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NILPOTENT ORBITS OF EXCEPTIONAL LIE ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS OF BAD CHARACTERISTIC

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Abstract

The classification of the nilpotent orbits in the Lie algebra of a reductive algebraic group (over an algebraically closed field) is given in all the cases where it was not previously known ($\mathbf{E}_7$ and $\mathbf{E}_8$ in bad characteristic, $\mathbf{F}_4$ in characteristic 3). The paper exploits the tight relation with the corresponding situation over a finite field. A computer is used to study this case for suitable choices of the finite field.

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Let $\mathfrak{g}$ be the Lie algebra of a connected reductive algebraic group G defined over an algebraically closed field k of characteristic p (0 or a prime). The classification of nilpotent orbits in $\mathfrak{g}$ reduces easily to the case where G is simple, and it is known in most cases. If p is good for G (that is, not bad [20, page 178]), Springer's correspondence [20, page 229] allows us to use the classification of unipotent elements in G ([20], [2], [7], [13], [11], [12]). If p is bad, the following cases are known: G classical [5], G_2 [23], E_6 [17], F_4 when $p = 2$ [18].

The remaining cases are

(a) $F_4, p = 3$,

(b) $E_7, p = 2, 3$,

(c) $E_8, p = 2, 3, 5$.

They are dealt with in this paper.

If k is an algebraic closure of a finite field $\mathbf{F}_q$ and G is defined over $\mathbf{F}_q$, it should be possible to use a computer in the study of the action of G^F on $\mathfrak{g}^F$, where F

denotes the Frobenius morphisms of G and $\mathfrak{g}$, and this would yield much information on the action of G on $\mathfrak{g}$. Let U be an F-stable maximal unipotent subgroup of G, with Lie algebra $\mathfrak{u}$. Instead of looking directly at the action of G^F on $\mathfrak{g}^F$, the computer is used in this paper to calculate the order of the stabilizer in U^F of various elements of $\mathfrak{u}^F$. As far as programming is concerned, this has the advantage that there are efficient algorithms for finite p-groups.

As a corollary of the classification, we get

THEOREM 1. *There are only finitely many nilpotent orbits in* $\mathfrak{g}$.

It would of course be desirable to have a unified proof, as for unipotent classes [8].

For $x \in \mathfrak{g}$, let G_x denote the stabilizer of x in G and let $\mathfrak{B}_x^G$ be the variety of all Borel subgroups of G whose Lie algebra contains x.

THEOREM 2. $\dim G_x = 2 \dim \mathfrak{B}_x^G + r$, *where r is the rank of G.*

We may assume that G is of type F_4, with $p = 3$, or of type E_7 or E_8, with p bad. We say that $x \in \mathfrak{g}$ is distinguished if it is nilpotent and $Z^0(G)$ is a maximal torus of G_x (this differs slightly from the definition given in [2]). As noticed in [15], it is sufficient to prove that the distinguished orbits can be obtained by the process of induction [10], and this is actually how we shall get hold of them.

The theory of Springer representations can be made to work in bad characteristic ([3], [9]), and Theorem 2 implies that the nilpotent orbits can be parametrized by irreducible representations of the Weyl group.

The method used here does not give the structure of G_x ($x \in \mathfrak{g}$ nilpotent), nor the inclusion relation between closures of nilpotent orbits. The results are therefore weaker than those obtained for unipotent elements by Mizuno [12].

The proof proceeds along the following lines. Let $\mathbf{F}_q$ be a finite field of characteristic p. We can assume that k is an algebraic closure of $\mathbf{F}_q$ (this is used by Lusztig [8] in his proof of the finiteness of the number of unipotent classes) and that G is defined and split over $\mathbf{F}_q$. Let $F : \mathfrak{g} \to \mathfrak{g}$ be the Frobenius morphism of $\mathfrak{g}$. Let C be an F-stable nilpotent orbit in $\mathfrak{g}$, let $x \in C$ and let S be a maximal torus of G_x. Then $|C^F|$ is given by a polynomial in q which depends only on $\dim C$ and the conjugacy class of S in G. The computer is used for the following two major steps:

(i) Let $C_1, \ldots, C_m$ be the non-distinguished nilpotent orbits in $\mathfrak{g}$. We compute $\dim C_i$, and hence $|C_i^F|$, for $1 \leqslant i \leqslant m$.

(ii) We construct distinguished orbits $C_{m+1}, \ldots, C_n$ such that $\Sigma_{1 \leqslant i \leqslant n} |C_i^F| = q^{2N}$ (as polynomials), where N is the number of positive roots of G. As the number of

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nilpotent elements in $\mathfrak{g}^F$ is q^{2N} [19], this shows that $C_1, \ldots, C_n$ are all the nilpotent orbits in $\mathfrak{g}$.

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0. Notation

In addition to that used in the introduction, we shall use the following notation.

If H is an affine algebraic group, we write U_H or $R_u(H)$ for its unipotent radical.

We fix a Borel subgroup B of G, with maximal unipotent subgroup U, and a maximal torus T in B. The Weyl group of G is $W = N_G(T)/T$. More generally, if $H \supset T$ is a connected subgroup of G, let $W_H = N_H(T)/T \subset W$. Let w_H be the element of maximal length in W_H (i.e. w_H is the unique element of W_H such that $H \cap B \cap {}^{w_H}B \subset TU_H$).

For the Lie algebra of an algebraic group we use the corresponding gothic letter, for example $\mathfrak{b}$, $\mathfrak{u}$, $\mathfrak{u}_H$ are the Lie algebras of B, U, U_H respectively. We write $g \cdot x$ for $\mathrm{Ad}(g)(x)$ ($g \in G$, $x \in \mathfrak{g}$). The nilpotent variety of $\mathfrak{g}$ is denoted $\mathscr{N}$.

Let $\Phi \subset \mathrm{Hom}(T, \mathbf{G}_m)$ be the root system of G, with Φ^+ the set of positive roots and Δ the basis corresponding to B. For each $\lambda \in \Phi$, let $x_\lambda\colon \mathbf{G}_a \to G$ be adapted to λ. Let $U_\lambda = x_\lambda(\mathbf{G}_a) \subset G$ and $X_\lambda = (dx_\lambda)_0(1) \in \mathfrak{g}$. Let also L_λ be the subgroup of G generated by U_λ and $U_{-\lambda}$. The roots are denoted as in [4]. For example, for F_4 the highest root is 2342, and for E_8 it is $\substack{2465432\\3}$.

Except in paragraph 1, k is an algebraic closure of a finite field $\mathbf{F}_q$. We assume that G, B, T and x_λ ($\lambda \in \Phi$) are defined over $\mathbf{F}_q$, and F denotes the Frobenius morphisms of G, $\mathfrak{g}$, etc. The conventions concerning Lang's theorem and its applications are the same as in [20]. For example, if T' is an F-stable maximal torus of G corresponding to $w \in W$, then F acts on T' as wF on T.

The classification of nilpotent orbits for E_7 and E_8 in bad characteristic was conjectured in [16], with a notation not adopted in this paper since it conflicts with the Bala-Carter method which we shall use constantly.

There are several references to results concerning unipotent classes in G instead of nilpotent orbits in $\mathfrak{g}$. In most cases the demonstrations can be transposed easily, but some care is needed in the use of the process of induction [10]. This is discussed in paragraph 5.

1. Reduction to $\overline{\mathbf{F}_p}$

For completeness we show why it is enough to consider the case of an algebraic closure of $\mathbf{F}_p$. Let k_0 be the algebraic closure of the prime field in k. Then G can be obtained by extension of scalars from a reductive algebraic group G_0 defined over k_0. The nilpotent variety $\mathcal{N}$ is likewise obtained by extension of scalars from the nilpotent variety $\mathcal{N}_0$ of $\mathfrak{g}_0$, as well as the adjoint action. Suppose that the number of G_0-orbits in $\mathfrak{g}_0$ is finite. Let $C_1, \ldots, C_n$ be these orbits and let $x_i \in C_i$ $(1 \leqslant i \leqslant n)$. Let X_0 be the disjoint union of n copies of G_0 and let f_0: $X_0 \to \mathcal{N}_0$ be the morphism which sends the element g of the ith copy to $g \cdot x_i$. Then f_0 is surjective. It is a general fact that if π_0: $Y_0 \to Z_0$ is a surjective morphism of k_0-varieties, then the morphism π: $Y \to Z$ obtained by extension of scalars to k is also surjective. Therefore the morphism f: $X \to \mathcal{N}$ from the disjoint union of n copies of G to $\mathcal{N}$, defined by $g \mapsto g \cdot x_i$ on the i^{th} copy, is also surjective. There are thus at most n nilpotent orbits in $\mathfrak{g}$. On the other hand, the closure $\overline{C}_i$ of C_i in $\mathfrak{g}$ is G_0-stable, hence G-stable since G_0 is dense in G, and if $x_j \in \overline{C}_i$ then x_j is also in the closure of C_i in $\mathfrak{g}_0$. This implies that $x_1, \ldots, x_n$ form a system of representatives for the G-orbits in $\mathfrak{g}$.

2. The polynomial $|C^F|$

Let $C \subset \mathfrak{g}$ be an F-stable nilpotent orbit. We want to show that $|C^F|$ is given by a polynomial in q of very special form.

Let $x \in C$, let S be a maximal torus of G_x and let $M = C_G(S)$. Then M is a Levi factor of some parabolic subgroup P of G, $S = Z^0(M)$ and x is distinguished in $\mathfrak{m}$ [2]. The finite group $W_{M,G} = N_G(M)/M \cong N_W(W_M)/W_M$ acts on M and on $\mathfrak{m}$.

Lemma 1. *$W_{M,G}$ acts trivially on the set of distinguished orbits in* $\mathfrak{m}$.

We can assume that G is simple. Then M has at most one factor which is not of type A. For type A the only distinguished orbit is the regular nilpotent orbit. If there is a factor of type other than A and $W_{M,G}$ acts by outer automorphisms on it, then it is of type D_n $(n \geqslant 4)$ or E_6. In both cases the nilpotent orbits are known, and the distinguished ones are stable under outer automorphisms.

Lemma 2. *In the situation above, it is possible to choose x and S both F-stable in such a way that P is also F-stable.*

We certainly can arrange to have S contained in T. Then S and P are F-stable. Moreover Lemma 1 shows that $C \cap \mathfrak{m}$ is a single distinguished orbit in $\mathfrak{m}$. As it is F-stable we can find $x \in (C \cap \mathfrak{m})^F$.

Let H be a connected algebraic group defined over $\mathbf{F}_q$, let T_0 be an F-stable maximal torus of H and let $W_H = N_H(T_0)/C_H(T_0)$ be the Weyl group of H. The following formula is due to Steinberg [22]:

$$\frac{1}{|H^F|} = \frac{1}{|W_H|} q^{\dim T_0 - \dim H} \sum_{w \in W_H} \frac{1}{|T_0^{wF}|}.$$

PROPOSITION 1. *Let x, S be as in Lemma 2 and let*

$$c_{M,G} = \frac{1}{|W_{M,G}|} \sum_{w \in W_{M,G}} \frac{q^{\dim S}}{|S^{wF}|}.$$

Then

$$|C^F| = c_{M,G} |G^F| q^{-\dim G_x}.$$

Let $A = G_x/G_x^0$. For each $a \in A$, let $x_a \in C^F$ correspond to the F-conjugacy class of a in A. It is easily seen that

$$|C^F| = \frac{1}{|A|} \sum_{a \in A} \frac{|G^F|}{\left|\left(G_{x_a}^0\right)^F\right|}.$$

Now $a \in A$ is a coset of the form gG_x^0, with $g \in G_x$. As all maximal tori of G_x are G_x^0-conjugate, we can assume that g normalizes S. Let $W_x = N_{G_x^0}(S)/C_{G_x^0}(S)$ be the Weyl group of G_x^0 and let $W_x' = N_{G_x}(S)/C_{G_x^0}(S)$. Then by Steinberg's formula,

$$\frac{1}{\left|\left(G_{x_a}^0\right)^F\right|} = \frac{1}{|W_x|} q^{\dim S - \dim G_x} \sum_{w \in W_x} \frac{1}{|S^{wgF}|},$$

and therefore

$$|C^F| = \frac{|G^F|}{|W_x'|} q^{\dim S - \dim G_x} \sum_{w \in W_x'} \frac{1}{|S^{wF}|}.$$

In order to prove the proposition, it remains only to check that

$$\frac{1}{|W_x'|} \sum_{w \in W_x'} \frac{1}{|S^{wF}|} = \frac{1}{|W_{M,G}|} \sum_{w \in W_{M,G}} \frac{1}{|S^{wF}|},$$

and this holds since Lemma 1 implies that the natural homomorphism $W_x' \to W_{M,G}$ is surjective.

REMARK. Consider $c_{M,G}$ as a rational function of q and $c_{M,G}|G^F|$ as a polynomial in q ($q = p^e$, $e \in \mathbf{N}^*$). Then they don't depend on the characteristic. For E_7, E_8 and F_4 they are therefore essentially contained in [12], [13].

3. Non-distinguished nilpotent orbits

Let $x \in \mathfrak{g}$ be nilpotent and let $C = G \cdot x$. Then all irreducible components of $C \cap \mathfrak{u}$ have the same dimension [14]. For $w \in W$, let $\mathfrak{u}_w = \mathfrak{u} \cap {}^w\mathfrak{u}$. The following properties are equivalent ([15], see also [21], [14]):

(1) $\dim G_x = 2 \dim \mathfrak{B}_x^G + \dim T$;

(2) $\dim C = 2 \dim(C \cap \mathfrak{u})$;

(3) $\dim \mathfrak{B}_x^G = \operatorname{codim}_{\mathfrak{u}}(C \cap \mathfrak{u})$;

(4) C contains a dense open subset of $\mathfrak{u}_w$ for some $w \in W$.

Moreover, if w is as in (4), then $\overline{B \cdot \mathfrak{u}_w}$ is an irreducible component of $\overline{C \cap \mathfrak{u}}$, all irreducible components of $\overline{C \cap \mathfrak{u}}$ are of this form, and they all have the same dimension.

Let $P \supset B$ be a parabolic subgroup of G, let $M \supset T$ be a Levi factor of P and let $U' = U \cap M$. Suppose that $x \in \mathfrak{m}$, and that $M \cdot x$ contains a dense open subset of $\mathfrak{u}' \cap {}^w\mathfrak{u}'$ for some $w \in W_M$. As $\mathfrak{u}' \cap {}^w\mathfrak{u}' = \mathfrak{u}_{w_G w_M w}$, $B \cdot (\mathfrak{u}' \cap {}^w\mathfrak{u}')$ is an irreducible component of $\overline{C \cap \mathfrak{u}}$. Therefore $\dim C = 2 \dim(C \cap \mathfrak{u}) = 2 \dim B \cdot (\mathfrak{u}' \cap {}^w\mathfrak{u}')$. But B is the semidirect product of $B \cap M$ and U_P. For any $y \in M \cdot x \cap \mathfrak{u}'$ we have $B_y = (B \cap M)_y (U_P)_y$ and $\dim(U_P)_y = \dim(U_P)_x$. Moreover, two elements y, y' of $M \cdot x \cap \mathfrak{u}'$ are in the same B-orbit if and only if they are in the same $(B \cap M)$-orbit. It follows that

$$\dim B \cdot (\mathfrak{u}' \cap {}^w\mathfrak{u}') = \dim(B \cap M) \cdot (\mathfrak{u}' \cap {}^w\mathfrak{u}') + \dim U_P \cdot x.$$

As $\dim M \cdot x = 2 \dim(B \cap M) \cdot (\mathfrak{u}' \cap {}^w\mathfrak{u}')$, we get finally.

PROPOSITION 2. *Suppose that* $x \in \mathfrak{m}$ *is nilpotent and that* $\dim M_x = 2 \dim \mathfrak{B}_x^M + \dim T$. *Then*: $\dim G \cdot x = \dim M \cdot x + 2 \dim U_P \cdot x$, $\dim G_x = \dim M_x + 2 \dim(U_P)_x$.

Assuming $\dim M_x$ known, we need therefore only to compute $\dim(U_P)_x$ to get $|C^F|$. For this we use the following result.

PROPOSITION 3. *For any* $x \in \mathfrak{m}$, $(U_P)_x$ *is connected.*

We may assume that $x \in \mathfrak{m} \cap \mathfrak{b}$ and that T contains a maximal torus S of M_x. Since $M \supset C_G(S)$, it is clear that S is also a maximal torus of G_x and B_x. Any irreducible component of B_x contains therefore an element which normalizes S, hence centralizes S since B is solvable and connected. Hence every irreducible component of B_x meets $C_G(S) \subset M$. But the semidirect decomposition $B = (B \cap M)U_P$ gives a similar decomposition $B_x = (B \cap M)_x(U_P)_x$. As every irreducible component of B_x meets M, $(U_P)_x$ must be connected.

COROLLARY. *Let* $x \in \mathfrak{m}^F$ *and let* $|(U_P)_x^F| = q^d$. *Then* $\dim(U_P)_x = d$.

APPLICATION. Using the Appendix with $q = p$, we can compute $\dim(U_P)_x$, and hence $\dim G_x$ and $|(G \cdot x)^F|$ for x nilpotent non-distinguished in $\mathfrak{g}$, assuming of course that the nilpotent orbits are known for Levi factors of proper parabolic subgroups of G. For this paper, this has been done with the help of a computer for E_7 and E_8. What was actually computed was $|U_x^F|$ and $|(U \cap M)_x^F|$. The semidirect decomposition with the help of a computer $U = (U \cap M)U_P$ gives then $|(U_p)_x^F| = |U_x^F|/|(U \cap M)_x^F|$.

4. The case of F_4

If G is of type F_4 and $p = 3$, there are 12 non-distinguished nilpotent orbits, as in characteristic 0. Let C be one of them. With the notation of the previous paragraph, we can find P ($\neq G$), M and $x \in \mathfrak{m} \cap C$ in such a way that $\dim(U_P)_x$ can be computed very easily (by hand), with the following exception for which some care is needed. For the class $\tilde{A}_2 + A_1$ ($\tilde{A}_2$ short, A_1 long), the computation requires an explicit knowledge of the constants in the commutation formulae. We can use those listed in [13]. In all cases we find that the non-distinguished orbits have the same dimension as the corresponding orbits in characteristic 0.

In particular, with the classes we have so far it is not possible to have $\dim \mathfrak{B}_x^G = 0$, 1, 2 or 4. But in this case we know that the number of nilpotent orbits in $\mathfrak{g}$ is finite [17]. Richardson orbits are therefore defined. Starting with a parabolic subgroup having a Levi factor of type $\varnothing$, $\tilde{A}_1$, $\tilde{A}_1 + A_1$ or $\tilde{A}_2 + A_1$, we get a nilpotent orbit C_P such that $\dim \mathfrak{B}_x^G = 0$, 1, 2, 4 respectively for $x \in C_P$. This gives 4 distinguished nilpotent orbits. Comparing with the case of large characteristic, we find that we have already q^{48} F-stable nilpotent elements. We have therefore all the nilpotent orbits in $\mathfrak{g}$. The distinguished orbits being Richardson orbits, we see also that theorem 2 holds for G.

5. Induction for nilpotent orbits

Induction for nilpotent orbits is the following operation. Let P be a parabolic subgroup of G and let M be a Levi factor of P. Let $x \in \mathfrak{m}$ be nilpotent and let $C = M \cdot x$. Assume that $\dim M_x = 2 \dim \mathfrak{B}_x^M + \dim T$. The induced orbit in $\mathfrak{g}$ is

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the nilpotent orbit $\tilde{C}$ which contains a dense open subset of $C + \mathfrak{u}_P$. If $x = 0$, $\tilde{C}$ is the Richardson orbit defined by P. In [10] the corresponding situation for unipotent elements is investigated, and the finiteness of the number of unipotent classes implies the existence of the induced class. We cannot use this argument here, but we still have:

(1) If $\tilde{C}$ exists, then all the properties described in [10] hold. In particular, if $\tilde{x} \in (x + \mathfrak{u}_P) \cap \tilde{C}$, then $P_{\tilde{x}} \subset M_x U_P$, $G_{\tilde{x}}^0 = P_{\tilde{x}}^0$ and $\dim G_{\tilde{x}} = \dim M_x = 2 \dim \mathfrak{B}_{\tilde{x}}^G + \dim T$.

(2) If there exists $\tilde{x} \in x + \mathfrak{u}_P$ such that $\dim P_{\tilde{x}} \leqslant \dim M_x$, then the induced orbit $\tilde{C}$ exists and $\tilde{x} \in \tilde{C}$.

If G is an exceptional group, induction can be computed explicitely as follows. We assume that Levi factors of proper parabolic subgroups of G have only finitely many nilpotent orbits, that these orbits are known, and that induction inside their Lie algebras is known.

Let S be a maximal torus of M_x and let $L = C_G(S)$. If $\tilde{C}$ exists, there is an element $\tilde{x} \in (x + \mathfrak{u}_P) \cap \tilde{C}$ such that S contains a maximal torus $\tilde{S}$ of $G_{\tilde{x}}$. Let $\tilde{G} = C_G(\tilde{S})$, $\tilde{P} = P \cap \tilde{G}$, $\tilde{M} = M \cap \tilde{G}$. Then $\tilde{P}$ is a parabolic subgroup of $\tilde{G}$, $\tilde{M}$ is a Levi factor of $\tilde{P}$ and the unipotent radical $U_{\tilde{P}}$ of $\tilde{P}$ is $U_P \cap \tilde{G}$. Now $\tilde{x} \in (x + \mathfrak{u}_P) \cap \tilde{\mathfrak{g}} = x + \mathfrak{u}_{\tilde{P}}$. Therefore $\tilde{C}$ contains a dense open subset of $x + \mathfrak{u}_{\tilde{P}}$. As $\tilde{C} \cap \tilde{\mathfrak{g}}$ is a single $\tilde{G}$-orbit, $\tilde{G} \cdot \tilde{x}$ must be the orbit of $\tilde{\mathfrak{g}}$ induced from $\tilde{M} \cdot x \subset \tilde{\mathfrak{m}}$.

There are only finitely many subgroups H of G which contain L and are Levi factors of proper parabolic subgroups of G. For each of them, let $\tilde{x}_H$ be an element in the orbit in $\mathfrak{h}$ induced from the orbit $(M \cap H) \cdot x \subset \mathfrak{m} \cap \mathfrak{h}$. If $\dim G_{\tilde{x}_H} = \dim M_x$ for some H, then $\tilde{C} = G \cdot \tilde{x}_H$. If no such H exists, the induced orbit is distinguished or does not exist.

We shall use this information to find the distinguished orbits in $\mathfrak{g}$. There will only be a finite number of them, and induced orbits will therefore exist. Moreover, it will turn out that with one exception, for each $d \in \mathbf{N}$ there is at most one distinguished orbit of codimension d in $\mathfrak{g}$ (G exceptional). If we are not in this special case we can then clearly find the induced orbit.

The exception occurs with E_8 when $p = 2$ and $d = 22$. We have then two distinguished orbits. As they will both be obtained by induction, the corresponding representations of W in Springer's parametrization can be determined [10], and this allows to compute induction in the remaining cases where $\dim M_x = 22$.

REMARK. The method described above works also for unipotent classes of exceptional groups, with the advantage that the difficulty with E_8, $p = 2$, $d = 22$ doesn't occur. This method seems to be more systematic than those devised so far (see e.g. [15]).

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6. Construction of distinguished orbits

For E_7 and E_8 in bad characteristic we don't know yet that the number of nilpotent orbits is finite.

Let $C_1, \ldots, C_m$ be the non-distinguished nilpotent orbits in $\mathfrak{g}$. The number $\Sigma_{1 \leqslant i \leqslant m} |C_i^F|$ can be viewed as a polynomial in q. The results obtained so far allow to compute it for E_7, and also for E_8 once we have representatives for the distinguished nilpotent orbits in the case of E_7. If the number of nilpotent orbits is finite and $C_{m+1}, \ldots, C_n$ are the distinguished ones, let $x_i \in C_i$ and $d_i = \dim G_{x_i}$ $(m < i \leqslant n)$. Thanks to a result of Springer [19], we have

$$\sum_{m < i \leqslant n} q^{-d_i} = |G^F|^{-1}\Big(q^{2N} - \sum_{1 \leqslant i \leqslant m} |C_i^F|\Big)$$

where $N = |\Phi^+|$. This gives $n - m$ and $d_{m+1}, \ldots, d_n$ (up to permutation).

Let $b_i = \frac{1}{2}(d_i - \dim T)$. We shall obtain the distinguished orbits by the process of induction, and this will ensure that $\dim G_{x_i} = 2 \dim \mathfrak{B}_{x_i}^G + \dim T$ $(m < i \leqslant n)$, so that $b_i = \dim \mathfrak{B}_{x_i}^G$.

If the characteristic is 0 and C is a distinguished orbit, there exist a parabolic subgroup P of G such that C contains a dense open subset of $\mathfrak{u}_P$ (that is C is a Richardson orbit). Moreover there is a canonical choice for P, up to conjugation [2]. In bad characteristic we can take the corresponding class of parabolic subgroups and the associated Richardson orbit. This orbit, the existence of which is not obvious at this stage, turns out to be distinguished. We call such orbits standard distinguished orbits. The corresponding values of b_i are all distinct (for exceptional groups). For E_7 and E_8 they are

E_7: 0, 1, 2, 3, 5, 7;
E_8: 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 16.

In bad characteristic we expect the following additional values:

$E_7, p = 2$: 6;
$E_7, p = 3$: none;
$E_8, p = 2$: 7, 9, 13 (notice that 7 is repeated);
$E_8, p = 3$: 11;
$E_8, p = 5$: none.

Comparing with characteristic 0, we find that for some non-distinguished nilpotent orbits $\dim G_x$ is larger than expected. For example, if G is of type E_7 and $p = 2$, the orbits $A_3 + A_2$ and A_6 give stabilizers of dimension 37 and 21 respectively, instead of 35, 19. For $A_3 + A_2$ the difference (in the number of

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F-stable nilpotent elements) is made up by an extra distinguished orbit in D_6, and the extra distinguished orbit in $\mathfrak{g}$ (with $b_i = 6$) should make up for A_6. In characteristic 0 the orbit A_6 is the Richardson orbit corresponding to the parabolic subgroups with Levi factors of type $A_2 + 3A_1$. In characteristic 2 the extra distinguished orbit should be the corresponding Richardson orbit in $\mathfrak{g}$.

In a similar way we find that for E_8, $p = 2$, the extra distinguished orbits in $\mathfrak{g}$ should take the place of D_7 ($b_i = 7$), $D_7(a_1)$ ($b_i = 9$) and $D_5 + A_2$ ($b_i = 13$). For $E_8, p = 3$, the extra distinguished orbit should take the place of A_7.

As in paragraph 3 we shall use the computer to calculate $|U_z^F|$ for various elements $z \in \mathfrak{u}^F$, in view to find $\dim U_z$. As U_z can now be disconnected, we get only the following

Test. Let $d \in \mathbf{N}$. If $|U_z^F| < q^{d+1}$, then $\dim U_z \leqslant d$.

On the other hand it is clear that if $\dim U_z = d$, then $|U_z^F| < q^{d+1}$ if q is large enough. In this paper it will be enough to take $q \leqslant 5$ to get the required results.

We describe now how to get distinguished orbits.

Choose a parabolic subgroup P of G, a Levi factor M of P and a nilpotent orbit C in $\mathfrak{m}$ in such a way that in characteristic 0 the induced orbit would be the expected one. We can take P maximal, $P \supset B$, $M \supset T$, and we choose $x \in C^F$ such that B and T contain respectively a Borel subgroup and a maximal torus of M_x.

Let $H = M_x U_P$. We find a suitable subset $E \subset \{\lambda \in \Phi | X_\lambda \in \mathfrak{u}_P\}$ such that $V = \sum_{\lambda \in E} kX_\lambda$ is H-stable (see however the remark at the end of the paragraph). Then H acts on $Y = (x + \mathfrak{u}_P)/V$. This action should be easy to work with. For $y \in x + \mathfrak{u}_P$, let $\bar{y} = y + V \in Y$. Then $P_y \subset H_{\bar{y}}$. We are interested in the following special cases.

Case 1. We can find $y \in x + \mathfrak{u}_P^F$ such that $H_{\bar{y}}^0 \subset U$ and $\dim U_y \leqslant \dim M_x$. Then the induced orbit $\tilde{C}$ exists, it is distinguished, $y \in \tilde{C}$ and $G_y^0 \subset U$.

Case 2. We can find $y \in x + \mathfrak{u}_P^F$ such that

(i) the orbit of $\bar{y}$ is dense in Y.

(ii) Let $K = H_{\bar{y}}$. Then $K^0/R_u(K^0)$ is either a torus or is of type A_1.

(iii) B, T contain respectively a Borel subgroup and a maximal torus of K.

Using the method described in paragraph 5, we check first that the induced orbit, if it exists, is distinguished. This shows that there exists a dense open subset of $y + V$ consisting of distinguished elements. Let $d = \dim M_x$.

(a) If $K^0/R_u(K^0)$ is a torus, it is enough to find $z \in y + V^F$ such that $\dim U_z \leqslant d$. Indeed, by semicontinuity the set $V_0 = \{v \in V | \dim U_{y+v} \leqslant d\}$ is

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then open dense in V. There exists therefore a distinguished element $\tilde{x} \in y + V_0$. Then $K_{\tilde{x}}^0 \subset G_{\tilde{x}}^0 \cap B \subset U$. Therefore $\dim K_{\tilde{x}} = \dim U_{\tilde{x}} \leqslant d$, as required.

(b) If $K^0/R_u(K^0)$ is of type A_1, we show first that there exists $z \in y + V^F$ such that $\dim U_z \leqslant d - 1$. Then $V_0 = \{v \in V | \dim U_{y+v} \leqslant d - 1\}$ is open dense in V. If $\tilde{x} \in y + V_0$ is distinguished, then $K_{\tilde{x}}^0$ is unipotent, and therefore $\dim K_{\tilde{x}} \leqslant \dim U_{\tilde{x}} + 1 \leqslant d$, as required.

(c) If $K^0/R_u(K^0)$ is simple of type A_1, we proceed as in (b). Let $e = \dim V$ and let $C_0 = K \cdot \tilde{x}$. Then $|C_0^F| = (q^2 - 1)q^{e-2}$. For $z' \in y + V$, $z' \notin C_0$, we must therefore have $\dim K_{z'} \geqslant d + 2$, and also $\dim U_{z'} \geqslant d$. This shows that $z \in G \cdot \tilde{x}$.

Notice that in cases 1 and 2(c) we get an explicit representative for the distinguished orbit, but not in cases 2(a) and 2(b).

It will be convenient to write $\overline{E} = \{\lambda \in \Phi | X_\lambda \in \mathfrak{u}_P \text{ and } \lambda \notin E\}$.

Example. If C is a distinguished orbit in $\mathfrak{m}$, let α be the unique element of Δ such that $X_\alpha \in \mathfrak{u}_P$. We take $\overline{E} = \{\alpha\}$. We are in case 1, with $y = x + X_\alpha$. The corresponding situation for unipotent elements is discussed in [10].

This gives already standard distinguished orbits for the following values of b_i:
E_7: 0, 1, 2, 3;
E_8: 0, 1, 2, 3, 5, 7 (assuming that the case of E_7 has been successfully dealt with).

Remark. If $E' \supset E$ and $V' = \sum_{\lambda \in E'} kX_\lambda$ have properties similar to those of E and V, we can choose $y' \in x + \mathfrak{u}_P^F$ in the same way as y is chosen in case 2, and then restrict our attention to the action of $K' = H_{\bar{y}'}$ on $y' + V'$ and $(y' + V')/V$, where $\bar{y}' = y' + V'$. In particular it is sufficient now for V to be K'-stable. We write also $\overline{E}' = \{\lambda \in \Phi | X_\lambda \in \mathfrak{u}_P \text{ and } \lambda \notin E'\}$.

7. Standard distinguished orbits

Let G be a group of type E_6 and consider the following elements of $\mathfrak{g}$:

$$x^{(1)} = X_{\substack{10000\\0}} + X_{\substack{00001\\0}} + X_{\substack{11000\\0}} + X_{\substack{00011\\0}} + X_{\substack{00100\\1}} + X_{\substack{01110\\0}}$$

$$x^{(2)} = X_{\substack{11000\\0}} + X_{\substack{01110\\0}} + X_{\substack{01100\\1}} + X_{\substack{00110\\1}} + X_{\substack{00111\\0}}$$

$$x^{(3)} = X_{\substack{11000\\0}} + X_{\substack{00011\\0}} + X_{\substack{00111\\1}} + X_{\substack{01210\\1}}$$

$$x^{(4)} = X_{\substack{11110\\0}} + X_{\substack{01111\\0}} + X_{\substack{11100\\1}} + X_{\substack{00111\\1}} + X_{\substack{01210\\1}}$$

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Their orbits are respectively $E_6(a_3)$, $A_4 + A_1$, $D_4(a_1)$ and $2A_2 + A_1$. Let $x = x^{(i)}$ $(1 \leqslant i \leqslant 4)$. Then B, T contain respectively a Borel subgroup and a maximal torus of G_x. Let $u = \prod_{\lambda > 0} x_\lambda(c_\lambda)$. Suppose that $u \in G_x$. Then:

(a) if $i = 2$,

$$c_{\substack{00001\\0}} = 0 \Rightarrow c_{\substack{00011\\0}} = 0.$$

(b) if $i = 3$,

$$c_{\substack{00001\\0}} = c_{\substack{00100\\0}} = c_{\substack{00110\\0}} = c_{\substack{00111\\0}} = 0.$$

Moreover, if $i = 1$, then $G_x^0 \subset U$. If $i = 4$, then $G_x^0/R_u(G_x^0)$ is simple of type A_1, and

$$\left(L_{\substack{01000\\0}} L_{\substack{00010\\0}} L_{\substack{00000\\1}}\right)_x \text{ is a subgroup of } G_x \text{ of type } A_1.$$

These statements follow easily from results in [17] and the commutation formulae.

If now G is of type E_7 or E_8, we consider the elements above as elements of $\mathfrak{g}$ in the obvious manner.

We have already constructed some standard distinguished orbits in the previous paragraph. The remaining ones correspond to $b_i = 5$, 7 for E_7, and $b_i = 4$, 6, 8, 10, 16 for E_8. The characteristic is assumed to be bad.

(A) For E_7 we must find representatives to be able to carry on with E_8. In both cases we take M of type E_6 and

$$\bar{E}' = \left\{\begin{matrix} 000001 \\ 0 \end{matrix}\right\}, \qquad y' = x + X_{\substack{000001\\0}}$$

(the notation is the same as in paragraph 6).

$b_i = 5$. Let $x = x^{(2)}$,

$$\bar{E} = \bar{E}' \cup \left\{\begin{matrix} 000011 \\ 0 \end{matrix}, \begin{matrix} 000111 \\ 0 \end{matrix}\right\}.$$

Taking

$$y = y' + X_{\substack{000111\\0}},$$

we are in case 1 of paragraph 6, as can be seen in particular from the remarks at the beginning of this paragraph. The inequality $\dim U_y \leqslant \dim M_x$ is obtained with $q = p$. Thus the induced class $\tilde{C}$ exists, $y \in \tilde{C}$ and $G_y^0 \subset U$.

$b_i = 7$. Let $x = x^{(3)}$,

$$\bar{E} = \bar{E}' \cup \left\{\begin{matrix} 000011 \\ 0 \end{matrix}, \begin{matrix} 000111 \\ 0 \end{matrix}, \begin{matrix} 001111 \\ 0 \end{matrix}\right\}.$$

Taking

$$y = y' + X_{\substack{000011\\0}} + X_{\substack{001111\\0}},$$

we are in case 1. We can use $q = p$. Thus $\tilde{C}$ exists, $y \in \tilde{C}$ and $G_y^0 \subset U$.

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It follows that B contains a Borel subgroup of M_x and $M_x^0/R_u(M_x^0)$ is of type $3A_1$. An easy computation with the commutation formulae shows that if $u = \prod_{\lambda>0} x_\lambda(c_\lambda) \in M_x$, then

$$c_{\substack{0010000\\0}} = c_{\substack{0000000\\1}} = c_{\substack{0010000\\1}} = 0.$$

It follows that we can take

$$\bar{E} = \left\{ \begin{matrix}0000001\\0\end{matrix}, \begin{matrix}0000011\\0\end{matrix}, \begin{matrix}0000111\\0\end{matrix}, \begin{matrix}0001111\\0\end{matrix}, \begin{matrix}0011111\\0\end{matrix}, \begin{matrix}0011111\\1\end{matrix} \right\}.$$

We are then in case 2(a). We use

$$y = x + X_{\substack{0001111\\0}} + X_{\substack{0011111\\0}} + X_{\substack{0011111\\1}},$$
$$z = y, \qquad q = p.$$

8. Non-standard distinguished orbits

For E_7 it remains to construct one non-standard distinguished orbit if $p = 2$; for E_8, three non-standard distinguished orbits if $p = 2$, and one if $p = 3$.

In each case the proof that B, T contain respectively a Borel subgroup and a maximal torus of M_x is omitted. One can use arguments similar to those invoked in the case of the standard distinguished orbit in E_8 for which $b_i = 16$.

(A) For E_7, we take M of type $D_5 + A_1$, with

$$x = X_{\substack{111000\\0}} + X_{\substack{011100\\0}} + X_{\substack{011000\\1}} + X_{\substack{001100\\1}},$$

in the orbit $(A_2 + 2A_1;\ \varnothing)$. Let L be the subgroup of M generated by

$$L_{\substack{100000\\0}}, \quad L_{\substack{010000\\0}}, \quad L_{\substack{000100\\0}} \quad \text{and} \quad L_{\substack{000000\\1}}.$$

Then L_x is simple of type A_1. We also have

$$L_{\substack{000001\\0}} \subset M_x,$$

and $M_x^0/R_u(M_x^0)$ is of type $2A_1$. Let

$$\bar{E} = \left\{ \begin{matrix}000010\\0\end{matrix}, \begin{matrix}000110\\0\end{matrix}, \begin{matrix}000011\\0\end{matrix}, \begin{matrix}000111\\0\end{matrix} \right\}.$$

We are then in case 2(c). We can use

$$y = x + X_{\substack{000110\\0}} + X_{\substack{000011\\0}},$$
$$z = y + X_{\substack{001111\\0}}, \qquad q = 2.$$

Thus $\tilde{C}$ exists and $z \in \tilde{C}$.

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(B) Suppose now that G is of type E_8 and $p = 2$.

$b_i = 7$. Let M be of type D_7 and let

$$x = X_{\substack{0110000\\1}} + X_{\substack{0011100\\0}} + X_{\substack{0001110\\0}} + X_{\substack{0111000\\0}} + X_{\substack{0011000\\1}} + X_{\substack{0000111\\0}},$$

in the orbit $2A_3$. Then

$$\left(L_{\substack{0010000\\0}} L_{\substack{0000010\\0}}\right)_x$$

is of type A_1, and $M_x^0/R_u(M_x^0)$ is of type A_1. We can take

$$\bar{E} = \left\{\begin{matrix}1000000\\0\end{matrix}\right\}.$$

We are in case 2(c). We can use

$$y = x + X_{\substack{1000000\\0}},$$

$$z = y + X_{\substack{1122100\\1}}, \qquad q = 2.$$

Thus $\tilde{C}$ exists and $z \in \tilde{C}$.

$b_i = 9$. Let M be of type $D_5 + A_2$ and let

$$x = X_{\substack{1111000\\0}} + X_{\substack{1110000\\1}} + X_{\substack{0121000\\1}} + X_{\substack{0000011\\0}},$$

in the orbit $(3A_1; A_1)$. Then

$$\left(L_{\substack{0001000\\0}} L_{\substack{0000000\\1}}\right)_x$$

is of type A_1,

$$L_{\substack{0100000\\0}} \subset M_x$$

and $M_x^0/R_u(M_x^0)$ is of type $2A_1$. We can take

$$\bar{E} = \left\{\begin{matrix}0000100\\0\end{matrix}, \begin{matrix}0001100\\0\end{matrix}\right\}.$$

We are in case 2(b). We use

$$y = x + X_{\substack{0001100\\0}},$$

$$z = y + X_{\substack{0000110\\0}} + X_{\substack{0011111\\0}} + X_{\substack{1232100\\1}},$$

$$q = 2.$$

$b_i = 13$. Let M be of type E_7 and let

$$x = X_{\substack{1111100\\0}} + X_{\substack{0111110\\0}} + X_{\substack{1111000\\1}}$$
$$+ X_{\substack{0111100\\1}} + X_{\substack{0011110\\1}} + X_{\substack{0121000\\1}},$$

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in the orbit $A_3 + A_2 + A_1$. Let $L' \supset T$ be a Levi factor of the parabolic subgroup of G generated by B and

$$L_{\substack{1110000\\1}}.$$

A direct computation shows that

$$\left(L'L_{\substack{0000110\\0}}\right)_x$$

is of type A_1, and $M_x^0/R_u(M_x^0)$ is also of type A_1. We can take

$$\bar{E} = \left\{ \begin{matrix}0000001\\0\end{matrix}, \begin{matrix}0000011\\0\end{matrix}, \begin{matrix}0000111\\0\end{matrix} \right\}$$

We are then in case 1 with

$$y = x + X_{\substack{0000011\\0}} + X_{\substack{0000111\\0}}.$$

The required inequality is obtained with $q = 2$.

(C) Let now G be of type E_8 with $p = 3$. We must find a distinguished orbit with $b_i = 11$. We take M of type $D_5 + A_2$ and

$$x = X_{\substack{1111000\\0}} + X_{\substack{1110000\\1}} + X_{\substack{0121000\\1}},$$

in the orbit $(3A_1;\ \varnothing)$. Then $M_x^0/R_u(M_x^0)$ is of type $A_2 + 2A_1$ and the factors A_1 are given by

$$L_{\substack{0100000\\0}}$$

and

$$\left(L_{\substack{0000000\\1}} L_{\substack{0001000\\0}}\right)_x.$$

We take first

$$\bar{E}' = \left\{ \begin{matrix}0000100\\0\end{matrix}, \begin{matrix}0001100\\0\end{matrix}, \begin{matrix}0000110\\0\end{matrix}, \begin{matrix}000111\\0\end{matrix}, \begin{matrix}0000111\\0\end{matrix}, \begin{matrix}0001111\\0\end{matrix} \right\},$$

with

$$y' = x + X_{\substack{0001110\\0}} + X_{\substack{0000111\\0}}.$$

We can take

$$\bar{E} = \bar{E}' \cup \left\{ \begin{matrix}0011100\\0\end{matrix}, \begin{matrix}0111100\\0\end{matrix}, \begin{matrix}0011100\\1\end{matrix} \right\}$$

and we are in case 2(c). We use

$$y = y' + X_{\substack{0111100\\0}} + X_{\substack{0011100\\1}}, \quad z = y + X_{\substack{0011111\\0}}, \quad q = 3.$$

Thus $\tilde{C}$ exists and $z \in \tilde{C}$.

REMARK. In the case of E_8, $p = 2$, it remains to show that the two distinguished orbits C, C' we have constructed for $b_i = 7$ are distinct.

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Let M be of type E_7 and let $C_1 \subset \mathfrak{m}$ be the distinguished M-orbit for which $b_i = 7$. In paragraph 6 we have constructed a distinguished orbit C in $\mathfrak{g}$ by inducing C_1 from $\mathfrak{m}$ to $\mathfrak{g}$. Since G/P is complete, $\overline{C} = G \cdot (\overline{C}_1 + \mathfrak{u}_P)$. The elements in $C_1 + \mathfrak{u}_P$ are all contained in C or in the closure of the distinguished orbit in $\mathfrak{g}$ for which $b_i = 8$ (this follows from results in [10] and the way this orbit is obtained in paragraph 7). If the orbit D_7 is contained in $\overline{C}$, it must then be induced from an orbit $C_2 \subset \overline{C}_1$ with C_2 of codimension 2 in $\overline{C}_1$. The only possibilities for C_2 are $D_6(a_2)$ and $E_6(a_3)$, and they cannot give D_7 by induction.

The orbit D_7 of G is therefore not contained in $\overline{C}$. But it is obviously in the closure of the distinguished orbit C' constructed in this paragraph. Thus $C \neq C'$.

9. The classification of nilpotent orbits

In paragraph 6 we have defined standard distinguished orbits. More generally, for $x \in \mathcal{N}$, let S be a maximal torus of G_x and let $M = C_G(S)$. We say that the orbit of x in $\mathfrak{g}$ is standard if $M \cdot x$ is a standard distinguished orbit in $\mathfrak{g}$. Let $d_x = \dim G_x$. If x is standard, we can also define d_x^0 to be the dimension of the centralizer of a corresponding element in characteristic 0.

We have proved

THEOREM 3. *Let G be of type E_7 or E_8 and let $x \in \mathcal{N}$. As above let S be a maximal torus of G_x and let $M = C_G(S)$. Then*:

(a) *If $G \cdot x$ is standard, then $d_x = d_x^0$, except for the orbit $D_4 + A_2$ if G is of type E_8 and $p = 2$, in which case $d_x = d_x^0 + 6$, and in the following cases where*

$d_x = d_x^0 + 2$:
$E_7, p = 2: A_3 + A_2, A_6;$
$E_8, p = 2: A_3 + A_2, A_6, D_5 + A_2, D_7, D_7(a_1);$
$E_8, p = 3: A_7.$

(b) If $G \cdot x$ is non-standard, we are in one of the following cases.

(i) $p = 2$, M is of type D_6, $d_x = 37$ (resp. 72) if G is of type E_7 (resp. E_8) and $M \cdot x$ is the only non-standard distinguished orbit in $\mathfrak{m}$.

(ii) $p = 2$, M is of type E_7, $d_x = 19$ (resp. 40) if G is of type E_7 (resp. E_8) and $M \cdot x$ is the only non-standard distinguished orbit in $\mathfrak{m}$.

(iii) $p = 2$, M is of type D_7, $d_x = 56$ and $M \cdot x$ is the only non-standard distinguished orbit in $\mathfrak{m}$.

(iv) $p = 2$, $M = G$ is of type E_8 and $G \cdot x$ is one of the three non-standard distinguished orbits in $\mathfrak{g}$. Moreover the possible values for d_x are 22, 26, 34.

(v) $p = 3$, $M = G$ is of type E_8, $d_x = 30$ and $G \cdot x$ is the only non-standard distinguished orbit in $\mathfrak{g}$.

REMARK. In [16] a different approach is used, and the orbits listed in part (a) of the theorem appear as the 'new' orbits, and those in part (b) as corresponding to orbits existing in characteristic 0.

We have proved also

THEOREM 4. *Let G be of type F_4, with $p = 3$. Then all nilpotent orbits in $\mathfrak{g}$ are standard, and $d_x = d_x^0$ for all $x \in \mathcal{N}$.*

Appendix

The Computation of Centralizers in Finite p-groups

The computational problem encountered above was the following. We are given a finite p-group G of order q^n, where q is a power of p, which acts in a prescribed manner on an elementary abelian p-group N, also of order q^n, and we want to calculate the order of the centralizer in G of various elements of N. In fact G is a unipotent subgroup of one of the finite Chevalley groups $E_7(q)$ and $E_8(q)$, in which case n is 63 or 120 respectively, and N is the underlying vector space of the corresponding Lie Algebra. The relevant values of q are 2, 4, 3 and 5. Equivalently, we are asking for the centralizer of an element of N in the semidirect product GN, and so we can consider this problem as a special case of the problem of computing centralizers of elements in finite p-groups. An efficient algorithm for doing this and for computing conjugacy classes in p-groups is described by Felsch and Neubüser in [6]. In fact, we used a somewhat simpler method, which would probably be slower in general, but was adequate for these particular examples. We shall now give a brief description of this method.

It is convenient to define our p-group P by means of a power-commutator presentation. This means that we have generators $x_1, x_2, \ldots, x_n$ of P, where $|P| = p^n$, such that

$$1 \subset \langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \subset \langle x_1, x_2, \ldots, x_n \rangle = P$$

is a central series for P, and we are given the values of the commutators $[x_i, x_j]$ $(i < j)$ and the powers x_i^p. Then every element $x \in P$ has a unique expression of

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the form

$$x = x_n^{i_n} \cdots x_2^{i_2} x_1^{i_1} \qquad (0 \leqslant i_j < p),$$

which we call the normal form of x. We assume that the commutators and powers are given in their normal form. Then these commutators and powers can be used in the so-called collection process (see, for example, [24]) to put an arbitrary word in the x_i into normal form. This collection process can be carried out very efficiently on a machine, and it forms the basis of most computer algorithms for dealing with nilpotent groups. In the normal form for x given above, we will call $x_j^{i_j}$ the leading term of x, where j is maximal such that $i_j \neq 0$. Note that the leading term of an element is not changed when we replace the element by a conjugate.

In our specific example, $P = GN$, the generators of G and N are in one-one correspondence with the positive roots of the Lie Algebra E_7 or E_8. If g and h are generators of G corresponding to roots ϕ and ψ respectively, then $[g, h] \neq 1$ if and only if $\phi + \psi$ is also a root, in which case $[g, h] = k^{\pm 1}$, where k corresponds to the root $\phi + \psi$. The same rule applies to the action of G on N. Furthermore, the p^{th} powers of all of the generators are trivial. This, together with the fact that all commutators have length 0 or 1, renders the collection process particularly easy in this case. (Of course, in the case $q = 4$, the number of power-commutator generators of G is really twice the number of roots, and commutators may have length 2 in these generators.) The chief difficulty pertaining to the input of this data was the sign in the expression $[g, h] = k^{\pm 1}$, which is only a problem when p is odd. The essential condition to be fulfilled here is that the relations in the presentation should be consistent, which means in effect that the associative law $(x_i x_j)x_k = x_i(x_j x_k)$ should be valid for all i, j and k. Values for these signs were originally taken from Table 12, at the end of Mizuno's paper [12]. The computer was then used to check consistency, and about six errors were found and corrected.

Now suppose that we wish to find generators of the centralizer of an element t in P. We assume that, at the ith step of the computation ($1 \leqslant i \leqslant n$), we have already found generators of the centralizer of t in $\langle x_1, \ldots, x_{i-1} \rangle$, and we want to find out whether or not there exists $x_i w \in C(t)$, for some $w \in \langle x_1, \ldots, x_{i-1} \rangle$. If so, then $[x_i, t]$ and $[w^{-1}, t]$ will have the same leading term, and so we will keep a record of the leading terms that can occur as commutators with t, at each stage. At the ith stage, we either find an element $x_i w \in C(t)$, or we find a commutator $[x_i w, t]$ for which the leading term generator has not occurred previously, and we record this commutator. We shall now write down this algorithm more precisely. C will be the set of generators of $C(t)$, and, for $1 \leqslant i \leqslant n$, $b[i]$ will be an element of P with $[b[i], t] = c[i]$, where $c[i]$ has some power of x_i as its leading term.

Begin Set $C = \varnothing$ *and* $c[i] = 1$ *for* $1 \leqslant i \leqslant n$;
For $i = 1$ *to* n *do*
Begin Put $y = x_i$;
Loop: *Put* $z = [y, t]$;
If $z = 1$ *then replace* C *by* $C \cup \{y\}$ *else*
Begin Let x_j^n *be the leading term of* z;
If $c[j] = 1$ *then put* $c[j] = z$ *and* $b[j] = y$ *else*
Begin Let x_j^m *be the leading term of* $c[j]$;
Put $l = -n/m \pmod p$ *and replace* y *by* $yb[j]^l$;
Goto Loop;
End;
End;
End;
End.

Using an implementation of this algorithm written in Burroughs Algol on the B6700 machine at Warwick University, a typical process time for the computation of the centralizer of an element in the case E_8, with $q = 3$ or 4 was 10 seconds. It was rather longer than this in a few bad cases in which the elements $b[j]$ and $c[j]$ grew unusually long, but, owing to the simple nature of the power-commutator presentation, this did not happen very often. The Felsch-Neubüser method is completely different, and works downwards through the successive factor groups in the central series for P, rather than upwards through the subgroups, as we are doing here. We suspect that the downwards method would be ultimately more efficient if more complex presentations were involved.

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