 0$, *i.e.* $a|_{A\setminus I(\mathfrak{b})} < b$.

CASE 3. $I(\mathfrak{a}) = I(\mathfrak{b})$ and $\epsilon(\mathfrak{b}) = +$. Then, by Lemma 2.17 (ii), $\epsilon(\mathfrak{a}) = +$.

Conversely, suppose that the condition stated above is satisfied. Let $c = b - (a|_{A\setminus I(\mathfrak{b})})$. By assumption, $c \geq 0$. Note also that $\mathrm{supp}(c) \cap I(\mathfrak{b}) = \emptyset$. We argue by cases.

CASE 1. $\epsilon(\mathfrak{a}) = +$ and $\epsilon(\mathfrak{b}) = -$. Put $\mathfrak{c} = c + I(\mathfrak{b})^-$. Then $\mathfrak{a} + \mathfrak{c} = \mathfrak{b}$. Moreover, by assumption, $c > 0$; thus $c \notin I(\mathfrak{b})$, whence $\mathfrak{c} \geq \mathbf{0}$.

CASE 2. $I(\mathfrak{a}) \neq I(\mathfrak{b})$ and $\epsilon(\mathfrak{b}) = -$. Put again $\mathfrak{c} = c + I(\mathfrak{b})^-$. Then $\mathfrak{a} + \mathfrak{c} = \mathfrak{b}$, and $c > 0$ by assumption, thus, once again, $\mathfrak{c} \geq \mathbf{0}$.

CASE 3. $I(\mathfrak{a}) = I(\mathfrak{b})$ and $\epsilon(\mathfrak{b}) = +$. By assumption, $\epsilon(\mathfrak{a}) = +$. Put $\mathfrak{c} = c + I(\mathfrak{b})^+$. Then $\mathfrak{a} + \mathfrak{c} = \mathfrak{b}$ and $\mathfrak{c} \geq \mathbf{0}$.

CASE 4. Neither Case 1, 2 or 3. Then we argue by subcases.

- $I(\mathfrak{a}) = I(\mathfrak{b})$. Thus $\epsilon(\mathfrak{b}) = -$ [Case 3], thus $\epsilon(\mathfrak{a}) = -$ [Case 1]. Take $\mathfrak{c} = c + I(\mathfrak{a})^+$. Then $\mathfrak{a} + \mathfrak{c} = \mathfrak{b}$ and $\mathfrak{c} \geq \mathbf{0}$.
- $I(\mathfrak{a}) \neq I(\mathfrak{b})$. Thus $\epsilon(\mathfrak{b}) = +$ [Case 2]. Let $\mathfrak{c} = c + I(\mathfrak{b})^+$. Then $\mathfrak{c} \geq \mathbf{0}$, and, by Lemma 2.17 (i,iii), $\mathfrak{a} + \mathfrak{c} = \mathfrak{b}$. ■

Now, let $\mathrm{In}^*(E)$ be the set of all nonempty intervals of E of the form $(-\infty, a)$ or $(-\infty, a]$ where $a \in E \cup \{+\infty\}$. It is trivial that $\mathrm{In}^*(E)$ is a submonoid of $\mathrm{In}\,(E)$. We equip it with the restriction of the ordering of $\mathrm{In}\,(E)$. Put $\infty = (-\infty, +\infty)$.

Further, let $\mathfrak{m}$ be an ideal of E_+. We put $\mathrm{In}_{\mathfrak{m}}(E) = \{\mathfrak{a} \in \mathrm{In}\,(E) : \mathfrak{m} \subseteq I(\mathfrak{a})\}$. Let $\rho_{\mathfrak{m}}\colon \mathrm{In}\,(E) \longrightarrow \mathrm{In}_{\mathfrak{m}}(E)$, $\mathfrak{a} \longmapsto \mathfrak{a} + \mathfrak{m}^+$. It is clear, using Lemma 2.18, that $\mathrm{In}_{\mathfrak{m}}(E)$ is a subsemigroup of $\mathrm{In}\,(E)$ with zero $\mathfrak{m}^+$, and that $\rho_{\mathfrak{m}}$ is a monoid homomorphism from

$\big(\mathrm{In}(E), +, \mathbf{0}\big)$ to $\big(\mathrm{In}_{\mathfrak{m}}(E), +, \mathfrak{m}^{+}\big)$. Define a map $e_{\mathfrak{m}}$ from $\mathrm{In}_{\mathfrak{m}}(E)$ to $\mathrm{In}^{*}(E)$ by putting, for all $\mathfrak{a}$ in $\mathrm{In}_{\mathfrak{m}}(E)$,

$$e_{\mathfrak{m}}(\mathfrak{a}) = \begin{cases} \infty & (\mathfrak{m} \subsetneq I(\mathfrak{a})) \\ \big(-\infty, \pi(\mathfrak{a})\big] & (\mathfrak{m} = I(\mathfrak{a}) \text{ and } \epsilon(\mathfrak{a}) = +) \\ \big(-\infty, \pi(\mathfrak{a})\big) & (\mathfrak{m} = I(\mathfrak{a}) \text{ and } \epsilon(\mathfrak{a}) = -). \end{cases}$$

LEMMA 2.20. *$e_{\mathfrak{m}}$ is a monoid homomorphism from* $\mathrm{In}_{\mathfrak{m}}(E)$ *to* $\mathrm{In}^{*}(E)$.

PROOF. It is trivial that $e_{\mathfrak{m}}(\mathbf{0}) = \mathfrak{m}^{+}$. Now, let $\mathfrak{a}$, $\mathfrak{b}$ in $\mathrm{In}_{\mathfrak{m}}(E)$. Put $a = \pi(\mathfrak{a})$ and $b = \pi(\mathfrak{b})$. Two cases can occur:

CASE 1. $\mathfrak{m} \subsetneq I(\mathfrak{a}) \cup I(\mathfrak{b})$:

Since E is linearly ordered, the initial segments (thus *a fortiori* the ideals) of E are linearly ordered under inclusion. It follows that $\mathfrak{m} \subsetneq I(\mathfrak{a})$ or $\mathfrak{m} \subsetneq I(\mathfrak{b})$. Hence, $e_{\mathfrak{m}}(\mathfrak{a} + \mathfrak{b}) = \infty = e_{\mathfrak{m}}(\mathfrak{a}) + e_{\mathfrak{m}}(\mathfrak{b})$.

CASE 2. $\mathfrak{m} = I(\mathfrak{a}) = I(\mathfrak{b})$.

Then we use Lemma 2.17 in all the occurring subcases: if $\epsilon(\mathfrak{a}) = \epsilon(\mathfrak{b}) = +$, then $\epsilon(\mathfrak{a}+\mathfrak{b}) = +$, whence $e_{\mathfrak{m}}(\mathfrak{a}) = (-\infty, a]$, $e_{\mathfrak{m}}(\mathfrak{b}) = (-\infty, b]$ and $e_{\mathfrak{m}}(\mathfrak{a}+\mathfrak{b}) = (-\infty, a+b]$; if $\epsilon(\mathfrak{a}) = +$ and $\epsilon(\mathfrak{b}) = -$, then $\epsilon(\mathfrak{a} + \mathfrak{b}) = -$, whence $e_{\mathfrak{m}}(\mathfrak{a}) = (-\infty, a]$, $e_{\mathfrak{m}}(\mathfrak{b}) = (-\infty, b)$ and $e_{\mathfrak{m}}(\mathfrak{a} + \mathfrak{b}) = (-\infty, a + b)$; similarly for $\epsilon(\mathfrak{a}) = -$ and $\epsilon(\mathfrak{b}) = +$; finally, if $\epsilon(\mathfrak{a}) = \epsilon(\mathfrak{b}) = -$, then $\epsilon(\mathfrak{a} + \mathfrak{b}) = -$, whence $e_{\mathfrak{m}}(\mathfrak{a}) = (-\infty, a)$, $e_{\mathfrak{m}}(\mathfrak{b}) = (-\infty, b)$ and $e_{\mathfrak{m}}(\mathfrak{a} + \mathfrak{b}) = (-\infty, a + b)$. In all these cases, we have $e_{\mathfrak{m}}(\mathfrak{a} + \mathfrak{b}) = e_{\mathfrak{m}}(\mathfrak{a}) + e_{\mathfrak{m}}(\mathfrak{b})$. ■

Now, let e be the map from $\mathrm{In}(E)$ to ${}^{\mathrm{Idl}(E)}\,\mathrm{In}^{*}(E)$ defined by $e(\mathfrak{a}) = \langle e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) : \mathfrak{m} \in \mathrm{Idl}(E)\rangle$. Equip the power ${}^{\mathrm{Idl}(E)}\,\mathrm{In}^{*}(E)$ with its componentwise ordered monoid structure.

LEMMA 2.21. *e is an ordered monoid embedding.*

PROOF. Let $\mathfrak{a}$, $\mathfrak{b}$ be two elements of $\mathrm{In}(E)$ such that $e(\mathfrak{a}) \leq e(\mathfrak{b})$, *i.e.* for all $\mathfrak{m} \in \mathrm{Idl}(E)$, $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) \leq e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{b})$. We prove that $\mathfrak{a} \leq \mathfrak{b}$. Put $a = \pi(\mathfrak{a})$ and $b = \pi(\mathfrak{b})$.

Suppose first that $I(\mathfrak{b}) \subsetneq I(\mathfrak{a})$. Put $\mathfrak{m} = I(\mathfrak{b})$. Then $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) = \infty$ while $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{b}) = e_{\mathfrak{m}}(\mathfrak{b}) = (-\infty, b]$, which contradicts $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) \leq e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{b})$. Thus, since the ideals of E_{+} are linearly ordered under inclusion, we have $I(\mathfrak{a}) \subseteq I(\mathfrak{b})$. Put $a' = a|_{A \setminus I(\mathfrak{b})}$. Two cases can occur:

CASE 1. $I(\mathfrak{a}) = I(\mathfrak{b})$.

Note that $a = a'$. Put $\mathfrak{m} = I(\mathfrak{b})$. Then we have

$$e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) = \begin{cases} (-\infty, a] & (\epsilon(\mathfrak{a}) = +) \\ (-\infty, a) & (\epsilon(\mathfrak{a}) = -) \end{cases}$$

$$e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{b}) = \begin{cases} (-\infty, b] & (\epsilon(\mathfrak{b}) = +) \\ (-\infty, b) & (\epsilon(\mathfrak{b}) = -). \end{cases}$$

Thus $a \leq b$. Furthermore, if $\epsilon(\mathfrak{b}) = +$, then $\epsilon(\mathfrak{a}) = +$, and if $\epsilon(\mathfrak{a}) = +$ and $\epsilon(\mathfrak{b}) = -$, then $a < b$.

CASE 2. $I(\mathfrak{a}) \subsetneq I(\mathfrak{b})$.

Put $\mathfrak{m} = I(\mathfrak{b})$. By Lemma 2.17 (iii) and Lemma 2.16, we have $\rho_{\mathfrak{m}}(\mathfrak{a}) = a + I(\mathfrak{b})^+ = a' + I(\mathfrak{b})^+$, whence $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{a}) = (-\infty, a']$. On the other hand, $e_{\mathfrak{m}}\rho_{\mathfrak{m}}(\mathfrak{b}) = e_{\mathfrak{m}}(\mathfrak{b}) = \begin{cases} (-\infty, b] & (\epsilon(\mathfrak{b}) = +) \\ (-\infty, b) & (\epsilon(\mathfrak{b}) = -) \end{cases}$. Thus $a' \leq b$, and if $\epsilon(\mathfrak{b}) = -$, then $a' < b$.

Now, it follows from Lemma 2.19 that in all these cases, we have $\mathfrak{a} \leq \mathfrak{b}$. ∎

To complete this section, we need some more information about the theory of linearly ordered vector spaces. Consider the (infinite) first-order language $\mathcal{L} = (+, 0, \underline{\lambda})_{\lambda \in \mathbf{Q}}$ where for all $\lambda \in \mathbf{Q}$, $\underline{\lambda}$ is a unary function symbol. The theory $\mathcal{T}$ of *non trivial linearly ordered* $\mathbf{Q}$-*vector spaces* is by definition the following:

$$\begin{cases} \text{All axioms of abelian groups for } (+,0) & \\ (\forall x, y)\big(\underline{\alpha}(x+y) = \underline{\alpha}x + \underline{\alpha}y\big) & (\text{all } \alpha \text{ in } \mathbf{Q}) \\ (\forall x)(\underline{\gamma}x = \underline{\alpha}x + \underline{\beta}x) & (\text{all } \alpha, \beta, \gamma \text{ in } \mathbf{Q} \text{ with } \gamma = \alpha + \beta) \\ (\forall x)\big(\underline{\gamma}x = \underline{\alpha}(\underline{\beta}x)\big) & (\text{all } \alpha, \beta, \gamma \text{ in } \mathbf{Q} \text{ such that } \gamma = \alpha\beta) \\ (\forall x)(\underline{1}x = x) & \\ \leq \text{ is a linear ordering} & \\ (\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z) & \\ (\forall x)(0 \leq x \Rightarrow 0 \leq \underline{\alpha}x) & (\text{all } \alpha \text{ in } \mathbf{Q}_+) \\ (\exists x)(x \neq 0) & \end{cases}.$$

Of course, common practice is to write αx instead of $\underline{\alpha}x$. We refer to [3] for the standard terminology and results about elimination of quantifiers.

LEMMA 2.22. *$\mathcal{T}$ admits the elimination of quantifiers.*

PROOF. Write $a < b$ instead of "$a \leq b$ and $a \neq b$". Consider a formula θ of $\mathcal{L}$ of the form $(\exists y)\big(\varphi_1(\vec{x}, y) \wedge \cdots \wedge \varphi_k(\vec{x}, y)\big)$ where $\vec{x} = (x_j)_{1 \leq j \leq n}$ and each φ_j is of the form $\alpha y + \sum_j \alpha_j x_j \leq 0$ (resp. $\alpha y + \sum_j \alpha_j x_j < 0$), $\alpha, \alpha_j \in \mathbf{Q}$. By separating cases $\alpha = 0$ and $\alpha \neq 0$ and by dividing by α in the second case, we obtain that θ is equivalent (modulo $\mathcal{T}$) to the conjunction of a quantifier-free formula and a formula θ' of the form $(\exists y)\big(\psi_1(\vec{x}, y) \wedge \cdots \wedge \psi_l(\vec{x}, y)\big)$, where for all i, ψ_i is of the form $z_i \leq y$ $(i \in I_1)$ or $z_i < y$ $(i \in I_2)$ or $y \leq z_i$ $(i \in I_3)$ or $y < z_i$ $(i \in I_4)$, $[1, l]$ being the disjoint union of I_1, I_2, I_3, I_4 and the z_i's being linear combinations of the x_i's. It follows easily that θ' is equivalent to the conjunction of all formulas of the following system:

$$\begin{cases} z_{i_1} \leq z_{i_3} & (\text{all } (i_1, i_3) \in I_1 \times I_3) \\ z_{i_1} < z_{i_4} & (\text{all } (i_1, i_4) \in I_1 \times I_4) \\ z_{i_2} < z_{i_3} & (\text{all } (i_2, i_3) \in I_2 \times I_3) \\ z_{i_2} < z_{i_4} & (\text{all } (i_2, i_4) \in I_2 \times I_4), \end{cases}$$

which is a quantifier-free formula. ∎

COROLLARY 2.23. *Every model of $\mathcal{T}$ embeds [elementarily] into some ultrapower of* $\mathbf{Q}$.

PROOF. Let E be a model of $\mathcal{T}$. It is clear that $\mathbf{Q}$ embeds into E. By Lemma 2.22, this embedding from $\mathbf{Q}$ into E is elementary. It follows [3] that E embeds [elementarily] into some ultrapower of $\mathbf{Q}$. ∎

Now, say that a P. O. M. A satisfies the *Riesz property* when it satisfies the axiom

$$(\forall x, y, z)\big(z \leq x+y \Rightarrow (\exists x' \leq x)(\exists y' \leq y)(z = x'+y')\big).$$

When it is the case, denote by $\operatorname{In}(A)$ the set of all nonempty initial segments of A, equipped with the addition defined by

$$\mathfrak{a}+\mathfrak{b} = \{a+b : a \in \mathfrak{a} \text{ and } b \in \mathfrak{b}\}$$

(note that the Riesz property ensures that $\mathfrak{a}+\mathfrak{b}$ belongs to $\operatorname{In} A$), and the *minimal* preordering (which is contained in the inclusion relation, thus is an *ordering*). Denote by $\operatorname{In}^*(A)$ the subset of $\operatorname{In}(A)$ consisting of all nonempty intervals of A of the form $[0,a]$ or $[0,a)$, $a \in A \cup \{+\infty\}$. Put $\mathbb{T} = \operatorname{In}^*(\mathbf{Q}_+)$. Thus $\mathbb{T}$ is isomorphic to the P. O. M. of all positive elements of $\operatorname{In}^*(\mathbf{Q})$. And we can now come to the conclusion of this section:

PROPOSITION 2.24. *Let A be a linearly ordered set. Then* $\operatorname{In}\mathbb{P}\langle\!\langle A\rangle\!\rangle$ *embeds into a reduced power of* $\mathbb{T}$.

PROOF. Note that the class of all P. O. M.'s isomorphic to some reduced power of $\mathbb{T}$ is closed under reduced power (one uses "iterated reduced powers", the classical method for ultrapowers [3] applies to this case). But by Lemma 2.21 (restricted to positive elements), $\operatorname{In}\mathbb{P}\langle\!\langle A\rangle\!\rangle$ embeds into a direct power of $\operatorname{In}^*(\mathbb{P}\langle\!\langle A\rangle\!\rangle)$. Thus it suffices to prove that $\operatorname{In}^*(\mathbb{P}\langle\!\langle A\rangle\!\rangle)$ embeds into a ultrapower of $\mathbb{T}$. So let $\rho\colon \mathbb{P}\langle\!\langle A\rangle\!\rangle \longrightarrow {}^*\mathbf{Q}_+$, $a \longmapsto \langle \rho_i(a) : i \in I\rangle_{\mathcal{U}}$ be an embedding from $\mathbb{P}\langle\!\langle A\rangle\!\rangle$ into some ultrapower ${}^*\mathbf{Q}_+ =^I \mathbf{Q}_+/\,\mathcal{U}$ of $\mathbf{Q}_+$ (Lemma 2.23) ($\mathcal{U}$ is a ultrafilter on the set I). Then it is routine to verify that the map $\bar{\rho}\colon \operatorname{In}^*(\mathbb{P}\langle\!\langle A\rangle\!\rangle) \longrightarrow^I \mathbb{T}/\,\mathcal{U}$ which sends each $[0,a]$ on $\langle [0,\rho_i(a)] : i \in I\rangle_{\mathcal{U}}$, each $[0,a)$ on $\langle \big[0,\rho_i(a)\big) : i \in I\rangle_{\mathcal{U}}$ and ∞ on $\langle \infty : i \in I\rangle_{\mathcal{U}}$ is an embedding. ∎

3. **Embeddings of rational** P. O. M.'s.

DEFINITION 3.1. Say that a P. O. M. A is *divisible* when for each $m \in \mathbb{N} \setminus \{0\}$, it satisfies the axiom $(\forall x)(\exists y)(my = x)$. A *rational* P. O. M. is a *minimal*, *antisymmetric*, *unperforated*, *divisible* P. O. M.

If A is a rational P. O. M., then one can define naturally an action of the multiplicative semigroup of $\mathbf{Q}_+$ on A, by putting, for all $p \in \mathbb{N}$, $q \in \mathbb{N} \setminus \{0\}$ and $a \in A$, $(p/q)a = pb$ for the unique $b \in A$ such that $qb = a$.

The *finite refinement property* is by definition the following axiom [20, 21, 22, 23]:

$$(\forall a_0, a_1, b_0, b_1)(\exists_{i,j<2} c_{ij})\Big(a_0+a_1 = b_0+b_1 \Rightarrow \bigwedge\!\!\!\bigwedge_{i<2}(a_i = c_{i0}+c_{i1} \text{ and } b_i = c_{0i}+c_{1i})\Big).$$

A *refinement* P. O. M. is by definition a minimal P. O. M. satisfying the finite refinement property [21, 22, 23].

For every P. O. M. A, let $\operatorname{Canc}(A) = \{a \in A : (\forall x, y)(x+a \leq y+a \Rightarrow x \leq y)\}$. Thus $\operatorname{Canc}(A)$ is a sub-P. O. M. of A. In addition, if A is preminimal, then $\operatorname{Canc}(A)$ is an *ideal* (*i.e.* an initial segment and a submonoid) of A. If $e\colon A \longrightarrow B$ is a P. O. M.-embedding, say that e is a *C-embedding* when $e[\operatorname{Canc}(A)] \subseteq \operatorname{Canc}(B)$. If A is a sub-P. O. M. of B, say that B *refines* A when for all a_0, a_1, b_0, b_1 in A such that $a_0+a_1 = b_0+b_1$, there are c_{ij} $(i,j < 2)$ in B such that for all $i < 2$, $a_i = c_{i0}+c_{i1}$ and $b_i = c_{0i}+c_{1i}$; similarly, say that B *divides* A when for all m in $\mathbb{N} \setminus \{0\}$ and all $a \in A$, there exists b in B such that $mb = a$.

LEMMA 3.2. *Every minimal subrational* P. O. M. *can be C-embedded into a rational refinement* P. O. M.

PROOF. Let A be a minimal subrational P. O. M. By [23, Corollary 2.7], there exists a refinement P. O. M. C_0 containing A; actually, by incorporating into the argument of [23, Corollary 2.7] the inclusion map $m\mathbb{N} \hookrightarrow \mathbb{N}$ (all $m \in \mathbb{N} \setminus \{0\}$), one may as well suppose C_0 divisible. At that point, it is not ensured that A C-embeds into C_0, nor that C_0 is unperforated. So define binary relations $\leq_*$, $\equiv_*$ on C_0 by

$$x \leq_* y \Leftrightarrow (\exists m \in \mathbb{N} \setminus \{0\})\big(\exists z \in \mathrm{Canc}(A)\big)(mx + z \leq my + z),$$
$$x \equiv_* y \Leftrightarrow (x \leq_* y \text{ and } y \leq_* x).$$

Put $C_1 = (C_0, +, 0, \leq_*)/ \equiv_*$. Then C_1 is minimal, unperforated, antisymmetric, refines and divides A and A C-embeds into C_1. To conclude, we define a sequence $(B_n)_{n\in\omega}$ by $B_0 = A$, and for all $n \in \omega$, B_{n+1} is unperforated, antisymmetric, refines and divides B_n. Take $B = \bigcup_{n\in\omega} B_n$. ∎

Now, let us give some important examples of rational P. O. M.'s.

EXAMPLE 3.3. Let G be an unperforated, divisible abelian ordered group. Then G_+ is a rational P. O. M. We will call this particular sort of rational P. O. M. a *rational cone*. Note also that $G_+ \cup \{+\infty\}$ is a rational P.O.M.; G_+ and $G_+ \cup \{+\infty\}$ are in fact *separative* (see Definition 1.1, and also [23]).

For any P. O. M. E, denote by $\mathrm{DirIn}(E)$ the set of all *directed* initial segments of E. If E satisfies the *Riesz property* (see previous section), then one can define an addition on $\mathrm{DirIn}(E)$ by putting

$$\mathfrak{a} + \mathfrak{b} = \{a + b : a \in \mathfrak{a} \text{ and } b \in \mathfrak{b}\}.$$

LEMMA 3.4. *Let E be a rational refinement* P. O. M. *Then* $\mathrm{DirIn}(E)$*, equipped with its minimal (pre)ordering, is a rational* P. O. M.

PROOF. Easy. ∎

EXAMPLE 3.5. The space $\mathbb{T} = \mathrm{In}^*(\mathbb{Q}_+)$ considered in Section 2, or $\mathrm{In}\,\mathbb{R}_+$, are rational P. O. M.'s. They are not separative: if $\mathfrak{a} = [0, 1]$ and $\mathfrak{b} = [0, 1)$, then $\mathfrak{a} + \mathfrak{b} = 2\mathfrak{b}$ but $\mathfrak{a} \not\leq \mathfrak{b}$.

LEMMA 3.6. *Let I be an ideal of a refinement* P. O. M. *A. Then for all $a \in A$, $\rho_I(a) = \{x \in I : x \leq a\}$ belongs to* $\mathrm{DirIn}(I)$*, and ρ_I is a monoid-homomorphism from* $(A, +, 0)$ *to* $\big(\mathrm{DirIn}(I), +, \{0\}\big)$.

PROOF. It is trivial that $\rho_I(0) = \{0\}$. Fix $a \in A$; Let x, y in $\rho_I(a)$. Since A is minimal, there are x' and y' in A such that $a = x+x' = y+y'$. Since A satisfies the finite refinement property, there are p, q, r, s in A such that $x = p+q$, $x' = r+s$, $y = p+r$ and $y' = q+s$. Thus p, q, r belong to I, whence $z = p+q+r$ belongs to I; in addition, $x, y \leq z \leq p+q+r+s = a$, whence $z \in \rho_I(a)$: so we have proved that $\rho_I(a) \in \mathrm{DirIn}(I)$.

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Now, let a, b in A. It is obvious that $\rho_I(a)+\rho_I(b) \subseteq \rho_I(a+b)$. Conversely, let $z \in \rho_I(a+b)$. Since A is a refinement P. O. M., it satisfies the Riesz property, and thus there are $x \leq a$ and $y \leq b$ such that $z = x + y$. Thus $z \in \rho_I(a) + \rho_I(b)$, so that $\rho_I(a + b) = \rho_I(a) + \rho_I(b)$. ∎

Note that if E is a linear cone (see Section 2), then $\operatorname{DirIn}(E)$ is nothing else as $\operatorname{In}(E)$.

DEFINITION 3.7. Let E, E_i ($i \in I$) be rational P. O. M.'s, let $e: E \longrightarrow \prod_{i\in I} E_i$, $x \mapsto \langle e_i(x) : i \in I\rangle$ be a P. O. M.-embedding. For all $i \in I$, let $\leq_i$ be the preordering of E defined by $x \leq_i y$ if and only if $e_i(x) \leq e_i(y)$. We will say that $(E_i)_{i\in I}$ is a *separating family* for E (*via* e) when $\{\leq_i : i \in I\}$ is a closed subset of $\mathcal{P}(E \times E)$ (equipped with the product topology of the discrete $\mathbf{2} = \{0, 1\}$ *via* the identification of $\mathcal{P}(E \times E)$ and ${}^{E\times E}\mathbf{2}$).

In that context, we have the

LEMMA 3.8. *For all $i \in I$, the map $\bar{e}_i: \operatorname{DirIn}(E) \longrightarrow \operatorname{DirIn}(E_i)$, $\mathfrak{a} \mapsto \downarrow e_i[\mathfrak{a}]$ is a monoid homomorphism from $\big(\operatorname{DirIn}(E), +, \{0\}\big)$ to $\big(\operatorname{DirIn}(E_i), +, \{0\}\big)$.*

PROOF. Straightforward. ∎

LEMMA 3.9. *Let $x \in E$ and $\mathfrak{a} \in \operatorname{DirIn}(E)$. Then $x \in \mathfrak{a}$ if and only if $(\forall i \in I)\big(e_i(x) \in \bar{e}_i(\mathfrak{a})\big)$.*

PROOF. The direct implication is trivial. Conversely, suppose that $(\forall i \in I)\big(e_i(x) \in \bar{e}_i(\mathfrak{a})\big)$ but $x \notin \mathfrak{a}$. By definition, there exists $(x_i)_{i\in I}$ such that for all $i \in I$, $x_i \in \mathfrak{a}$ and $x \leq_i x_i$. Put $\Omega = \{\leq_i : i \in I\}$. For all $i \in I$, put $F_i = \{\sqsubseteq \in \Omega : x \not\sqsubseteq x_i\}$. Thus F_i is a clopen subset of Ω. Let p be a finite subset of I. Since $\mathfrak{a}$ is directed, there exists y in $\mathfrak{a}$ such that $(\forall i \in p)(x_i \leq y)$. If $(\forall \sqsubseteq \in \Omega)(x \sqsubseteq y)$, *i.e.* $(\forall j \in I)(x \leq_j y)$, then $x \leq y$ since e is a P. O. M.-embedding, whence $x \in \mathfrak{a}$, a contradiction. Thus $\bigcap_{i\in p} F_i \neq \emptyset$. By compactness of Ω, $\bigcap_{i\in I} F_i \neq \emptyset$. Let $\sqsubseteq$ be an element of $\bigcap_{i\in I} F_i$. There exists $i \in I$ such that $\sqsubseteq = \leq_i$. Thus $x \leq_i x_i$; but $\leq_i \in F_i$, whence $x \not\leq_i x_i$, a contradiction. ∎

The following corollary is now obvious:

LEMMA 3.10. *The map $\bar{e}: \operatorname{DirIn}(E) \longrightarrow \prod_{i\in I} \operatorname{DirIn}(E_i)$, $\mathfrak{a} \mapsto \langle \bar{e}_i(\mathfrak{a}) : i \in I\rangle$ is a one-to-one monoid homomorphism. In addition, for all $\mathfrak{a}$, $\mathfrak{b}$ in $\operatorname{DirIn}(E)$, $\mathfrak{a} \subseteq \mathfrak{b}$ if and only if $(\forall i \in I)\big(\bar{e}_i(\mathfrak{a}) \subseteq \bar{e}_i(\mathfrak{b})\big)$.* ∎

LEMMA 3.11. *Every linearly ordered rational cone embeds into a ultrapower of $\mathbb{R}_+$.*

PROOF. Let E be a linearly ordered rational cone. If $E = \{0\}$ then the conclusion is trivial. If $E \neq \{0\}$, then it is a model of the theory $\mathcal{T}$ considered in Section 2. Thus, by Corollary 2.23, E embeds into a ultrapower of $\mathbf{Q}_+$, hence *a fortiori* into a ultrapower of $\mathbb{R}_+$. ∎

LEMMA 3.12. *Every rational cone admits a separating family of the form $\langle \mathbb{P}\langle\!\langle A_i \rangle\!\rangle : i \in I\rangle$ where the A_i's are linearly ordered sets.*

PROOF. Let E be a rational cone. Thus $E = G_+$ for some directed, abelian, unperforated, divisible ordered group G. The analogue of Corollary 2.3 for linearly ordered $\mathbf{Q}_+$-vector spaces is still valid (with a similar proof). Thus if $\Omega = \{\leq_i : i \in I\}$ is the set of all group linear orderings on G containing $\leq$, then $\leq\; = \bigcap_{i\in I} \leq_i$. For all $i \in I$, let E_i be the positive cone of $(G, \leq_i)$. By Lemma 3.11, every E_i embeds into the positive cone E_i' of some ($\mathbb{R}$-) vector line. By Proposition 2.11, E_i' embeds into $F_i = \mathbb{P}\langle\!\langle A_i\rangle\!\rangle$ for some linearly ordered set A_i. Let $e: E \longrightarrow \prod_{i\in I} F_i$ be the diagonal map. Since $\leq\; = \bigcap_{i\in I} \leq_i$, e is a P. O. M.-embedding. For all $i \in I$ and all $x, y \in E$, $x \leq_{F_i} y$ if and only if $x \leq_i y$; since $\{\leq_i \cap (E\times E) : i \in I\}$ is closed, $(F_i)_{i\in I}$ is a separating family for E *via* e. ∎

Let now A be an arbitrary rational P. O. M. For each $a \in A$, consider the semigroup-congruence $\equiv_a$ on A defined by $x \equiv_a y \Leftrightarrow x + a = y + a$; for all $x \in A$, denote by $[x]_a$ the equivalence class of x modulo $\equiv_a$. Equip the monoid $\frac{A}{a} = (A, +, 0)/\equiv_a$ with its minimal ordering (so that $[x]_a \leq [y]_a \Leftrightarrow x + a \leq y + a$). It is clear that $e_a: A \longrightarrow \frac{A}{a}$, $x \longmapsto [x]_a$ is a P. O. M.-homomorphism. We refer to [21, 23] for more about this construction.

LEMMA 3.13. *$\frac{A}{a}$ is a rational* P. O. M., Canc$\left(\frac{A}{a}\right)$ *is a rational cone containing $\frac{A|a}{a}$ (where $A|a = \{x \in A : (\exists n \in \mathbb{N})(x \leq na)\}$).*

PROOF. Only the last assertion is not trivial. So let z in $A|a$, we must prove that $[z]_a$ is cancellable in $\frac{A}{a}$. So let x, y in A such that $[x]_a + [z]_a \leq [y]_a + [z]_a$. There exists $n \in \mathbb{N}$ such that $z \leq na$. It follows that $x + (n+1)a \leq y + (n+1)a$, whence, by an easy induction, $(n+1)x + (n+1)a \leq (n+1)y + (n+1)a$. Using $n+1$-unperforation of A, it follows that $x + a \leq y + a$, *i.e.* $[x]_a \leq [y]_a$. ∎

Now, for each $a \in A$, there exists by Lemma 3.2 a rational refinement P. O. M. A_a such that $\frac{A}{a}$ C-embeds into A_a. Thus, by Lemma 3.13, $E_a = \mathrm{Canc}(A_a)$ is a rational cone containing $\frac{A|a}{a}$. By Lemma 3.12, there exists a separating family $\langle E_{ai} : i \in I_a\rangle$ (*via* $\epsilon_a: x \longmapsto \langle \epsilon_{ai}(x) : i \in I_a\rangle$) for E_a such that for all $i \in I_a$, $E_{ai} = \mathbb{P}\langle\!\langle A_{ai}\rangle\!\rangle$ for some linearly ordered set A_{ai}. Let $\bar\epsilon_a$ be the corresponding monoid-embedding from $\left(\mathrm{DirIn}(E_a), +, \{0\}\right)$ into $\prod_{i\in I_a}\left(\mathrm{DirIn}(E_{ai}), +, \{0\}\right)$ as in Lemma 3.12.

PROPOSITION 3.14. *A embeds into the direct product*

$$\prod_{a\in A}\left(\frac{A|a}{a} \cup \{\infty\}\right) \times \prod_{a\in A, i\in I_a} \mathrm{In}\, E_{ai}.$$

PROOF. For all $a \in A$, let $e_a': A \longrightarrow \frac{A|a}{a} \cup \{\infty\}$ be defined by $e_a'(x) = [x]_a$ if $x \in A|a$, ∞ otherwise. Thus e_a' is a P. O. M.-homomorphism from A to $\frac{A|a}{a} \cup \{\infty\}$ (see [23]). Put $S = \prod_{a\in A}\left(\frac{A|a}{a} \cup \{\infty\}\right)$, and let $e: A \longrightarrow S$, $x \longmapsto \langle e_a'(x) : a \in A\rangle$.

CLAIM 1. *Let a, b in A. Then $e(a) \leq e(b)$ implies $a + b \leq 2b$.*

PROOF OF CLAIM. $e(a) \leq e(b)$ implies $e_b'(a) \leq e_b'(b)$. Thus $e_b'(a) \neq \infty$, *i.e.* $a \preceq b$, whence $[a]_b \leq [b]_b$. The conclusion follows. ∎

Now, for each $a \in A$ and $i \in I_a$, let $\leq_{ai}$ be the minimal ordering of $\mathrm{In}\, E_{ai}$. Define a P. O. M.-preordering $\leq_a$ on $\mathrm{DirIn}(E_a)$ by $\mathfrak{a} \leq_a \mathfrak{b}$ if and only if $(\forall i \in I_a)\left(\bar\epsilon_{ai}(\mathfrak{a}) \leq_{ai} \bar\epsilon_{ai}(\mathfrak{b})\right)$. By Lemma 3.10, $\leq_a$ is intermediate between the minimal ordering and the inclusion ordering on $\mathrm{DirIn}(E_a)$.

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For all a in A, let $\rho_a = \rho_{E_a}: A_a \longrightarrow \mathrm{DirIn}(E_a)$ as in Lemma 3.6. Since A_a is minimal, ρ_a is a P. O. M.-homomorphism. Thus we have the following P. O. M.-homomorphisms:

$$A \xrightarrow{e_a} \frac{A}{a} \hookrightarrow A_a \xrightarrow{\rho_a} \mathrm{DirIn}(E_a).$$

Let $\rho: A \longrightarrow \prod_{a\in A} \mathrm{DirIn}(E_a)$, $x \longmapsto \langle \rho_a e_a(x) \,:\, a \in A\rangle$. Thus ρ is a P. O. M.-homomorphism.

CLAIM 2. *Let a, b in A. Then $\rho(a) \leq \rho(b)$ implies $2a \leq a + b$.*

PROOF OF CLAIM. If $\rho(a) \leq \rho(b)$, then $\rho_a e_a(a) \leq_a \rho_a e_a(b)$, thus, since $\leq_a$ is contained into the inclusion, $\rho_a e_a(a) \subseteq \rho_a e_a(b)$. But $\rho_a e_a(a) = \{x \in E_a \,:\, x \leq e_a(a)\}$ contains $e_a(a)$ as an element (because $e_a(a) \in \frac{A|a}{a} \subseteq E_a$), thus $e_a(a) \in \rho_a e_a(b)$, thus $e_a(a) \leq e_a(b)$, *i.e.* $2a \leq a + b$. ■

Now, it follows immediately from Claims 1 and 2 and 2-unperforation of A that the P. O. M.-homomorphism $a \longmapsto \big(e(a), \rho(a)\big)$ is a P. O. M.-embedding. ■

COROLLARY 3.15. *Every rational* P. O. M. *embeds into a reduced power of* $\mathbb{T}$.

PROOF. Let A be a rational P. O. M. By Proposition 2.24, every $\mathrm{In}\, E_{ai}$ embeds into a reduced power of $\mathbb{T}$. Thus, by Proposition 3.14, it suffices to prove that for all $a \in A$, $\frac{A|a}{a} \cup \{\infty\}$ embeds into a reduced power of $\mathbb{T}$. Since $\frac{A|a}{a} \cup \{\infty\}$ embeds into $\prod_{i\in I_a}(E_{ai} \cup \{\infty\})$, it suffices to prove the conclusion for $E_{ai} \cup \{\infty\}$ for all $a \in A$, $i \in I_a$. But E_{ai} embeds into an ultrapower of $\mathbb{Q}_+$ (Corollary 2.23), and $\mathbb{Q}_+ \cup \{\infty\}$ obviously embeds into $\mathbb{T}$ (*via* $a \longmapsto [0, a] \cap \mathbb{Q}_+$). ■

It remains to drop the minimality assumption on A. This is the goal of the next section.

4. **Embeddings into minimal** P. O. M.'s. In this section, we shall prove that every subrational P. O. M. embeds into a rational P. O. M. The main difficulty is to embed a given preminimal P. O. M. into a minimal P. O. M. (it is *not* always possible—see next example—thus stronger sufficient conditions have to be found). Since it does not make the proof of Proposition 4.3 more complicated, we shall work in this general context. It can be shown for example that if a and b are elements of a given P. O. M. A, then A can be embedded into a P. O. M. B satisfying $(\exists x)(a + x = b)$ if and only if A satisfies $a \leq b$ *and* the following formula $\theta(a, b)$:

$$(\forall x, y)\big((x + a \leq y + a \Rightarrow x + b \leq y + b) \text{ and } (x + a = y + a \Rightarrow x + b = y + b)\big).$$

On the other hand, if one tries to consider all pairs (a, b) *simultaneously* (in the case where $(\forall a, b)\big(a \leq b \Rightarrow \theta(a, b)\big)$ holds), then it may not be possible to find an extension of A in which $(\forall a, b \in A)\big(a \leq b \Rightarrow (\exists x \in B)(a + x = b)\big)$.

EXAMPLE 4.1. Let A be the free antisymmetric preminimal P. O. M. with generators a, b, c, u, v, w and relations $a \leq b \leq c$, $b + u = a + v$, $c + u = a + w$. Suppose that there is an extension B of A containing as elements x, y such that $a + x = b$ and $b + y = c$. Then $c + v = a + x + y + v = b + u + y + x = c + u + x = a + w + x = b + w$. However, it is possible to prove that in A, $c + v \neq b + w$. Unfortunately, the proof of this fact is not short, thus we will not write it here, since it would digress too much from the main topic.

DEFINITION 4.2. A P. O. M. A is *strongly preminimal* when it satisfies both following axioms:

$$(\forall x, y, z)\big((x + z = y + z \text{ and } z \leq x, y) \Rightarrow x = y\big);$$
$$(\forall x, y, z)\big((x + z \leq y + z \text{ and } z \leq x, y) \Rightarrow x \leq y\big);$$

Thus, it is obvious that [separative] $\Rightarrow$ [strongly preminimal] $\Rightarrow$ [preminimal]. The main result of this section is the following

PROPOSITION 4.3. *Every strongly preminimal* P. O. M. *embeds into a minimal* P. O. M.

PROOF. Let A be a strongly preminimal P. O. M. Put $D = \{(a, b) \in A \times A : a \leq b\}$. For all $(a, b) \in D$, there exists a unique increasing function τ_{ab}, defined on $a+A$, such that $(\forall x \in A)(\tau_{ab}(x+a) = x+b)$ (this comes from the preminimality of A). Denote by $\mathbb{N}^{(D)}$ the free commutative monoid over D: elements of $\mathbb{N}^{(D)}$ are families $(m_{ab})_{(a,b)\in D}$ with finite support such that $(\forall (a, b) \in D)(m_{ab} \in \mathbb{N})$. Let $(\delta_{ab})_{(a,b)\in D}$ be the canonical basis for $\mathbb{N}^{(D)}$. For all $(a, b) \in D$, consider the function $\xrightarrow{ab}$ whose domain is the set of all $(x, \vec{m} + \delta_{ab})$ for $x \in a + A$ and $\vec{m} \in \mathbb{N}^{(D)}$, sending $(x, \vec{m} + \delta_{ab})$ on $\big(\tau_{ab}(x), \vec{m}\big)$. View the $\xrightarrow{ab}$'s as relations, and consider the monoid-congruence $\equiv$ on $A \times \mathbb{N}^{(D)}$ generated by all $\xrightarrow{ab}$'s, $(a, b) \in D$. Note that all $\xrightarrow{ab}$'s are compatible with the addition (*i.e.* $s_0 \xrightarrow{ab} s_1$ implies $s_0 + s \xrightarrow{ab} s_1 + s$), whence $\equiv$ is nothing else as the transitive closure of the union of all the $\xrightarrow{ab}$'s and their inverses, the $\xleftarrow{ab}$'s. Equip the quotient monoid $\mathrm{M}(A) = A \times \mathbb{N}^{(D)}/\equiv$ with its minimal preordering; for all $(a, \vec{m})$ in $A \times \mathbb{N}^{(D)}$, let $[a, \vec{m}]$ denote the equivalence class of $(a, \vec{m})$ modulo $\equiv$. Let $e : a \longmapsto [a, 0]$ be the natural homomorphism $A \longrightarrow \mathrm{M}(A)$. The aim of that proof is to show that e is a P. O. M.-embedding.

Say that a *path* is a word $w = w_1 \cdots w_l$ where for all $i \in [1, l]$, w_i is either α^+ or α^- for some $\alpha \in D$. We will call l the *length* of w. For example, for all α, β, γ, δ in D, $\alpha^+\beta^+\gamma^-\alpha^-\alpha^+\delta^+$ is a path of length 6. If w is a path and $s, t \in A \times \mathbb{N}^{(D)}$, then one defines the relation $s \xrightarrow{w} t$ by induction on the length of w, the natural way: if $w = w'\alpha^+$, then $s \xrightarrow{w} t$ if and only if $(\exists u)(s \xrightarrow{w'} u \xrightarrow{\alpha} t)$; if $w = w'\alpha^-$, then $s \xrightarrow{w} t$ if and only if $(\exists u)(s \xrightarrow{w'} u \xleftarrow{\alpha} t)$. Clearly, $s \equiv t$ if and only if $(\exists w)(s \xrightarrow{w} t)$. Note that at this point, we have just used preminimality of A.

CLAIM 1. *Let x, y in A, let $\vec{m}$ in $\mathbb{N}^{(D)}$. Then $(x, \vec{m}) \equiv (y, 0)$ implies $x \leq y$.*

PROOF OF CLAIM. By induction on the length of a path w from $(x, \vec{m})$ to $(y, 0)$. If w has length 0 then it is trivial. If w has length 1, then $w = \alpha^-$ is not possible ($\xrightarrow{\alpha}$ decreases strictly the second coordinate), thus $(x, \vec{m}) \xrightarrow{\alpha} (y, 0)$ for some $\alpha \in D$, thus $y = \tau_\alpha(x) \geq x$. Suppose that the claim is proved for all paths of length $< l$ where $l \geq 2$ is the length of w. If $w = \alpha^+w'$ for some $\alpha \in D$ and some path w', then there exists $(z, \vec{n})$ such that

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$(x, \vec{m}) \xrightarrow{\alpha} (z, \vec{n}) \xrightarrow{w'} (y, 0)$. By induction hypothesis, $z \leq y$. But $z = \tau_\alpha(x) \geq x$, whence $x \leq y$. So now, suppose that $w = \alpha^- w'$ for some $\alpha \in D$ and some path w'. Necessarily, α^+ appears at least once in w', because $\xrightarrow{\alpha}$ decreases strictly the α^{th} component of the second coordinate. Thus, one can write $w = \alpha^- u \alpha^+ v$, where u is some path without any occurrence of α^+ (it may have α^-) and v is a path. Thus we can write

$$(x, \vec{m}) \xleftarrow{\alpha} (x_1, \vec{m}_1) \xrightarrow{u} (x_2, \vec{m}_2) \xrightarrow{\alpha} (x_3, \vec{m}_3) \xrightarrow{v} (y, 0)$$

for some $(x_i, \vec{m}_i)$ $(i = 1, 2, 3)$ in $A \times \mathbb{N}^{(D)}$. Furthermore, since α^+ does not appear in u, every intermediate step (boundary included) $(z, \vec{n})$ in $(x_1, \vec{m}_1) \xrightarrow{u} (x_2, \vec{m}_2)$ satisfies $\vec{n} \geq \delta_\alpha$. Therefore, one also has

$$(x_1, \vec{m}_1 - \delta_\alpha) \xrightarrow{u} (x_2, \vec{m}_2 - \delta_\alpha), \text{ thus, } \textit{a fortiori},$$

$$(x_1 + x, \vec{m}_1 - \delta_\alpha) \xrightarrow{u} (x_2 + x, \vec{m}_2 - \delta_\alpha).$$

Furthermore, we have $x_2 + x = x_2 + \tau_\alpha x_1 = \tau_\alpha x_2 + x_1 = x_3 + x_1$ and $\vec{m}_2 - \delta_\alpha = \vec{m}_3$, thus we have $(x_2 + x, \vec{m}_2 - \delta_\alpha) = (x_3 + x_1, \vec{m}_3)$. It follows finally that

$$(x_1 + x, \vec{m}_1 - \delta_\alpha) \xrightarrow{u} (x_2 + x, \vec{m}_2 - \delta_\alpha) \xrightarrow{v} (y + x_1, 0).$$

Since uv has length l—2, the induction hypothesis yields $x_1 + x \leq x_1 + y$. But $x_1 \leq \tau_\alpha x_1 = x$, and furthermore, $u\alpha^+ v$ is a path from $(x_1, \vec{m}_1)$ to $(y, 0)$ whence, by induction hypothesis, $x_1 \leq y$. Since A is strongly preminimal, we get $x \leq y$. ■

CLAIM 2. *Let x, y in A such that* $(x, 0) \equiv (y, 0)$. *Then* $x = y$.

PROOF OF CLAIM. Let w be a path from $(x, 0)$ to $(y, 0)$. Again, we argue by induction on the length l of w. If $l = 0$ it is trivial. Otherwise $l \geq 2$, and there are $\alpha \in D$ and paths u, v such that $w = \alpha^- u \alpha^+ v$ and α^+ does not appear in u. Thus again, there are $(x_i, \vec{m}_i)$ $(i = 1, 2, 3)$ such that

$$(x, 0) \xleftarrow{\alpha} (x_1, \vec{m}_1) \xrightarrow{u} (x_2, \vec{m}_2) \xrightarrow{\alpha} (x_3, \vec{m}_3) \xrightarrow{v} (y, 0)$$

(thus $\vec{m}_1 = \delta_\alpha$). Again, as in the proof of Claim 1, for all intermediate stages $(z, \vec{n})$ in $(x_1, \vec{m}_1) \xrightarrow{u} (x_2, \vec{m}_2)$, we have $\vec{n} \geq \delta_\alpha$. Since $\vec{m}_1 - \delta_\alpha = 0$, we get $(x_1, 0) \xrightarrow{u} (x_2, \vec{m}_2 - \delta_\alpha)$, and thus, as in the proof of Claim 1,

$$(x_1 + x, 0) \xrightarrow{u} (x_2 + x, \vec{m}_2 - \delta_\alpha) \xrightarrow{v} (y + x_1, 0).$$

Thus, by the induction hypothesis, $x_1 + x = x_1 + y$. But $x_1 \leq \tau_\alpha x_1 = x$, and $x_1 \leq y$ by the result of Claim 1. Since A is strongly preminimal, we get $x = y$. ■

But Claim 1 and Claim 2 together imply that e is an embedding, which concludes the proof. ■

We can now prove the following

PROPOSITION 4.4. *Every subrational* P. O. M. *embeds into a rational* P. O. M.

PROOF. Let A be a subrational P. O. M.

CLAIM. *A is strongly preminimal.*

PROOF OF CLAIM. Let a, b, c in A such that $a+c \leq b+c$ and $c \leq a, b$. Since A is preminimal and $c \leq a$, we have $2a \leq a+b$; since A is preminimal and $c \leq b$, we have $a+b \leq 2b$. Thus $2a \leq 2b$, but A is 2-unperforated, thus $a \leq b$. Since A is antisymmetric, we can conclude. ∎

By Proposition 4.3 and the claim, A embeds into a minimal P. O. M. M. As in the proof of Lemma 3.2, M can be embedded into a minimal, divisible P. O. M., so that one can assume without loss of generality that M is divisible. Define on M binary relations $\leq_*$ and $\equiv_*$ by

$$x \leq_* y \Leftrightarrow (\exists m \in \mathbb{N} \setminus \{0\})(mx \leq my),$$
$$x \equiv_* y \Leftrightarrow x \leq_* y \text{ and } y \leq_* x.$$

Let B the P. O. M. $(M, +, 0, \leq_*)/\equiv_*$. By definition, B is antisymmetric and unperforated. Since M is divisible and minimal, B is divisible and minimal, so that B is a rational P. O. M. Furthermore, since A is antisymmetric and unperforated, the natural homomorphism from A to B is an embedding, which concludes the proof. ∎

Now we can state the

THEOREM 4.5. *A* P. O. M. *embeds into a reduced power of* $\mathbb{T}$ *if and only if it is a subrational* P. O. M.

PROOF. It is immediate that reduced powers of $\mathbb{T}$ are subrational P. O. M.'s. Conversely, we conclude by Proposition 4.4 and Corollary 3.15. ∎

QUESTION 4.6. It is proved in [23] that every separative P. O. M. embeds into a minimal, separative P. O. M. *Does every strongly preminimal* P. O. M. *embed into a minimal, strongly preminimal* P. O. M.*?*

5. **Embeddings into full type spaces.** At that point, full type spaces do not play any role in the proof of Theorem 4.5. As to embeddability into type spaces, the following question comes up:

Does $\mathbb{T}$ *embed into a full type space?*

A crucial intermediary result is the following proposition, due to M. Laczkovich [14]; we reproduce the proof here, with the authorization of the author:

PROPOSITION 5.1 (M. LACZKOVICH). *There are a full type space A and two elements a, b of A such that $a+b=2b$ but $a \not\leq b$.*

PROOF. Put $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Put $X = \mathbb{N}^+ \times \{0,1,2,3\}$, $A = \mathbb{N}^+ \times \{0\}$, $B_i = \mathbb{N}^+ \times \{i\}$ $(i = 1,2,3)$. We define two bijections, a and b of X onto itself as follows. First we define for every $k, m \in \mathbb{N}$,

$$(*) \qquad a\big((2k+1,0)\big) = (2k+1,2), \quad a\big((2k+2,0)\big) = (2k+1,3),$$

and

$$(**)\qquad a\big((2^{m+2}k+2^m,1)\big)=(2^{m+2}k+2^{m+1},2),$$

$$(***)\qquad a\big((2^{m+2}k+2^{m+1}+2^m,1)\big)=(2^{m+2}k+2^{m+1},3).$$

Then the map a is a bijection from $A\cup B_1$ onto $B_2\cup B_3$. We extend a to $B_2\cup B_3$ such that $a=a^{-1}$ holds on X. Next we define for every $n\in\mathbb{N}^+$

$$b\big((n,0)\big)=(n,0),\quad b\big((n,1)\big)=(n,2),\quad b\big((n,2)\big)=(n,3),\quad b\big((n,3)\big)=(n,1).$$

Then b is a bijection of X onto itself such that b^3 is the identity map. Let G denote the group generated by a and b. It is clear that $A\cup B_1\equiv_G B_2\cup B_3$ and $B_1\equiv_G B_2\equiv_G B_3$. However, we shall prove that A is not G-equidecomposable to any subset of B_1. For every $s\in\mathbb{N}^+$, we shall put

$$G_s=\big\{b^{n_s}ab^{n_{s-1}}a\cdots b^{n_1}ab^{n_0}:n_0,n_s\in\{0,1,2\},n_i\in\{1,2\}\text{ (all }i\in[1,s-1])\big\}.$$

Since $a^2=b^3=\mathrm{id}$, we have $G\setminus\{\mathrm{id},b,b^2\}=\bigcup_{s=1}^{+\infty}G_s$.

CLAIM. *Let $s\in\mathbb{N}^+$, $p,n\in\mathbb{N}$, $f\in G_s$ such that $f\big((n,0)\big)=(p,i)$. Then we have*

(i) $|p-n|\leq 2^s$;

(ii) If $i\neq 0$, then there are m, k in $\mathbb{N}$ such that $p=2^{m+1}k+2^m$ and $m<s$.

PROOF OF CLAIM. By induction on s. If $f\in G_1$ then $f=b^{n_1}ab^{n_0}$ for some $n_0,n_1\in\{0,1,2\}$. If $f\big((n,0)\big)=(p,i)$, then $a\big((n,0)\big)=(p,j)$ since b does not move the points of A and does not change the first coordinate of any point. Then, by $(*)$, $|p-n|\leq 1$ and $p=2k+1$. Thus the statement of the claim is true for $s=1$.

Let $s>1$ and suppose that the claim is true for $s-1$. If $f\in G_s$, then $f=b^{n_s}ag$, where $g\in G_{s-1}$. Let $f\big((n,0)\big)=(p,i)$ and $g\big((n,0)\big)=(q,j)$. Then we have $a\big((q,j)\big)=(p,i')$ for some $i'\in\{0,1,2,3\}$. By the induction hypothesis, $|q-n|\leq 2^{s-1}$.

If $j=0$ then $|p-q|\leq 1$ by $(*)$ and hence $|p-n|\leq 2^{s-1}+1<2^s$. Also, $p=2k+1$ and hence (ii) holds.

Suppose $j\neq 0$. Then, by the induction hypothesis, $q=2^{m+1}k+2^m$, where $m<s-1$. Assume first that $m=0$. If $j\in\{2,3\}$ then $a\big((q,j)\big)=(p,0)$ and $|p-q|\leq 1$ by $(*)$. If $j=1$ then $p=q\pm 1$ by $(**)$ and $(***)$. In both cases $|p-n|\leq 2^{s-1}+1<2^s$. Also, if $i\neq 0$ then $i'\neq 0$ and it follows from the definition of a that only $j=1$ is possible. Then $p=4l+2$ for some l and thus (ii) holds.

Next suppose $m>0$. Then $a\big((q,j)\big)=(p,i')$ implies $|p-q|\leq 2^m<2^{s-1}$ by $(**)$ and $(***)$, and hence $|p-n|\leq|p-q|+|q-n|<2^{s-1}+2^{s-1}=2^s$. Also, if $i\neq 0$ then $i'\neq 0$, and there is l such that we have either $p=2^{m+2}l+2^{m+1}$ (if $j=1$) or $p=2^ml+2^{m-1}$ (if $j\in\{2,3\}$). Since $m+1<s$, we have (ii) in each case and this completes the proof of the claim. ∎

Now suppose that $A\equiv_G A'\subseteq B_1$. Then there are partitions $A=\bigcup_{i<t}A_i$ and $A'=\bigcup_{i<t}A'_i$ and maps f_i $(i<t)$ in G such that for all $i<t$, $f_i[A_i]=A'_i$. Obviously, the f_i's cannot be powers of b and hence there is an N in $\mathbb{N}^+$ such that $\{f_i:i<t\}\subseteq\bigcup_{s=1}^N G_s$. Let

$U = [1, 2^{2N}] \times \{0\}$ and $V = \bigcup_{i<t} f_i[U \cap A_i]$. Then $|V| = |U| = 2^{2N}$. On the other hand, if $(p, 1) \in V$ then $(p, 1) = f\big((n, 0)\big)$ for some $n \leq 2^{2N}$ and $f \in G_s$ with $s \leq N$. Thus, by the claim, $p \leq 2^{2N} + 2^N$ and $p = 2^{m+1}k + 2^m$ for some $m < N$. But for a fixed $m < N$ the number of k's satisfying $2^{m+1}k + 2^m \leq 2^{2N} + 2^N$ is $2^{2N-m-1} + 2^{N-m-1} - 1$. Thus,

$$|V| \leq \sum_{m<N} (2^{2N-m-1} + 2^{N-m-1} - 1) < 2^{2N}(1 - 2^{-N}) + 2^N = 2^{2N},$$

a contradiction. Thus the conclusion holds with $a = [A]_G$, $b = [B_1]_G$. ∎

This allows us to answer affirmatively the question whether $\mathbb{T}$ is a full measure P.O.M.:

COROLLARY 5.2. *$\mathbb{T}$ is a full measure* P. O. M.

PROOF. By Proposition 5.1, there are a full type space $A = S(X)/G$ and elements a, b of A such that $a+b = 2b$ but $a \not\leq b$. In addition, using Lemma 1.7 (and taking $c = [\hat{X}]_{\hat{G}}$), it is easy to see that one can suppose that there is c in A such that $a+c = b+c = c = 2c$.

Put again $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. For all m, n in $\mathbb{N}^+$ such that $m|n$ (*i.e.* m divides n), let $e^n_m : A \longrightarrow A, x \longmapsto (n/m)x$. Then it is immediate that ($p|q$ and $q|r$) implies that $e^r_p = e^r_q \circ e^q_p$; furthermore, $e^n_n = \mathrm{id}$ for all $n \in \mathbb{N}^+$. Thus, we can form the *direct limit* of this system, that is, $(B, e_n)_{n\in\mathbb{N}^+} = \varinjlim(A_m, e^n_m)_{m,n\in\mathbb{N}^+, m|n}$ (so that $B = \bigcup_{n\in\mathbb{N}^+} e_n[A]$). By Corollary 1.4, the e^n_m's are all embeddings. Therefore, all the e_n's are embeddings. We have the

CLAIM 1. *B is divisible.*

PROOF OF CLAIM. Let $x \in B$, $m \in \mathbb{N}^+$. There exists $n \in \mathbb{N}^+$ such that $x \in e_n[A]$, so that $x = e_n(t)$, $t \in A$. Put $y = e_{mn}(t)$. Then we have

$$my = e_{mn}(mt) = e_{mn}e^{mn}_n(t) = e_n(t) = x,$$

whence the conclusion holds. ∎

But B is, by Corollary 1.10, a full measure P. O. M. Since it is divisible (and unperforated), the multiplicative semigroup of $\mathbb{Q}_+$ acts on B. Furthermore, since e_1 is an embedding, we can identify A with $e_1[A]$.

CLAIM 2. *Let r, s in $\mathbb{Q}_+$. Then the following holds:*
(i) $ra \leq sa$ if and only if $rb \leq sb$ if and only if $r \leq s$;
(ii) $ra \leq sb$ if and only if $r = s = 0$ or $r < s$;
(iii) $ra + sb = (r+s)b$ if $s > 0$;
(iv) $rb \leq sa$ if and only if $r = 0$.

PROOF OF CLAIM. Since B is unperforated, it suffices to prove (i)–(iv) for $r, s \in \mathbb{N}$. If $2b = b$, then $a \leq a+b = 2b = b$, a contradiction; if $2a = a$, then $2a \leq 2a+b = a+b = 2b$, whence $a \leq b$, a contradiction. Since B is a subrational P. O. M., it follows that for all $n \in \mathbb{N}$, $(n+1)a \not\leq na$ and $(n+1)b \not\leq nb$; (i) follows. (iii) results immediately from $a + b = 2b$. Furthermore, it implies the right-to-left implication in (ii); if $(r, s) \neq (0, 0)$ and $ra \leq sb$, suppose $r \geq s$. Thus $r > 0$ and $ra \leq rb$, whence $a \leq b$, contradiction; (ii) follows. To prove (iv), it suffices to prove that for all $n \in \mathbb{N}$, $b \not\leq na$; suppose

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otherwise; since $a+b=b+b$ and $b \leq na$, we have $a+na \leq b+na$, *i.e.* $(n+1)a \leq (n+1)b$, whence $a \leq b$, a contradiction; (iv) follows. ∎

Now, define a map $\varphi\colon \mathbb{T} \longrightarrow B$ by putting, for all $r \in \mathbb{Q}_+$,

$$\begin{cases} \varphi\big([0,r]\big) = ra, & \\ \varphi\big([0,r)\big) = rb & \text{if } r > 0, \\ \varphi\big([0,+\infty)\big) = c. & \end{cases}$$

It is immediate, using Claim 2, that φ is an embedding. Thus, $\mathbb{T}$ *embeds into* B. ∎

Before stating the main theorem, recall that $\mathbb{T}$ is the space of all intervals $\mathfrak{a}$ of $\mathbb{Q}_+$ with rational (possibly infinite) endpoints such that $0 \in \mathfrak{a}$, equipped with the addition defined by $\mathfrak{a}+\mathfrak{b} = \{a+b : a \in \mathfrak{a} \text{ and } b \in \mathfrak{b}\}$ and the ordering $\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow (\exists \mathfrak{c})(\mathfrak{a}+\mathfrak{c} = \mathfrak{b})$.

THEOREM 5.3. *Let S be a preordered semigroup. Then the following are equivalent:*

(1) S embeds into a full type space;

(2) S embeds into a reduced power of $\mathbb{T}$;

(3) S satisfies the following list of axioms:

(i) $(\forall x, y)(x \leq x+y)$;

(ii) $(\forall x, y)\big((x \leq y \text{ and } y \leq x) \Rightarrow x = y\big)$;

(iii) $(\forall x, y, u, v)\big((x+u \leq y+u \text{ and } u \leq v) \Rightarrow x+v \leq y+v\big)$;

(iv) $(\forall x, u, v)\big((x+u = u \text{ and } u \leq v) \Rightarrow x+v = v\big)$;

(v) $(\forall x, y)(mx \leq my \Rightarrow x \leq y)$ *(all $m \in \mathbb{N} \setminus \{0\}$).*

PROOF. (2) ⇒ (3) is trivial. Conversely, assume (3). Let A be the commutative monoid obtained by adjoining a zero element 0 to S; extend the preordering from S to A by saying that $0 < x$ for all $x \in S$. Since S satisfies axiom (i), A is a P. O. M. Using the remaining axioms, it is easy to prove that A is a subrational P. O. M. We conclude by Theorem 4.5. (1) ⇒ (2) comes from Theorem 4.5 and the fact that full type spaces are subrational P. O. M.'s. (2) ⇒ (1) comes from Corollary 5.2 and Lemma 1.9. ∎

COROLLARY 5.4. *The classes of subrational* P. O. M.*'s and full measure* P. O. M.*'s coincide.* ∎

COROLLARY 5.5. *The universal theory of all full type spaces,* i.e. *the set of all universal formulas of the language* $(+, \leq)$ *that hold in every full type space, is decidable.*

PROOF. By Theorem 5.3, the universal theory of full type spaces is the set of universal formulas φ such that $\mathcal{T} \vdash \varphi$, where $\mathcal{T}$ is the theory consisting of the list of axioms (i)–(v) above. Writing the quantifier-free part of φ in normal conjunctive form, one sees that one can without loss of generality restrict attention to formulas φ of the form $(\forall \vec{x})\Big(A(\vec{x}) \Rightarrow \big(B_1(\vec{x}) \vee \cdots \vee B_n(\vec{x})\big)\Big)$ where A is a conjunction of atomic formulas and the B_i's are atomic. But we have the

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CLAIM. *Let φ be $(\forall\vec{x})\big(A(\vec{x}) \Rightarrow \big(B_1(\vec{x}) \vee \cdots \vee B_n(\vec{x})\big)\big)$ as above with $n \neq 0$. Then $\mathcal{T} \vdash \varphi$ if and only if there exists i in $[1,n]$ such that $\mathcal{T} \vdash (\forall\vec{x})\big(A(\vec{x}) \Rightarrow B_i(\vec{x})\big)$.*

PROOF OF CLAIM. We prove the non trivial direction. So suppose that for all i, $\mathcal{T} \nvdash (\forall\vec{x})\big(A(\vec{x}) \Rightarrow B_i(\vec{x})\big)$. Thus for all i, there exists a model S_i of $\mathcal{T}$ and a list $\vec{a}_i$ of elements of S_i such that $S_i \models A(\vec{a}_i)$ and $\neg B_i(\vec{a}_i)$. Let $S = \prod_{i=1}^n S_i$. Then S is still a model of $\mathcal{T}$; denote by π_i the i^{th} projection from S onto S_i. If $\vec{a}$ is the list of elements of S such that for all i, $\pi_i\vec{a} = \vec{a}_i$, then $S \models A(\vec{a})$ but for all i, $S \not\models B_i(\vec{a})$. Thus $\mathcal{T} \nvdash \varphi$. ■

Thus it suffices to be able to decide if formulas φ of the form $(\forall\vec{x})\big(A(\vec{x}) \Rightarrow B(\vec{x})\big)$ or $(\forall\vec{x})\neg A(\vec{x})$, where A is a conjunction of atomic formulas and B is an atomic formula, are consequences of $\mathcal{T}$. But then, φ is a universal Horn formula, thus preserved under submodels of reduced products ([3]; a direct proof is easy). Thus, by Theorem 4.5, $\mathcal{T} \vdash \varphi$ if and only if $\mathbb{T} \models \varphi$. Now, elements of $\mathbb{T}$ have always the form $[0,r]$ ($r \in \mathbf{Q}_+$) or $[0,r)$ ($r \in (\mathbf{Q}_+ \setminus \{0\}) \cup \{+\infty\}$), and modulo this representation, the interpretations of + and 0 in $\mathbb{T}$ are easily seen to be definable (without quantifiers) in $(+_{\mathbf{Q}}, \leq_{\mathbf{Q}})$. Thus, the problem reduces to know whether $\mathbf{Q}_+ \models \psi$ where ψ is some universal formula (constructed recursively from φ). But this is known to be decidable (see *e.g.* Lemma 2.22). ■

REMARK 5.6. It is easy to prove that in fact, $\mathbb{T}$ is a *refinement algebra* [18, Definition 11.26]; this provides us with a simple example of refinement algebra that does not embed into any strong refinement P. O. M. (see [21, Definition 1.12] and also [16]). Also, every reduced power of $\mathbb{T}$ is also a refinement algebra. Full type spaces are also refinement algebras, but the proof is much less easy [18, Theorem 11.12]. It follows that *every subrational* P. O. M. *embeds into a rational refinement algebra.*

QUESTION 5.7. Ketonen's Theorem [9] states that every countable commutative semigroup can be embedded into the monoid of isomorphism types of *countable* Boolean algebras. Here, Theorem 5.3 provides us an analogue for full type spaces of this theorem, namely that *every subrational* P. O. M. *embeds into a full type space.* A natural way to refine this result would be the following (recall that F_2 denotes the free group on two generators):

Does every countable subrational P. O. M. *embed into* $\mathrm{S}(F_2)/F_2$*? (F_2 acts on itself by left translations).*

QUESTION 5.8. In this paper, we have considered full type spaces, with arbitrary group actions. It is also known [18] that if an exponentially bounded group G acts on a set Ω, then the corresponding full type space satisfies the axiom $(\forall a,b,c)(a + c = b + 2c \Rightarrow a = b + c)$, which is not the case for $\mathbb{T}$. The question is:

Is there any analogue of Theorem 5.3 for full type spaces with exponentially bounded groups?

QUESTION 5.9. It is not difficult to prove (just by algebraic methods) that the space X of the proof of Proposition 5.1 is not G-paradoxical. This suggests a positive answer to the following question:

Does every subrational P. O. M. *embed into some* $S(\Omega)/G$ *where G is* amenable*?*

QUESTION 5.10. Say that an embedding from A into B is *pure* [6] whenever for every positive existential formula φ with parameters from A, $B \models \varphi$ implies $A \models \varphi$. Theorem 5.3 shows that any subrational P. O. M. embeds into a full type space. Under which conditions can a subrational P. O. M. be *purely* embedded into a full type space? For example, it has to be a *refinement algebra* [18]. Is this also sufficient? That is, *does every subrational refinement algebra admit a pure embedding into some full type space?*

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