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Almost proximal extensions of minimal flows

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Abstract. In this paper, we study almost proximal extensions of minimal flows. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then π is called an almost proximal extension if there is some $N \in \mathbb{N}$ such that the cardinality of any almost periodic subset in each fiber is not greater than N. When $N = 1$, π is proximal. We will give the structure of π and give a dichotomy theorem: any almost proximal extension of minimal flows is either almost finite to one, or almost all fibers contain an uncountable strongly scrambled subset. Using the category method, Glasner and Weiss showed the existence of proximal but not almost one-to-one extensions [On the construction of minimal skew products. *Israel J. Math.* **34** (1979), 321–336]. In this paper, we will give explicit such examples, and also examples of almost proximal but not almost finite to one extensions.

Key words: proximal extensions, minimal flows, chaos, weak mixing, uniform rigidity
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1. *Introduction*

The structure theory of minimal flows originated in Furstenberg's seminal work [13] for distal minimal flows, and the structure theorem for the general minimal flows was built by Ellis, Glasner, and Shapiro [11], McMahon [21], Veech [28], and Glasner [17]. Roughly speaking, the class of minimal flows is the smallest class of flows containing the trivial flow and closed under homomorphisms, inverse limits, and has three 'building blocks' which are equicontinuous extensions, proximal extensions, and topologically weakly mixing extensions. In this paper, we mainly study proximal extensions and almost proximal extensions. Refer to [17] for a systematical study of proximal flows.

In this paper, a *flow* (X, T) is a compact metric space X with an infinite countable discrete group T acting continuously on X. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. The proximal relation $P(X, T)$ is defined by

$$P(X, T) = \left\{(x, x') \in X^2 : \inf_{t \in T} d(tx, tx') = 0\right\}.$$

Any pair in $P(X, T)$ is called a *proximal pair* and π is *proximal* if any pair (x, x') in the same fiber is proximal, that is, $(x, x') \in P(X, T)$ whenever $\pi(x) = \pi(x')$. A subset A of X is called an *almost periodic set* if every finite subset $\{x_1, x_2, \ldots, x_n\}$ of A,

$(x_1, x_2, \ldots, x_n)$ is a minimal point of the product flow (X^n, T). Note that any pair (x, x') with $x \neq x'$ will not be proximal if (x, x') is minimal in (X^2, T). By this fact, it is easy to show that π is proximal if and only if each almost periodic subset in the fiber of π is a singleton. Inspired by this, we call π an *almost proximal extension* if there is some $N \in \mathbb{N}$ such that the cardinality of any almost periodic subset in each fiber is not greater than N. When $N = 1$, π is proximal.

An extension $\pi : (X, T) \to (Y, T)$ of minimal flows is *almost finite to one* if some fiber is finite, that is, there is some y where $\pi^{-1}(y)$ is finite. It is not difficult to see that any almost finite-to-one extension is almost proximal (Proposition 3.8). After we study the structure of almost proximal extensions, we show that an extension $\pi : X \to Y$ of minimal flows is almost finite-to-one if and only if it is almost proximal and point distal.

It is an open question [**2**, Problem 5.23] that: if a minimal flow (X, T) is not point distal (that is, for any point $x \in X$, there is $x' \neq x$ such that (x, x') is proximal), is it chaotic in the sense of Li–Yorke? It was showed in [**2**, Theorem 5.17] that if a minimal flow (X, T) is a proximal but not an almost one-to-one extension of some flow (Y, T), then (X, T) is not point distal and it is Li–Yorke chaotic. In this paper, we generalize this result, and show that any almost proximal extension has the following dichotomy theorem: any almost proximal extension of minimal flows is either almost finite-to-one, or almost all fibers contain an uncountable strongly scrambled subset. In particular, if a minimal flow (X, T) is an almost proximal but not almost finite-to-one extension of some flow (Y, T), then (X, T) is not point distal and it is Li–Yorke chaotic.

Since there are no non-trivial proximal minimal flows under abelian group actions [**17**, Theorem 3.4], it is not easy to give a minimal $\mathbb{Z}$-flow which is proximal but not almost one-to-one extension of its maximal equicontinuous factor. In fact, this was a question by Furstenberg several years ago. Using the category method, Glasner and Weiss showed the existence of proximal but not almost one-to-one extensions [**19**, Theorem 3]. In this paper, using methods in [**7**], we will give such explicit examples, and also examples of almost proximal but not almost finite-to-one extensions. In addition, all examples constructed are uniformly rigid.

1.1. *Organization of the paper.* We organize the paper as follows. In §2, we introduce some basic notions and results needed in the paper. In §3, we introduce the notion of almost proximal and give its structure. In §4, we study chaotic properties of proximal but not almost one-to-one extensions. In §5, we will give a dichotomy theorem: any almost proximal extension of minimal flows is either almost finite-to-one, or almost all fibers contain a strongly scrambled subset. In §6, we will give explicit examples of almost proximal but not almost finite-to-one extensions. In the final section, we will give some questions.

2. *Basic facts about abstract topological dynamics*

In this section, we recall some basic definitions and results in abstract topological flows. For more details, see [**4**, **8**, **10**, **17**, **28**]. In the article, integers, non-negative integers, and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$, respectively.

2.1. *Topological transformation groups.* A *flow* or a *topological dynamical system* is a triple $\mathcal{X} = (X, T, \Pi)$, where X is a compact Hausdorff space, T is a Hausdorff topological group, and $\Pi : T \times X \to X$ is a continuous map such that $\Pi(e, x) = x$ and $\Pi(s, \Pi(t, x)) = \Pi(st, x)$, where e is the unit of T, $s, t \in T$, and $x \in X$. We shall fix T and suppress the action symbol.

In this paper, we always assume that T is infinite countable and discrete, unless we state it explicitly in some places. Moreover, we always assume that X is a compact metric space with metric $d(\cdot, \cdot)$.

When $T = \mathbb{Z}$, (X, T) is determined by a homeomorphism f, that is, f is the transformation corresponding to 1 of $\mathbb{Z}$. In this case, we usually denote $(X, \mathbb{Z})$ by (X, f), and also call it a *discrete flow*.

Let (X, T) be a flow and $x \in X$. Let $\mathcal{O}(x, T) = \{tx : t \in T\}$ be the *orbit* of x, which is also denoted by Tx. We usually denote the closure of $\mathcal{O}(x, T)$ by $\overline{\mathcal{O}}(x, T)$ or $\overline{Tx}$. A subset $A \subseteq X$ is called *invariant* if $ta \subseteq A$ for all $a \in A$ and $t \in T$. When $Y \subseteq X$ is a closed and invariant subset of the flow (X, T), we say that the flow (Y, T) is a *subflow* of (X, T). If (X, T) and (Y, T) are two flows, their *product flow* is the flow $(X \times Y, T)$, where $t(x, y) = (tx, ty)$ for any $t \in T$ and $x, y \in X$. For $n \geq 2$, we write (X^n, T) for the n-fold product flow $(X \times \cdots \times X, T)$.

A flow (X, T) is called *minimal* if X contains no proper non-empty closed invariant subsets. A point $x \in X$ is called a *minimal point* or an *almost periodic point* if $(\overline{\mathcal{O}}(x, T), T)$ is a minimal flow.

A flow (X, T) is called *transitive* if every invariant open subset of X is dense; and it is *point transitive* if there is a point with a dense orbit (such a point is called a *transitive point*). It is easy to verify that a flow is minimal if and only if every orbit is dense.

The flow (X, T) is *weakly mixing* if the product flow $(X \times X, T)$ is transitive.

A *factor map* $\pi : X \to Y$ between the flow (X, T) and (Y, T) is a continuous onto map which intertwines the actions; we say that (Y, T) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S). The flows are said to be *isomorphic* if π is bijective.

Let (X, T) be a flow. Fix $(x, y) \in X^2$. It is a *proximal* pair if $\inf_{t \in T} d(tx, ty) = 0$; it is a *distal* pair if it is not proximal. Denote by $P(X, T)$ or $P(X)$ the set of proximal pairs of (X, T). Here, $P(X, T)$ is also called the *proximal relation* of (X, T). A flow (X, T) is *distal* if $P(X, T) = \Delta_X$, where $\Delta_X = \{(x, x) \in X^2 : x \in X\}$ is the diagonal of $X \times X$. A flow (X, T) is *equicontinuous* if for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, then $d(tx, ty) < \epsilon$ for all $t \in T$. Any equicontinuous flow is distal.

Let (X, T) be a flow. There is a smallest invariant equivalence relation S_{eq} such that the quotient flow $(X/S_{\mathrm{eq}}, T)$ is equicontinuous [12, Theorem 1]. The equivalence relation S_{eq} is called the *equicontinuous structure relation* and the factor $(X_{\mathrm{eq}} = X/S_{\mathrm{eq}}, T)$ is called the *maximal equicontinuous factor* of (X, T).

2.2. *Enveloping semigroups.* Given a flow (X, T), its *enveloping semigroup* or *Ellis semigroup* $E(X, T)$ is defined as the closure of the set $\{t : t \in T\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). For an enveloping semigroup, $E(X, T) \to E(X, T) : q \mapsto qp$ and $p \mapsto tp$ is continuous for all $p \in E(X, T)$ and $t \in T$. Note that (X^X, T) is a flow and $(E(X, T), T)$ is its subflow.

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Definition 2.1. A set E is an $\mathcal{E}$*-semigroup* if it satisfies the following three conditions:

(1) E is a semigroup;

(2) E has a compact Hausdorff topology;

(3) the right translation map $R_p : E \longrightarrow E, q \longmapsto qp$ is continuous for every $p \in E$.

It is easy to see that for a flow (X, T), the enveloping semigroup $E(X, T)$ is an $\mathcal{E}$-semigroup.

For a semigroup, the element u with $u^2 = u$ is called *idempotent*. Ellis–Numakura theorem says that for any $\mathcal{E}$-semigroup E, the set $J(E)$ of idempotents of E is not empty (see [10, Corollary 2.10] or [17, Ch. I, Lemma 2.2]).

Let E be an $\mathcal{E}$-semigroup. A non-empty subset $I \subseteq E$ is a *left ideal* if $EI \subseteq I$. A *minimal left ideal* is a left ideal that does not contain any proper left ideal of E. Every left ideal is a semigroup and every left ideal contains some minimal left ideal.

2.3. *Universal point transitive flow and universal minimal flow.* For a fixed T, there exists a universal point transitive flow (S_T, T) such that T can densely and equivariantly be embedded in S_T (see [4, Ch. 8], [17, Ch. I]). The multiplication on T can be extended to a multiplication on S_T, and S_T is an $\mathcal{E}$-semigroup. The universal minimal flow $\mathfrak{M} = (\mathbf{M}, T)$ is isomorphic to any minimal left ideal in S_T and $\mathbf{M}$ is also an $\mathcal{E}$-semigroup. Hence $J = J(\mathbf{M})$ of idempotents in $\mathbf{M}$ is non-empty.

PROPOSITION 2.2. [17, Ch. I, Proposition 2.3]

(1) *For $v \in J$ and $p \in \mathbf{M}$, $pv = p$.*

(2) *For each $v \in J$, $v\mathbf{M} = \{vp : p \in \mathbf{M}\} = \{p \in \mathbf{M} : vp = p\}$ is a subgroup of $\mathbf{M}$ with identity element v. For every $w \in J$, the map $p \mapsto wp$ is a group isomorphism of vI onto wI.*

(3) *$\{v\mathbf{M} : v \in J\}$ is a partition of $\mathbf{M}$. Thus, if $p \in \mathbf{M}$, then there exists a unique $v \in J$ such that $p \in v\mathbf{M}$.*

Since $v\mathbf{M}$ is a group ($v \in J$), for $p \in v\mathbf{M}$, we denote p^{-1} as the inverse of p in $v\mathbf{M}$, that is, $p^{-1} \in v\mathbf{M}$ such that $p^{-1}p = pp^{-1} = v$. If we choose an arbitrary idempotent $u \in J$ and denote $G = u\mathbf{M}$, then every element p of $\mathbf{M}$ has unique representation $p = v\alpha$ for $v \in J$ and $\alpha \in G$. Moreover, $p^{-1} = v\alpha^{-1}$ (see [17, Ch. I, Proposition 2.3, Corollary 2.4]).

The sets S_T and $\mathbf{M}$ act on X as semigroups and $S_T x = \overline{Tx}$, while for a minimal flow (X, T), we have $\mathbf{M}x = \overline{Tx} = X$ for every $x \in X$ (see [17, Ch. I, Proposition 3.1] for details). For $x \in X$, set $J_x = \{u \in J : ux = x\}$.

PROPOSITION 2.3. [17, Ch. I, Proposition 3.1] *Let (X, T) be a flow and $x \in X$. A necessary and sufficient condition for x to be minimal is that $ux = x$ for some $u \in J$.*

Thus, for a closed invariant subset A of X, $JA = \{ua : u \in J, a \in A\}$ is the set of all minimal points contained in A.

PROPOSITION 2.4. [17, Ch. I, Proposition 3.2] *Let (X, T) be a flow, $x, y \in X$.*

(1) *For each $u \in J(S_T)$, $(x, ux) \in P(X, T)$.*

(2) *A pair $(x, y) \in P(X, T)$ if and only if $px = py$ for some $p \in S_T$, if and only if there is some minimal left ideal I of S_T such that $px = py$ for every $p \in I$.*

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(3) *If (X, T) is a minimal flow, then $(x, y) \in P(X)$ if and only if there is a minimal idempotent $u \in J(S_T)$ such that $y = ux$.*

2.4. *Hyperspace flow and circle operation.* Let X be a compact metric space. Let 2^X be the collection of non-empty closed subsets of X endowed with the Hausdorff topology. Let (X, T) be a flow. We can induce a flow on 2^X. The action of T on 2^X is given by $tA = \{ta : a \in A\}$ for each $t \in T$ and $A \in 2^X$. Then $(2^X, T)$ is a flow and it is called the *hyperspace flow*.

As $(2^X, T)$ is a flow, S_T acts on 2^X too. To avoid ambiguity, we denote the action of S_T on 2^X by the *circle operation* as follows. Let $p \in S_T$ and $D \in 2^X$. Define $p \circ D = \lim_{2^X} t_i D$ for any net $\{t_i\}_i$ in T with $t_i \to p$. Moreover,

$$p \circ D = \{x \in X : \text{there are } d_i \in D \text{ with } x = \lim_i t_i d_i\}$$

for any net $t_i \to p$ in S_T. We always have $pD \subseteq p \circ D$.

Note that if $A \in 2^X$ is finite and $p \in S_T$, then $pA = p \circ A$.

2.5. *Almost one-to-one extensions and O-diagram.* Let (X, T) and (Y, T) be flows and let $\pi : X \to Y$ be a factor map. One says that:

(1) π is an *open extension* if it is open as a map;

(2) π is an *almost one-to-one extension* if there exists a dense G_δ set $X_0 \subseteq X$ such that $\pi^{-1}(\{\pi(x)\}) = \{x\}$ for any $x \in X_0$.

The following is a well-known fact about open mappings (see [8, Appendix A.8], for example).

THEOREM 2.5. *Let $\pi : (X, T) \to (Y, T)$ be a factor map of flow. Then the map $\pi^{-1} : Y \to 2^X$, $y \mapsto \pi^{-1}(y)$ is continuous if and only if π is open.*

Every extension of minimal flows can be lifted to an open extension by almost one-to-one modifications ([27, Theorem 3.1], [5, Lemma III.6] or [8, Ch. VI]). To be precise, we have the following theorem.

THEOREM 2.6. *For every extension $\pi : X \to Y$ of minimal flows, there exists a commutative diagram of extensions (called the O-diagram)*

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X^* \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi*} \\ Y & \xleftarrow[\tau]{} & Y^* \end{array}$$

with the following properties:

(a) *σ and τ are almost one-to-one;*

(b) *π^* is an open extension;*

(c) *X^* is the unique minimal set in $R_{\pi\tau} = \{(x, y) \in X \times Y^* : \pi(x) = \tau(y)\}$ and σ and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^*, respectively.*

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We sketch the construction of these factors. Let $x \in X$, $u \in J_x$ and $y = \pi(x)$. Let $y^* = u \circ \pi^{-1}(y)$. One has that y^* is a minimal point of $(2^X, T)$ and define $Y^* = \{p \circ y^* : p \in \mathbf{M}\}$ as the orbit closure of y^* in 2^X for the action of T. Finally, $X^* = \{(px, p \circ y^*) \in X \times Y^* : p \in \mathbf{M}\}$, $\tau(p \circ y^*) = py$ and $\sigma((px, p \circ y^*)) = px$. It can be proved that $X^* = \{(\tilde{x}, \tilde{y}) \in X \times Y^* : \tilde{x} \in \tilde{y}\}$.

There is another equivalent way to get an O-diagram. Let $\pi^{-1} : Y \to 2^X$, $y \mapsto \pi^{-1}(y)$. Then π^{-1} is a u.s.c. map, and the set Y_c of continuous points of π^{-1} is a dense G_δ subset of Y. Let

$$\widetilde{Y} = \overline{\{\pi^{-1}(y) : y \in Y\}} \quad \text{and} \quad Y^* = \overline{\{\pi^{-1}(y) : y \in Y_c\}},$$

where the closure is taken in 2^X. It is obvious that $Y^* \subseteq \widetilde{Y} \subseteq 2^X$. Note that for each $A \in \widetilde{Y}$, there is some $y \in Y$ such that $A \subseteq \pi^{-1}(y)$, and hence $A \mapsto y$ define a map $\tau : \widetilde{Y} \to Y$. It is easy to verify that $\tau : (\widetilde{Y}, T) \to (Y, T)$ is a factor map. One can show that if (Y, T) is minimal, then (Y^*, T) is a minimal flow and it is the unique minimal subflow in $(\widetilde{Y}, T)$, and $\tau : Y^* \to Y$ is an almost one-to-one extension such that $\tau^{-1}(y) = \{\pi^{-1}(y)\}$ for all $y \in Y_c$ (see [28, §2.3]). When Y^* and τ are defined, X^*, σ, and π^* are defined as above.

2.6. *Proximal extensions and RIC-diagram.* Let (X, T) and (Y, T) be flows and let $\pi : X \to Y$ be a factor map. One says that:

(1) π is a *distal extension* if $\pi(x_1) = \pi(x_2)$ and $x_1 \neq x_2$ implies $(x_1, x_2) \notin P(X, T)$;

(2) π is an *equicontinuous or isometric extension* if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\pi(x_1) = \pi(x_2)$ and $d(x_1, x_2) < \delta$ imply $d(tx_1, tx_2) < \epsilon$ for any $t \in T$;

(3) π is a *weakly mixing extension* if (R_π, T) as a subflow of the product flow $(X \times X, T)$ is transitive, where $R_\pi = \{(x_1, x_2) \in X^2 : \pi(x_1) = \pi(x_2)\}$.

Let $\pi : (X, T) \to (Y, T)$ be a factor map of minimal flows, and $x_0 \in X$, $y_0 = \pi(x_0)$, and $u \in J_{x_0}$. We say that π is an *RIC* (relatively incontractible) extension if for every $y = py_0 \in Y$, $p \in \mathbf{M}$,

$$\pi^{-1}(y) = p \circ u\pi^{-1}(y_0).$$

One can show that $\pi : X \to Y$ is RIC if and only if it is open and for every $n \geq 1$, the minimal points are dense in the relation

$$R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j) \quad \text{for all } 1 \leq i \leq j \leq n\}.$$

Note that every distal extension is RIC.

Every factor map between minimal flows can be lifted to an RIC extension by proximal extensions (see [11, Theorem 5.13] or [8, Ch. VI]).

THEOREM 2.7. *Given a factor map $\pi : X \to Y$ of minimal flows, there exists a commutative diagram of factor maps (called an RIC-diagram)*

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X' \arrow[l, "\sigma'"'] \arrow[d, "\pi'"] \\ Y & Y' \arrow[l, "\tau'"] \end{tikzcd}$$

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such that:

(a) *τ' and σ' are proximal extensions;*

(b) *π' is an RIC extension;*

(c) *X' is the unique minimal set in $R_{\pi\tau'} = \{(x, y) \in X \times Y' : \pi(x) = \tau'(y)\}$, and σ' and π' are the restrictions to X' of the projections of $X \times Y'$ onto X and Y', respectively.*

We sketch the construction of these factors. Let $x \in X$, $u \in J_x$, and $y = \pi(x)$. Let $y' = u\circ u\pi^{-1}(y)$, then y' is a minimal point in 2^X. Define $Y' = \{p\circ y' : p \in \mathbf{M}\}$ to be the orbit closure of y' and $X' = \{(px, p\circ y') \in X \times Y' : p \in \mathbf{M}\}$, and factor maps are given by $\tau'(p\circ y') = py$ and $\sigma'((px, p\circ y')) = px$. It can be proved that $X' = \{(\tilde{x}, \tilde{y}) \in X \times Y' : \tilde{x} \in \tilde{y}\}$.

3. *Almost proximal extensions*

In this section, we introduce almost proximal extensions. We will give the structure of almost proximal extensions, and study its relationship with almost finite to one extensions.

3.1. *Almost proximal extensions.*

An *almost periodic set* for (X, T) is a subset A of X such that if $z \in X^{|A|}$ with range$(z) = A$, then z is a minimal point of the flow $(X^{|A|}, T)$. (Here, $|A|$ denotes the cardinality of A.) For example, a finite set $A = \{x_1, x_2, \ldots, x_n\}$ is an almost periodic set if and only if $(x_1, x_2, \ldots, x_n)$ is a minimal point of (X^n, T). Using the basis for the Tychonoff topology, we see that a set A is an almost periodic set if and only if every finite subset of A is an almost periodic set. The notion of almost periodic sets was introduced by Auslander, refer to [4, Ch. 5] for more information.

Definition 3.1. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then π is called an *almost proximal extension* if there is some $N \in \mathbb{N}$ such that for each $y \in Y$, the cardinality of any almost periodic subset in the fiber $\pi^{-1}(y)$ is not greater than N.

Let A be an almost periodic set of (X, T). Then for each $z \in X^{|A|}$ with range$(z) = A$, z is a minimal point of $(X^{|A|}, T)$. By Proposition 2.3, there is some $u \in J$ such that $uz = z$. It follows that $uA = \{ua : a \in A\} = A$. Thus, for a subset Z of X, a subset $A \subseteq Z$ is an almost periodic subset of Z if and only if $A \subseteq uZ$ for some $u \in J$.

LEMMA 3.2. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Let $y, y' \in Y$, $u \in J_y$, and $v \in J_{y'}$. Then, $|u\pi^{-1}(y)| = |v\pi^{-1}(y')|$.*

Proof. We only show the case when $u\pi^{-1}(y)$, $v\pi^{-1}(y')$ are finite. The same proof works for the general case. Let $u\pi^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$ and $v\pi^{-1}(y') = \{x'_1, x'_2, \ldots, x'_m\}$ for some $n, m \in \mathbb{N}$. Then, $(x_1, x_2, \ldots, x_n) \in X^n$ is a minimal point of (X^n, T) as $u(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$. Since (X, T) is minimal, there is $p \in \mathbf{M}$ such that $x'_1 = px_1$. Note that $x'_1 = vx'_1 = vpx_1$ and $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$. It follows that

$$y' = \pi(x'_1) = \pi(vpx_1) = \cdots = \pi(vpx_n) = vp\pi(x_1) = vpy.$$

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Since $x_1, x_2, \ldots, x_n$ are distinct and $(x_1, x_2, \ldots, x_n)$ is minimal, $vpx_1, vpx_2, \ldots, vpx_n$ are also distinct. As

$$vpx_1, vpx_2, \ldots, vpx_n \in v\pi^{-1}(y') = \{x'_1, x'_2, \ldots, x'_m\},$$

we have $n \leq m$. Similarly, we have $m \leq n$. Thus, $|u\pi^{-1}(y)| = |v\pi^{-1}(y')|$. $\square$

By Lemma 3.2 and the fact that A is an almost periodic subset of subset Z of (X, T) if and only if $A \subseteq uZ$ for some $u \in J$, we have the following proposition readily.

PROPOSITION 3.3. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then the following are equivalent.*

(1) *π is almost proximal.*

(2) *For some $y \in Y$, there is some $N \in \mathbb{N}$ such that the cardinality of any almost periodic subset in the fiber $\pi^{-1}(y)$ is not greater than N.*

(3) *There is some $N \in \mathbb{N}$ such that for some $y \in Y$ and $u \in J_y$, $|u\pi^{-1}(y)| = N < \infty$.*

(4) *There is some $N \in \mathbb{N}$ such that for each $y \in Y$ and $u \in J_y$, $|u\pi^{-1}(y)| = N < \infty$.*

When $N = 1$, π is proximal.

For $x \in X$, $P[x] = \{x' \in X : (x, x') \in P(X, T)\}$ is called the *proximal cell* of x. It is clear that π is proximal if and only if for any $y \in Y$ and any $x \in \pi^{-1}(y)$, $\pi^{-1}(y) \subseteq P[x]$.

COROLLARY 3.4. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. If π is almost proximal, then for each $y \in Y$, there is some finite subset F of $\pi^{-1}(y)$ such that each point of $\pi^{-1}(y)$ is proximal to some point of F, that is, $\pi^{-1}(y) \subseteq \bigcup_{x\in F} P[x]$.*

Proof. Let $y \in Y$ and $u \in J_y$. Then $F = u\pi^{-1}(y)$ is finite by Proposition 3.3. For any $x \in \pi^{-1}(y)$, by Proposition 2.4, $(x, ux) \in P(X)$ and $ux \in F$. The proof is complete. $\square$

Let $\pi : (X, T) \to (Y, T)$ be an extension of flows (X, T) and (Y, T) and $n \in \mathbb{N}$. Let

$$2^X_\pi = \{A \in 2^X : A \subseteq \pi^{-1}(y) \text{ for some } y \in Y\},$$

and

$$2^X_{\pi,n} = \{A \in 2^X_\pi : |A| \leq n\}, \quad \text{and} \quad 2^X_{\pi,*} = \{A \in 2^X_\pi : |A| < \infty\} = \bigcup_{n=1}^{\infty} 2^X_{\pi,n}.$$

It is clear that $(2^X_\pi, T)$, $(2^X_{\pi,n}, T)$, and $(2^X_{\pi,*}, T)$ are subflows of $(2^X, T)$. Let

$$\widetilde{\pi} : (2^X_\pi, T) \to (Y, T), \quad A \mapsto \pi(A).$$

Then $\widetilde{\pi}$ is an extension. Note that $2^X_{\pi,1} = \{\{x\} \in 2^X : x \in X\}$ is isomorphic to X and $\widetilde{\pi}|_{2^X_{\pi,1}}$ is the same to π. Thus, $\pi : (X, T) \to (Y, T)$ is proximal if and only if $\widetilde{\pi}|_{2^X_{\pi,1}} : (2^X_{\pi,1}, T) \to (Y, T)$ is proximal.

PROPOSITION 3.5. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then π is almost proximal if and only if there is some $n \in \mathbb{N}$ such that there are no minimal points in $2^X_{\pi,*} \setminus 2^X_{\pi,n}$.*

Proof. If π is almost proximal, then there is some $n \in \mathbb{N}$ such that $|u\pi^{-1}(y)| = n$ for all $y \in Y$, $u \in J_y$. Let $A \in 2^X_{\pi,*} \setminus 2^X_{\pi,n}$ and $y = \pi(A)$. Then $|A| \geq n + 1$. For each $u \in J_y$, $|uA| \leq n$ since π is almost proximal. Thus, $u \circ A = uA \neq A$, in particular, by Proposition 2.3, A is not minimal.

Conversely, assume that there is some $n \in \mathbb{N}$ such that there are no minimal points in $2^X_{\pi,*} \setminus 2^X_{\pi,n}$. If π is not almost proximal, then there is some $y \in Y$ and $u \in J_y$ such that $|u\pi^{-1}(y)| = \infty$. We choose $A \subseteq u\pi^{-1}(y)$ with $|A| = n + 1$. Then

$$u \circ A = uA = A,$$

which means A is a minimal point of $2^X_{\pi,*} \setminus 2^X_{\pi,n}$, which is a contradiction. The proof is complete. □

Remark 3.6. It is well known that each flow has a minimal subflow. Since $2^X_{\pi,*} \setminus 2^X_{\pi,n}$ may not be compact, it is an invariant subset of 2^X but maybe not a subflow of 2^X. So it is possible that $2^X_{\pi,*} \setminus 2^X_{\pi,n}$ contains no minimal points.

3.2. *Almost finite-to-one extensions*

Definition 3.7. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then π is called an *almost finite-to-one* extension if some fiber is finite.

PROPOSITION 3.8. [**24**, Proposition 3.5], [**20**, Proposition 2.15] *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. The following statements are equivalent:*

(1) *π is almost finite to one, that is, some fiber is finite;*
(2) *there exists $N \in \mathbb{N}$ such that $Y_0 = \{y \in Y : |\pi^{-1}(y)| = N\}$ is a residual subset of Y;*
(3) *there exist $N \in \mathbb{N}$ and $y_0 \in Y$ such that $|\pi^{-1}(y_0)| = N$ and $\pi^{-1}(y_0)$ is an almost periodic set;*
(4) *the cardinality of each minimal point of $(2^X, T)$ in 2^X_π is not greater than some fixed integer N.*

Remark 3.9

(1) By Proposition 3.8, almost finite-to-one extensions of minimal flows are almost proximal. In §6, we will give examples which are almost proximal but not almost finite-to-one extensions.
(2) By definition, it is obvious that a finite-to-one extension (that is, every fiber is finite) is almost finite-to-one. However, in general, an almost finite-to-one extension may not be finite-to-one. For example, for Rees' example [**22**], $\pi : (X, T) \to (X_{\mathrm{eq}}, T)$ is an almost one-to-one extension (X_{eq} is the maximal equicontinuous factor of X), and for any $y \in X_{\mathrm{eq}}$, either $|\pi^{-1}(y)| = 1$ or $|\pi^{-1}(y)| = \infty$.
(3) For more discussion about finite-to-one and almost finite-to-one extensions, refer to [**20**].

By Proposition 3.8, it is easy to check the following corollary.

COROLLARY 3.10. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. If π is proximal but not almost one-to-one, then every fiber is infinite, that is, π is not almost finite-to-one.*

3.3. *Structure of almost proximal extensions.* The following result is well known and it is easy to be verified (one may find a proof in [**20**, Lemma 2.20]).

LEMMA 3.11. *Let $\pi : X \to Y$ be a finite-to-one extension (that is, $\pi^{-1}(y)$ is finite for all $y \in Y$) of the minimal flows (X, T) and (Y, T). Then the following conditions are equivalent:*

(1) *π is open;*
(2) *π is distal;*
(3) *π is equicontinuous;*

In this case, there exists $N \in \mathbb{N}$ such that π is an N-to-one map.

THEOREM 3.12. *Let $\pi : (X, T) \to (Y, T)$ be an almost proximal extension of minimal flows. Then it has the following structure:*

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X' \arrow[l, "\sigma'"'] \arrow[d, "\pi'"] \\ Y & Y' \arrow[l, "\tau'"] \end{tikzcd}$$

where σ' and τ' are proximal, π' is a finite-to-one equicontinuous extension.

Moreover, π is almost finite-to-one if and only if τ', σ' are almost one-to-one.

Proof. In light of the construction of the RIC-diagram (Theorem 2.7), σ' and τ' are proximal extensions and π' is an RIC extension. We show that if π is almost proximal, then π' is a finite-to-one equicontinuous extension. Recall the construction of RIC-diagram. Let $x \in X$, $u \in J_x$, and $y = \pi(x)$. Let $y' = u \circ u\pi^{-1}(y) \in 2^X$. Then $Y' = \{p \circ y' : p \in \mathbf{M}\}$ is the orbit closure of y' and $\tau'(p \circ y') = py$. And $X' = \{(px, p \circ y') \in X \times Y' : p \in \mathbf{M}\} = \{(\tilde{x}, \tilde{y}) \in X \times Y' : \tilde{x} \in \tilde{y}\}$ is the unique minimal set in $R_{\pi\tau'} = \{(x, y) \in X \times Y' : \pi(x) = \tau'(y)\}$, and σ' and π' are the restrictions to X' of the projections of $X \times Y'$ onto X and Y', respectively. Since π is almost proximal, we have that $|u\pi^{-1}(y)| < \infty$. Thus, $y' = u \circ u\pi^{-1}(y)$ is finite, and for every $p \in \mathbf{M}$, $|p \circ y'| < \infty$. Hence, for every $p \in \mathbf{M}$, $|(\pi')^{-1}(p \circ y')| = |\{(\tilde{x}, p \circ y') : \tilde{x} \in p \circ y'\}| = |p \circ y'|$ is finite. It follows that π' is finite-to-one and open. By Lemma 3.11, π' is equicontinuous.

Clearly, if τ', σ' are almost one-to-one, then π is almost finite-to-one. Conversely, when π is almost finite-to-one, by Proposition 3.8, there exists y_0 such that $|\pi^{-1}(y_0)| < \infty$ and $\pi^{-1}(y_0)$ is a minimal point of $(2^X, T)$, that is, $u \circ \pi^{-1}(y_0) = \pi^{-1}(y_0)$. Hence, the construction of the RIC-diagram coincides with the O-diagram. Therefore, τ', σ' are almost one-to-one. □

If, in addition, the extension is regular (see definition below), an almost proximal extension has a succinct structure. Let $\mathrm{Aut}(X, T)$ be the group of automorphisms of the flow (X, T), that is, the group of all self-homeomorphisms ψ of X such that $\psi \circ t = t \circ \psi$ for all $t \in T$. For an extension $\pi : (X, T) \to (Y, T)$, let

$$\mathrm{Aut}_\pi(X, T) = \{\chi \in \mathrm{Aut}(X, T) : \pi \circ \chi = \pi\},$$

that is, elements of $\mathrm{Aut}(X, T)$ mapping every fiber of π into itself.

Definition 3.13. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. One says π is *regular* if for any point $(x_1, x_2) \in R_\pi$, there exists $\chi \in \mathrm{Aut}_\pi(X, T)$ such that $(\chi(x_1), x_2) \in P(X, T)$. It is equivalent to: for any minimal point (x_1, x_2) in R_π, there exists $\chi \in \mathrm{Aut}_\pi(X, T)$ such that $\chi(x_1) = x_2$.

The notion of regularity was introduced by Auslander [3]. Examples of regular extensions are proximal extensions, group extensions. For more information about regularity, refer to [3, 4, 15].

THEOREM 3.14. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. If π is regular and almost proximal, then there exists a flow $(Y^\#, T)$ with the following commutative diagram:*

$$\begin{tikzcd}
X \arrow[dd, "\pi"'] \arrow[dr, "\pi^\#"] & \\
& Y^\# \arrow[dl, "\tau^\#"] \\
Y &
\end{tikzcd}$$

where $\tau^\#$ is proximal and $\pi^\#$ is a finite-to-one equicontinuous extension. And π is almost finite-to-one if and only if $\tau^\#$ is almost one-to-one.

Proof. Let $y_0 \in Y$ and $u \in J_{y_0}$, that is, $uy_0 = y_0$. Since π is almost proximal, by Proposition 3.3, there exits an $n \in \mathbb{N}$ such that $|v\pi^{-1}(y)| = n$ for all $y \in Y$, $v \in J_y$. Thus, $|v\pi^{-1}(py_0)| = n$ for all $p \in \mathbf{M}$ and all $v \in J(\mathbf{M})$ with $vp = p$.

Let $\mathbf{y}_0 = u\pi^{-1}(y_0) = \{x_1, x_2, \ldots, x_n\}$. As $\{x_1, x_2, \ldots, x_n\}$ is an almost periodic set, $(x_1, \ldots, x_n)$ is a minimal point of (X^n, T), and it follows that for all $p \in \mathbf{M}$, $|p\mathbf{y}_0| = |\{px_1, px_2, \ldots, px_n\}| = n$. Let

$$Y^\# = \{p\mathbf{y}_0 : p \in \mathbf{M}\}.$$

It is clear that $X = \bigcup_{\mathbf{y} \in Y^\#} \mathbf{y}$. We show that $Y^\#$ is a partition of X, that is, for all $p, q \in \mathbf{M}$, either $p\mathbf{y}_0 = q\mathbf{y}_0$ or $p\mathbf{y}_0 \cap q\mathbf{y}_0 = \emptyset$.

Assume that there are $p, q \in \mathbf{M}$ such that $p\mathbf{y}_0 = \{px_1, \ldots, px_n\} \neq \{qx_1, \ldots, qx_n\} = q\mathbf{y}_0$ and $p\mathbf{y}_0 \cap q\mathbf{y}_0 \neq \emptyset$. Without loss of generality, we assume that $qx_1 = px_j \in p\mathbf{y}_0$ for some $j \in \{1, 2, \ldots, n\}$ and $qx_2 \notin p\mathbf{y}_0$.

Since $(x_1, \ldots, x_n)$ is a minimal point of (X^n, T), $(qx_1, \ldots, qx_n) = q(x_1, \ldots, x_n)$ is also a minimal point of (X^n, T). In particular, $(qx_1, qx_2) \in R_\pi$ is a minimal point. As π is regular, there is some $\chi \in \mathrm{Aut}_\pi(X, T)$ such that $qx_2 = \chi(qx_1)$. Let $v \in J(\mathbf{M})$ such that $vp = p$ (Proposition 2.2). Then,

$$qx_2 = \chi(qx_1) = \chi(px_j) = \chi(vpx_j) = v\chi(px_j) = v\chi(qx_1) = vqx_2.$$

Thus, $qx_2 \in v\pi^{-1}(py_0)$ and $v\{px_1, px_2, \ldots, px_n, qx_2\} = \{px_1, px_2, \ldots, px_n, qx_2\}$. It follows that $\{px_1, px_2, \ldots, px_n, qx_2\}$ is an almost periodic subset of $\pi^{-1}(py_0)$, whose cardinality is $n + 1$. In particular, $|v\pi^{-1}(py_0)| \geq n + 1$, which contradicts with the fact $|v\pi^{-1}(py_0)| = n$. Thus, $Y^\#$ is a partition of X.

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Since $Y^{\#}$ is a partition of X, it induces a map

$$\pi^{\#} : X \to Y^{\#},\ x \mapsto p\mathbf{y}_0,\ (x \in p\mathbf{y}_0),\ p \in \mathbf{M}.$$

It is easy to check that π is open, and by Lemma 3.11, it is an n-to-1 equicontinuous extension.

Let

$$\tau^{\#} : Y^{\#} \to Y,\ p\mathbf{y_0} \mapsto py_0 \quad \text{for all } p \in \mathbf{M}.$$

We show that it is proximal. Let $\tau^{\#}(p\mathbf{y}_0) = \tau^{\#}(q\mathbf{y}_0)$ for some $p, q \in \mathbf{M}$. Then, $py_0 = qy_0$. There are minimal idempotents $u_1, u_2 \in J(\mathbf{M})$ such that

$$p = u_1 p, q = u_2 q.$$

Since $|\mathbf{y}_0| < \infty$, we have

$$p\mathbf{y}_0 = u_1 p\mathbf{y}_0 = u_1 pu\pi^{-1}(y_0) = u_1\pi^{-1}(py_0).$$

Similarly, one has $q\mathbf{y}_0 = u_2\pi^{-1}(qy_0)$. Then,

$$u_1 q\mathbf{y}_0 = u_1 u_2\pi^{-1}(qy_0) = u_1\pi^{-1}(py_0) = p\mathbf{y}_0.$$

So $p\mathbf{y}_0$ and $q\mathbf{y}_0$ are proximal. That is, $\tau^{\#}$ is a proximal extension. $\square$

3.4. *Relations between almost proximal extensions and almost one-to-one extensions.* It is clear that any almost finite-to-one extension is an almost proximal extension. In this subsection, we show that if, in addition, the extension is point distal, then the converse holds.

Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. A point $x \in X$ is called a *π-distal point* whenever $P_\pi[x] \triangleq \{x' \in \pi^{-1}(\pi(x)) : (x, x') \in P(X, T)\} = \{x\}$, that is, (x, x') is a distal pair for every $x' \in \pi^{-1}(\pi(x)) \setminus \{x\}$. The extension π is said to be *point distal* when there exists a π-distal point in X. Note that π is distal if and only if every point is π-distal. It is easy to see that a point x is π-distal if and only if $ux = x$ for all $u \in J_{\pi(x)}$ [8, Ch. VI(4.3)].

THEOREM 3.15. [28, Theorem 2.3.5] *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then RIC-diagram and O-diagram coincide if and only if there is some $y \in Y$ such that $\bigcap_{u\in J_y} u \circ u\pi^{-1}(y) \neq \emptyset$.*

THEOREM 3.16. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then the following are equivalent:*

(1) *π is almost proximal and point distal;*
(2) *π is almost finite-to-one.*

Proof. If π is almost finite-to-one, it is almost proximal. By Proposition 3.8, there exist $N \in \mathbb{N}$ and $y_0 \in Y$ such that $|\pi^{-1}(y_0)| = N$ and $\pi^{-1}(y_0)$ is a minimal point of $(2^X, T)$. Each point in $\pi^{-1}(y_0)$ is π-distal.

Conversely, assume that π is almost proximal and point distal. Let $x_0 \in X$ be a π-distal point and $y_0 = \pi(x_0)$. Since x_0 is π-distal, $ux_0 = x_0$ for all $u \in J_{y_0}$.

Thus, $x_0 \in \bigcap_{u \in J_{y_0}} u \circ u\pi^{-1}(y_0)$. By Theorem 3.15, the RIC-diagram and O-diagram of π coincide. That is, in the RIC-diagram,

$$\begin{tikzcd} X & X' \arrow[l, "\sigma'"'] \arrow[d, "\pi'"] \\ Y \arrow[from=u, "\pi"'] & Y' \arrow[l, "\tau'"] \end{tikzcd}$$

Here, σ' and τ' are almost one-to-one, and π' is a finite-to-one extension. Thus, π is almost finite-to-one. □

COROLLARY 3.17. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then π is proximal and point distal if and only if π is almost one-to-one.*

4. *Proximal extensions and weakly mixing extensions*

In this section, we will study the chaotic properties of proximal but not almost one-to-one extensions.

4.1. *Weakly mixing extensions.*

Let (X, T) be a flow. A point $x \in X$ is *recurrent* if for each neighborhood U of x, the set of return times of x to U, $N(x, U) = \{t \in T : tx \in U\}$ is infinite. A point x is recurrent if and only if there exists $p \in S_T \setminus T$ such that $px = x$, if and only if there exists an idempotent $u \in S_T \setminus T$ such that $ux = x$ [9, Lemma 5.18].

Definition 4.1. Let (X, T) be a flow.

(1) A pair $(x, y) \in X^2$ is called a *strong Li–Yorke pair* if it is proximal and it is also a recurrent point of (X^2, T).

(2) A subset S of X is called *strongly scrambled* if every pair of distinct points in S is a strong Li–Yorke pair.

(3) The flow (X, T) is said to be *strongly Li–Yorke chaotic* if it contains an uncountable strongly scrambled set.

The following result says that each non-trivial weakly mixing extension contains plenty of strongly scrambled subsets.

THEOREM 4.2. [2, Theorem 5.10] *Let (X, T), (Y, T) be a flow and $\pi : (X, T) \to (Y, T)$ an open non-trivial weakly mixing extension. Then there is a residual subset $Y_0 \subseteq Y$ such that for every point $y \in Y_0$, the set $\pi^{-1}(y)$ contains a dense uncountable strongly scrambled subset K such that*

$$K \times K \setminus \Delta_X \subseteq \mathrm{Trans}(R_\pi),$$

where $\mathrm{Trans}(R_\pi)$ *denotes the set of all transitive points in R_π, and in particular, $\pi^{-1}(y)$ is perfect, and it has the cardinality of the continuum.*

Definition 4.3. Let $\pi : (X, T) \to (Y, T)$ be an extension of flows.

(1) π is called *n-weakly mixing* ($n \geq 2$) if (R_π^n, T) is transitive.

(2) If π is n-weakly mixing for every $n \geq 2$, then π is called *totally weakly mixing*.

Note that a 2-weakly mixing extension is just weakly mixing. Totally weakly mixing extensions will present stronger chaotic properties than weakly mixing extensions, see [2, 23] for more details. For RIC extensions, weak mixing is equivalent to total weak mixing.

THEOREM 4.4. [6, Theorem 2.12], [16, Theorem 2.7] *Let $\pi : (X, T) \to (Y, T)$ be an RIC extension of minimal flows. Then the following conditions are equivalent:*

(1) *π is weakly mixing;*
(2) *π is n-weakly mixing for some $n \geq 2$;*
(3) *π is totally weakly mixing.*

In general, the fact that $\pi : (X, T) \to (Y, T)$ is weakly mixing does not imply that π is totally weakly mixing. In [16, Theorem 4.1], for (Y, T) being trivial, the author gave a weakly mixing flow (X, T) such that (X^3, T) is not transitive. By Theorem 4.4, such T is not abelian. In §6, we will show that there exists $\pi : (X, \mathbb{Z}) \to (Y, \mathbb{Z})$ which is weakly mixing but not 3-weakly mixing. In fact, we have the following example.

Example 4.5. There exists a discrete flow $(X, \mathbb{Z})$ such that $\pi : (X, \mathbb{Z}) \to (X_{\mathrm{eq}}, \mathbb{Z})$ is a proximal extension, and it is weakly mixing but not 3-weakly mixing. (See Theorem 6.12.)

4.2. *Proximal but not almost one-to-one extensions.* Let $P^{(n)}(X)$ be the subset of X^n of all points whose orbit closure intersects the diagonal of X^n. Thus, $P(X) = P^{(2)}(X)$. Glasner showed that for a minimal flow (X, T), if $P^{(n)}(X)$ is dense in X^n for every $n \geq 2$, then X is weakly disjoint from all minimal flows, that is, $(X \times Y, T)$ is transitive for all minimal flow (Y, T) [17, Ch. II, Proposition 2.1]. In particular, any minimal proximal flow is weakly mixing. This was extended by van der Woude [26, Ch. VII, Corollary 2.14] as the following theorem (see also [16, Corollary 6.4]).

THEOREM 4.6. *A non-trivial open proximal extension of minimal flows is a weakly mixing extension.*

Thus, by Theorem 4.2, we have the following theorem.

THEOREM 4.7. [2, Theorem 5.17] *Let $\pi : X \to Y$ be a proximal but not almost one-to-one extension of minimal flows. Then there is a residual subset $Y_0 \subseteq Y$ such that for each $y \in Y_0$, $\pi^{-1}(y)$ contains an uncountable strongly scrambled subset K.*

In fact, we have the following result which is slightly stronger than Theorem 4.7.

THEOREM 4.8. *Let $\pi : X \to Y$ be a proximal but not almost one-to-one extension of minimal flows. Then there is a residual subset $Y_0 \subseteq Y$ such that for each $y \in Y_0$, $\pi^{-1}(y)$ contains an uncountable strongly scrambled subset K satisfying that for any $x_1 \neq x_2 \in K$,*

$$\pi^{-1}(y) \times \pi^{-1}(y) \subseteq \overline{\mathcal{O}}((x_1, x_2), T).$$

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Proof. Let $\pi : X \to Y$ be a proximal but not almost one-to-one extension of minimal flows. We consider its O-diagram:

$$\begin{tikzcd} X & X^* \arrow[l, "\sigma"'] \arrow[d, "\pi*"] \\ Y \arrow[from=u, "\pi"'] & Y^*. \arrow[l, "\tau"] \end{tikzcd}$$

Recall the construction of an O-diagram (see Theorem 2.6):

$$Y^* = \overline{\{\pi^{-1}(y) : y \in Y_c\}} \subseteq 2^X, \quad X^* = \{(x, \mathbf{y}) \in X \times Y^* : x \in \mathbf{y}\},$$

where Y_c is the set of continuous points of $\pi^{-1} : Y \to 2^X$ and it is a residual subset of Y, X^* is the unique minimal set in $R_{\pi\tau} = \{(x, y) \in X \times Y^* : \pi(x) = \tau(y)\}$, and σ and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^*, respectively. Also $\tau : Y^* \to Y$ is an almost one-to-one extension such that $\tau^{-1}(y) = \{\pi^{-1}(y)\}$ for all $y \in Y_c$.

Since π is proximal but not almost one-to-one, π^* is open proximal but not almost one-to-one. By Theorem 4.6, π^* is weakly mixing, and hence by Theorem 4.2, there is a residual subset Y_0^* of Y^* such that for each $\mathbf{y} \in Y_0^*$, $(\pi^*)^{-1}(\mathbf{y})$ contains a dense uncountable strongly scrambled subset K^* such that

$$K^* \times K^* \setminus \Delta_{X^*} \subseteq \mathrm{Trans}(R_{\pi^*}).$$

Since τ is almost one-to-one, $\tau(Y_0^*)$ is a residual subset of Y. Let

$$Y_0 = Y_c \cap \tau(Y_0^*).$$

Then Y_0 is a residual subset of Y. We need to verify that for each $y \in Y_0$, $\pi^{-1}(y)$ contains an uncountable strongly scrambled subset K satisfying that for any $x_1 \neq x_2 \in K$,

$$\pi^{-1}(y) \times \pi^{-1}(y) \subseteq \overline{\mathcal{O}}((x_1, x_2), T).$$

Let $y \in Y_0$. Since $y \in Y_c \cap \tau(Y_0^*) \subseteq Y_c$, by the property of τ, we have that $\tau^{-1}(y) = \{\pi^{-1}(y)\}$. Thus, $\mathbf{y} = \pi^{-1}(y) \in Y_0^*$. So by the definition of Y_0^*, $(\pi^*)^{-1}(\mathbf{y})$ contains a dense uncountable strongly scrambled subset K^* such that $K^* \times K^* \setminus \Delta_{X^*} \subseteq \mathrm{Trans}(R_{\pi^*})$. Let

$$K^* = \{(x_\alpha, \mathbf{y}) : \alpha \in \Lambda, x_\alpha \in \mathbf{y}\},$$

where Λ is an uncountable index set. Now let $K = \sigma(K^*) = \{x_\alpha\}_{\alpha\in\Lambda}$. First note that

$$\pi(K) = \pi\sigma(K^*) = \tau\pi^*(K^*) = \tau(\mathbf{y}) = y,$$

and hence $K \subseteq \pi^{-1}(y)$.

Let $x_1, x_2 \in K$ with $x_1 \neq x_2$. We show that $\pi^{-1}(y) \times \pi^{-1}(y) \subseteq \overline{\mathcal{O}}((x_1, x_2), T)$. Let $y_1, y_2 \in \pi^{-1}(y) = \mathbf{y}$. Then

$$(y_1, \mathbf{y}), (y_2, \mathbf{y}) \in (\pi^*)^{-1}(\mathbf{y}).$$

Since $((x_1, \mathbf{y}), (x_2, \mathbf{y})) \in K^* \times K^* \setminus \Delta_{X^*} \subseteq \mathrm{Trans}(R_{\pi^*})$,

$$((y_1, \mathbf{y}), (y_2, \mathbf{y})) \in \overline{\mathcal{O}}(((x_1, \mathbf{y}), (x_2, \mathbf{y})), T).$$

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It follows that $(y_1, y_2) \in \overline{\mathcal{O}}((x_1, x_2), T)$. Thus, $\pi^{-1}(y) \times \pi^{-1}(y) \subseteq \overline{\mathcal{O}}((x_1, x_2), T)$. The proof is complete. □

5. *A dichotomy theorem on almost proximal extensions*

In this section, we show that any almost proximal extension has the following dichotomy theorem: any almost proximal extension of minimal flows is either almost finite-to-one, or almost all fibers contain an uncountable strongly scrambled subset.

LEMMA 5.1. *Let $\pi : (X, T) \to (Y, T)$ be a distal extension of minimal flows. For any strongly scrambled set K of Y, there is a strongly scrambled set K' of X such that $\pi|_{K'} : K' \to K$ is one-to-one.*

Proof. Let $K = \{y_\alpha\}_{\alpha\in\Lambda}$, where Λ is an index set. Fix a point $y_0 \in K$. By the assumption, $K \subseteq P[y_0]$. Then by Proposition 2.4 for each $\alpha \in \Lambda$, there is a minimal idempotent $v_\alpha \in J(\mathbf{M})$ such that $y_\alpha = v_\alpha y_0$. Thus,

$$K = \{v_\alpha y_0\}_{\alpha\in\Lambda} \subseteq P[y_0].$$

Since K is a strongly scrambled set, any pair (y_α, y_β) is a recurrent point of $(Y \times Y, T)$ and there is some idempotent $u_{\alpha\beta} \in J(S_T \setminus T)$ such that

$$u_{\alpha\beta}(y_\alpha, y_\beta) = (y_\alpha, y_\beta).$$

Now choose a point $x_0 \in \pi^{-1}(y_0)$, and let

$$K' = \{v_\alpha x_0\}_{\alpha\in\Lambda}.$$

For any $\alpha, \beta \in \Lambda$, since v_α, v_β are minimal idempotents, we have

$$v_\beta(v_\alpha x_0, v_\beta x_0) = (v_\beta v_\alpha x_0, v_\beta^2 x_0) = (v_\beta x_0, v_\beta x_0),$$

and hence $(v_\alpha x_0, v_\beta x_0) \in P(X, T)$. By $u_{\alpha\beta} y_\alpha = y_\alpha$, we have $\pi(u_{\alpha\beta} v_\alpha x_0) = u_{\alpha\beta} y_\alpha = y_\alpha = \pi(v_\alpha x_0)$. Since π is distal, it follows that $u_{\alpha\beta} v_\alpha x_0 = v_\alpha x_0$. Similarly, we have $u_{\alpha\beta} v_\beta x_0 = v_\beta x_0$. So

$$u_{\alpha\beta}(v_\alpha x_0, v_\beta x_0) = (v_\alpha x_0, v_\beta x_0).$$

That is, $(v_\alpha x_0, v_\beta x_0)$ is a recurrent point of $(X \times X, T)$. Thus, K' is a strongly scrambled subset of X. As π is distal and each pair in K' is proximal, $\pi|_{K'} : K' \to K$ is one-to-one. The proof is complete. □

THEOREM 5.2. *Let $\pi : (X, T) \to (Y, T)$ be an almost proximal extension of minimal flows. Then one of the following holds:*

(1) *π is almost finite-to-one;*

(2) *there is a residual subset $Y_0 \subseteq Y$ such that for every $y \in Y_0$, the fiber $\pi^{-1}(y)$ contains an uncountable strongly scrambled set.*

Proof. We show that if π is not almost finite-to-one, then there is a residual subset $Y_0 \subseteq Y$ such that for every $y \in Y_0$, the fiber $\pi^{-1}(y)$ contains an uncountable strongly scrambled set.

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Let $y_0 \in Y$ and $u \in J(\mathbf{M})$ such that $uy_0 = y_0$ and let $u\pi^{-1}(y_0) = \{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$. Let $z_0 = (x_1, x_2, \ldots, x_n) \in X^n$. Then z_0 is a minimal point of (X^n, T). Let

$$X^{\#} = \overline{\mathcal{O}}(z_0, T) = \{pz_0 \in X^n : p \in \mathbf{M}\} \subseteq X^n.$$

Then $(X^{\#}, T)$ is a minimal flow. Let

$$\psi : X^{\#} \to X,\ \ pz_0 \mapsto px_1,\ \ \widetilde{\pi} = \pi \circ \psi : X^{\#} \to Y,\ pz_0 \mapsto py_0, \quad \text{for all } p \in \mathbf{M}.$$

$$\begin{tikzcd} & X^{\#} \arrow[dl, "\psi"'] \arrow[dd, "\widetilde{\pi}"] \\ X \arrow[dr, "\pi"'] & \\ & Y \end{tikzcd}$$

We divide the proof into several steps.

Step 1. $\widetilde{\pi}$ is regular. Let $p_1, p_2 \in \mathbf{M}$ such that $\widetilde{\pi}(p_1z_0) = \widetilde{\pi}(p_2z_0)$ and (p_1z_0, p_2z_0) is minimal. Let $v \in J(\mathbf{M})$ such that $v(p_1z_0, p_2z_0) = (p_1z_0, p_2z_0)$. Since $\widetilde{\pi}(p_1z_0) = \widetilde{\pi}(p_2z_0)$, we have that $p_1y_0 = p_2y_0$. Hence, $up_1^{-1}p_2y_0 = y_0$, where p_1^{-1} is defined after Proposition 2.2. Now let

$$\chi : X^{\#} \to X^{\#}, \quad pz_0 \mapsto p(up_1^{-1}p_2)z_0.$$

First, we verify that χ is well defined. Let $p, q \in \mathbf{M}$ such that $pz_0 = qz_0$. Since $z_0 = (x_1, x_2, \ldots, x_n)$, it follows that $px_j = qx_j$, $1 \leq j \leq n$. By $up_1^{-1}p_2\pi^{-1}(y_0) = u\pi^{-1}(y_0)$,

$$\{up_1^{-1}p_2x_1, up_1^{-1}p_2x_2, \ldots, up_1^{-1}p_2x_n\} = \{x_1, x_2, \ldots, x_n\}.$$

Thus,

$$p(up_1^{-1}p_2)x_j = q(up_1^{-1}p_2)x_j, \quad 1 \leq j \leq n,$$

that is, $\chi(pz_0) = p(up_1^{-1}p_2)z_0 = q(up_1^{-1}p_2)z_0 = \chi(qz_0)$. That is, χ is well defined. As $\widetilde{\pi}(\chi(pz_0)) = p(up_1^{-1}p_2)y_0 = py_0 = \widetilde{\pi}(pz_0)$, $\chi \in \mathrm{Aut}_{\widetilde{\pi}}(X^{\#}, T)$.

Finally, note that

$$\chi(p_1z_0) = \chi(vp_1z_0) = vp_1(up_1^{-1}p_2)z_0 = vp_2z_0 = p_2z_0.$$

Thus, $\widetilde{\pi}$ is regular.

Step 2. $\widetilde{\pi}$ is almost proximal and not almost finite-to-one. Since π is not almost finite-to-one, it is easy to see that $\widetilde{\pi}$ is also not almost finite-to-one. Next we show it is almost proximal. We show that $|u\widetilde{\pi}^{-1}(y_0)| < \infty$. Let $pz_0 \in u\widetilde{\pi}^{-1}(y_0)$. Then, $up(x_1, x_2, \ldots, x_n) = upz_0 = pz_0 = (px_1, px_2, \ldots, px_n)$. It follows that

$$\{px_1, px_2, \ldots, px_n\} = \{x_1, x_2, \ldots, x_n\} = u\pi^{-1}(y_0).$$

Thus, $(px_1, px_2, \ldots, px_n)$ is a permutation of $(x_1, x_2, \ldots, x_n)$. Hence, $|u\widetilde{\pi}^{-1}(y_0)| \leq |S_n| < \infty$, where S_n is the symmetric group on $\{x_1, x_2, \ldots, x_n\}$. That is, $\widetilde{\pi}$ is almost proximal.

Step 3. There is a residual subset $Y_0 \subseteq Y$ such that for every $y \in Y_0$, the fiber $\pi^{-1}(y)$ contains an uncountable strongly scrambled set. Since $\widetilde{\pi}$ is regular and almost proximal, by Theorem 3.14, there is a finite-to-one equicontinuous extension $\phi : X^{\#} \to Y^{\#}$ and a proximal extension $\pi^{\#} : Y^{\#} \to Y$ such that $\widetilde{\pi} = \pi^{\#} \circ \phi$.

$$\begin{tikzcd} & X^{\#} \arrow[dl, "\psi"'] \arrow[dr, "\phi"] \arrow[dd, "\widetilde{\pi}"] & \\ X \arrow[dr, "\pi"'] & & Y^{\#} \arrow[dl, "\pi^{\#}"] \\ & Y & \end{tikzcd}$$

Since $\widetilde{\pi}$ is not almost finite-to-one, $\pi^{\#}$ is proximal but not almost one-to-one. By Theorem 4.8, there is a residual subset $Y_0 \subseteq Y$ such that for every $y \in Y_0$, the fiber $(\pi^{\#})^{-1}(y)$ contains a countable strongly scrambled set K'_y. Since ϕ is distal, by Lemma 5.1, there is a countable strongly scrambled set $K^{\#}_y$ such that $\phi(K^{\#}_y) = K'_y$. Note that $K^{\#}_y \subseteq X^{\#} \subseteq X^n$. Let $K_j = \pi_j(K^{\#}_y)$, $1 \le j \le n$, where π_j is the projection from X^n to jth coordinate. Since

$$K^{\#}_y \subseteq K_1 \times K_2 \times \cdots \times K_n,$$

there is some $j_0 \in \{1, 2, \ldots, n\}$ such that $|K_{j_0}|$ is uncountable. Since $K^{\#}_y$ is a strongly scrambled set of $X^{\#}$, K_{j_0} is a strongly scrambled set of X. As the diagram is commutative, it is easy to check that $K_{j_0} \subseteq \pi^{-1}(y)$. The proof is complete. □

6. *Examples*

Since there are no non-trivial proximal minimal flows under abelian group actions [17, Ch. II, Theorem 3.4], it is not easy to construct a proximal but not almost one-to-one extension of minimal flows. In fact, this was a question by Furstenberg several years ago. Using category method, Glasner and Weiss constructed the first of this kind of extensions of minimal flows and gave a positive answer to this question. In this section, using methods in [7], we give explicit examples of proximal but not almost one-to-one extensions. And examples constructed are uniformly rigid.

In this section, first we briefly introduce Glasner and Weiss' results, which will be used later. For Glasner and Weiss' methods, refer to [19] for more details. Then we show how to give explicit examples of almost proximal but not almost finite-to-one extensions.

6.1. *Glasner and Weiss' results.* Let (Y, f) be a minimal discrete flow and Z be a compact metric space with metric d_Z. Let $H(Z, Z)$ be the space of all homeomorphisms of Z equipped with the metric

$$\rho_Z(g, h) = \sup_{z \in Z} d_Z(g(z), h(z)) + \sup_{z \in Z} d_Z(g^{-1}(z), h^{-1}(z)).$$

With this metric, $H(Z, Z)$ is a complete metric space and a topological group. Let $X = Y \times Z$. Let $H_s(X, X)$ be the subspace of $H(X, X)$ which consists of homeomorphisms which fix all subspace of X of the form $\{y\} \times Z$, $y \in Y$. Such a homeomorphism G is

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determined by a continuous map $\alpha : Y \to H(Z, Z)$, by $G(y, z) = (y, \alpha(y)(z))$. Put

$$\mathcal{S}(f) = \{G^{-1} \circ f \circ G : G \in H_s(X, X)\}.$$

(Here, f is identified with $f \times \mathrm{id}_Z$, where id_Z is the identity map on Z.)

If $\mathcal{G}$ is a subgroup of $H(Z, Z)$, let $\mathcal{G}_s \subseteq H_s(X, X)$ be the subgroup of those elements of $H_s(X, X)$ which come from cocycles (see definition below) of $\mathcal{G}$. That is,

$$\mathcal{G}_s = \{G \in H_s(X, X) : G(y, z) = (y, \alpha(y)(z)), \alpha \in C(Y, \mathcal{G})\}.$$

Let

$$\mathcal{S}_{\mathcal{G}}(f) = \{G^{-1} \circ f \circ G : G \in \mathcal{G}_s\}.$$

THEOREM 6.1. [19, Theorem 1] *Let $\mathcal{G}$ be a pathwise connected subgroup of $H(Z, Z)$ such that $(Z, \mathcal{G})$ is a minimal flow. If (Y, f) is minimal, then for a residual subset $\mathcal{R} \subseteq \overline{\mathcal{S}_{\mathcal{G}}(f)}$, (X, F) is a minimal flow for every $F \in \mathcal{R}$.*

THEOREM 6.2. [19, Theorem 3] *Let $\mathcal{G}$ be a pathwise connected subgroup of $H(Z, Z)$ with the following property: for every pair of points $z_1, z_2 \in Z$, there exist neighborhoods U and V of z_1 and z_2, respectively, such that for every $\epsilon > 0$, there exists $h \in \mathcal{G}$ with* $\mathrm{diam}\,(h(U \cup V)) < \epsilon$. *Then for a residual subset $\mathcal{R} \subseteq \overline{\mathcal{S}_{\mathcal{G}}(f)}$, (X, F) is a proximal extension of (Y, f) for every $F \in \mathcal{R}$.*

According to [19, §1], if we choose $Z = \mathbf{P}^n$, $n \geq 1$ to be the projective n-space, and let $\mathcal{G}$ be a pathwise connected component of id_Z in $H(Z, Z)$, then $\mathcal{G}$ satisfies the conditions of Theorems 6.1 and 6.2. Thus, for an arbitrary minimal infinite flow (Y, f), there are many minimal homeomorphisms of $Y \times Z$ which are proximal but not almost one-to-one extensions of (Y, f).

6.2. *Skew-product.* Let (Y, f) be a discrete flow and (Z, d_Z) a compact metric space. Denote the set of all the homeomorphisms of Z to Z with $H(Z, Z)$. For $\varphi_1, \varphi_2 \in H(Z, Z)$, set

$$D_Z(\varphi_1, \varphi_2) = \sup_{z \in Z} d_Z(\varphi_1(z), \varphi_2(z)).$$

Assume $X = Y \times Z$ and let ρ_X denote the max-metric on the product space $Y \times Z$,

$$\rho_X((y_1, z_1), (y_2, z_2)) = \max\{d_Y(y_1, y_2), d_Z(z_1, z_2)\},$$

where d_Y is the metric of Y.

A continuous map $\sigma : Y \to H(Z, Z)$ is called a *cocycle*. By a given cocycle σ, one can define

$$f_\sigma : X \to X, (y, z) \mapsto (f(y), \sigma(y)(z)) \quad \text{for all } (y, z) \in X.$$

The new flow (X, f_σ) is called a *skew-product flow*.

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For $(y, z) \in X$, set $f_\sigma^n(y, z) = (f^n(y), \sigma_n(y)(z))$, where

$$\sigma_n(y) = \begin{cases} \sigma(f^{n-1}y)\cdots\sigma(fy)\sigma(y), & n \geq 1; \\ \mathrm{id}_Z, & n = 0; \\ \sigma(f^n y)^{-1}\cdots\sigma(f^{-2}y)^{-1}\sigma(f^{-1}y)^{-1}, & n < 0. \end{cases}$$

6.3. *A cocycle.* Let $Y = Z(2) = \{0, 1\}^{\mathbb{N}}$ with the metric

$$d(\alpha, \beta) = \frac{1}{\min\{i \in \mathbb{N} : \alpha_i \neq \beta_i\}}, \quad \alpha = (\alpha_i)_{i=1}^\infty, \quad \beta = (\beta_i)_{i=1}^\infty \in Z(2).$$

The map

$$\tau : Z(2) \to Z(2)$$

is defined as follows: for every $\alpha \in Z(2)$, $\tau(\alpha) = \alpha + 10000\ldots$, where the addition is modulo 2 from the left to right. Obviously, τ is continuous. Moreover, it can be shown that τ is invertible and $(Z(2), \tau)$ is an equicontinuous minimal flow. The flow $(Z(2), \tau)$ is called an *adding machine* or odometer. Similarly, one can define $(Z(k) = \{0, 1, \ldots, k-1\}^{\mathbb{N}}, \tau)$.

In the following, an increasing sequence $\{n_i\}_{i=1}^\infty \subseteq \mathbb{N}$ is fixed. For any $\alpha \in Z(2)$, α can be written as

$$\alpha = \alpha^1\alpha^2\alpha^3\ldots, \quad \text{where } \alpha^k \text{ is a block of } n_k \text{ digits of } \alpha. \tag{6.1}$$

Let $e : \bigcup_{j=1}^\infty \{0, 1\}^j \to \mathbb{Z}_+$ be the evaluation function: for $x = x_1x_2\ldots x_j \in \{0, 1\}^j$,

$$e(x) = x_1 + 2x_2 + 2^2x_3 + \cdots + 2^{j-1}x_j.$$

For $n \in \mathbb{Z}_+$, if $x = x_1x_2\ldots x_j \in \{0, 1\}^j$ such that $e(x) = n$, then let

$$\underline{n} = x_1x_2\ldots x_j0^\infty \in Z(2) = \{0, 1\}^{\mathbb{N}},$$

where $0^\infty = 000\ldots$. For example, $\underline{0} = 0^\infty$, $\underline{2} = 010^\infty$, $\underline{7} = 1110^\infty$.

Let $Z = \mathbb{S}^1 = \{\exp(i2\pi x) : 0 \leq x \leq 1\}$ and

$$\Phi = \{\varphi_k^j : 0 \leq j \leq 2^{n_k} - 2\}_{k=1}^\infty \subseteq H(\mathbb{S}^1, \mathbb{S}^1). \tag{6.2}$$

We use Φ to construct a cocycle $\sigma : Z(2) \to H(\mathbb{S}^1, \mathbb{S}^1)$:

$$\sigma(\alpha) = \begin{cases} \varphi_k^{e(\alpha^k)}, & \alpha \neq 1^\infty \text{ and } \alpha^k \text{ is the first block in (6.1) containing at least one zero} \\ & \text{digit;} \\ \mathrm{id}_{\mathbb{S}^1}, & \alpha = 1^\infty. \end{cases} \tag{6.3}$$

LEMMA 6.3. *If Φ satisfies the following condition:*

$$\lim_{k\to\infty} \max_{0 \leq j \leq 2^{n_k}-2} D_{\mathbb{S}^1}(\varphi_k^j, \mathrm{id}_{\mathbb{S}^1}) = 0, \tag{C1}$$

then $\sigma : Z(2) \to H(\mathbb{S}^1, \mathbb{S}^1)$ is continuous and hence it is a cocycle.

Proof. Let $\beta_n, \alpha \in Z(2)$ such that $\beta_n \to \alpha, n \to \infty$. We show that $\lim_{n\to\infty} D_{\mathbb{S}^1}(\sigma(\beta_n), \sigma(\alpha)) = 0$.

If $\alpha \neq 1^\infty$, then one has $\sigma(\beta_n) = \sigma(\alpha)$ when n is large enough and $D_{\mathbb{S}^1}(\sigma(\beta_n), \sigma(\alpha)) = 0$. Now assume that $\alpha = 1^\infty$. For β_n, we have that $\beta_n = \beta_n^1\beta_n^2\beta_n^3\ldots$, where β_n^k is a block of n_k digits of β_n. Thus, $\sigma(\beta_n) = \varphi_k^{e(\beta_n^k)}$, where β_n^k is the first block containing at least one zero digit. Since $\beta_n \to \alpha, n \to \infty$, and $\lim_{k\to\infty} \max_{0\le j\le 2^{n_k}-2} D_{\mathbb{S}^1}(\varphi_k^j, \mathrm{id}_{\mathbb{S}^1}) = 0$, we have that

$$\lim_{n\to\infty} D_{\mathbb{S}^1}(\sigma(\beta_n), \mathrm{id}_{\mathbb{S}^1}) = 0.$$

That is, σ is continuous. □

Thus, when (C1) is satisfied, σ is a cocycle. Hence, we have a skew product flow $(X = Z(2) \times \mathbb{S}^1, F_\Phi)$:

$$F_\Phi \triangleq \tau_\sigma : X \to X,\ F_\Phi(\alpha, y) = (\tau(\alpha), \sigma(\alpha)y) = \begin{cases} (\tau(\alpha), \varphi_k^{e(\alpha^k)}(y)), & \alpha \neq 1^\infty; \\ (\underline{0}, y), & \alpha = 1^\infty, \end{cases} \tag{6.4}$$

where α^k is the first block in (6.1) containing at least one zero digit when $\alpha \neq 1^\infty$.

6.4. *The form of F_Φ^n.* Let $\alpha = \alpha^1\alpha^2\ldots \in Z(2)$ as in (6.1) and $z_0 \in \mathbb{S}^1$. Let

$$F_\Phi^n(\alpha, z_0) = (\tau^n\alpha, z_n).$$

We need to know the formula of z_n for some special n.

For simplicity, we always assume the following conditions hold:

$$\varphi_k^0 = \mathrm{id}_{\mathbb{S}^1} \quad \text{for all } k \in \mathbb{N}, \tag{C2}$$

and

$$\Psi_k = \varphi_k^{2^{n_k}-2} \circ \varphi_k^{2^{n_k}-3} \circ \cdots \circ \varphi_k^1 \circ \varphi_k^0 = \mathrm{id}_{\mathbb{S}^1} \quad \text{for all } k \in \mathbb{N}. \tag{C3}$$

First we have a formula for $\alpha = \underline{0}$. By an easy induction, we have the following lemma.

Lemma 6.4. *Let $(\underline{0}, y_0) \in Z(2) \times \mathbb{S}^1$ and $(\underline{n}, y_n) = F_\Phi^n(\underline{0}, y_0)$. Let $m_k = 2^{n_1+n_2+\cdots+n_k}$ for all $k \in \mathbb{N}$. Then*

$$y_{m_k} = \varphi_{k+1}^0 \circ \psi_k(y_0) = y_0, \tag{6.5}$$

and

$$y_{s\cdot m_k} = \varphi_{k+1}^{s-1} \circ \varphi_{k+1}^{s-2} \circ \cdots \circ \varphi_{k+1}^1 \circ \varphi_{k+1}^0(y_0), \quad 1 \le s < 2^{n_{k+1}}. \tag{6.6}$$

Now we see the general case. Let $\alpha = \alpha^1\alpha^2\ldots \in Z(2)$ as in (6.1) and $z_0 \in \mathbb{S}^1$. We denote $F_\Phi^n(\alpha, z_0) = (\tau^n\alpha, z_n)$. Let

$$\psi_k^+ = \varphi_k^{e(\alpha^k)-1} \circ \varphi_k^{e(\alpha^k)-2} \circ \cdots \circ \varphi_k^1 \circ \varphi_k^0 \quad \text{for all } k \in \mathbb{N}, \tag{6.7}$$

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and

$$\psi_k^- = \varphi_k^{2^{n_k}-2} \circ \varphi_k^{2^{n_k}-3} \circ \cdots \circ \varphi_k^{e(\alpha^k)+1} \circ \varphi_k^{e(\alpha^k)} \quad \text{for all } k \in \mathbb{N}. \tag{6.8}$$

By the assumption (C3), one has that $(\psi_k^+)^{-1} = \psi_k^-$.

By an easy induction, we have the following lemma.

LEMMA 6.5. *Let $\alpha = \alpha^1\alpha^2 \ldots \in Z(2)$ as in (6.1) and $z_0 \in \mathbb{S}^1$. Let $F_\Phi^n(\alpha, z_0) = (\tau^n\alpha, z_n)$. Let $m_k = 2^{n_1+n_2+\cdots+n_k}$ for all $k \in \mathbb{N}$. Then*

$$F_\Phi^{m_k - e(\alpha^1\alpha^2\ldots\alpha^k)}(\alpha, z_0) = (0^{n_1+\cdots+n_k}\tau(\alpha^{k+1}\alpha^{k+2}\cdots), z_{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)}), \tag{6.9}$$

where $z_{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)} = \varphi_{k+1}^{e(\alpha^{k+1})} \circ \psi_k^- \circ \psi_{k-1}^- \circ \cdots \circ \psi_1^-(z_0)$. And

$$F_\Phi^{m_k}(\alpha, z_0) = (\alpha^1\alpha^2 \ldots \alpha^k\tau(\alpha^{k+1}\alpha^{k+2}\ldots), z_{m_k}), \tag{6.10}$$

where $z_{m_k} = \psi_1^+ \circ \psi_2^+ \circ \cdots \circ \psi_k^+ \circ \varphi_{k+1}^{e(\alpha^{k+1})} \circ \psi_k^- \circ \cdots \circ \psi_1^-(z_0)$.

Remark 6.6. (i) To simplify the calculation, if we require that

$$\varphi_k^1 = \varphi_k^2 = \cdots = \varphi_k^{2^{n_k-1}-1} \triangleq \varphi_k,$$
$$\varphi_k^{2^{n_k-1}} = \varphi_k^{2^{n_k-1}+1} = \cdots = \varphi^{2^{n_k}-2} \triangleq \varphi_k^- = (\varphi_k)^{-1},$$

then $\psi_k^+ = (\varphi_k)^{c_k}$, $\psi_k^- = (\varphi_k^-)^{c_k}$, where $c_k = 2^{n_k-1} - 1 - |e(\alpha^k) - 2^{n_k-1}|$. Thus, (6.10) will be

$$z_{m_k} = (\varphi_1)^{c_1} \circ (\varphi_2)^{c_2} \circ \cdots \circ (\varphi_k)^{c_k} \circ \varphi_{k+1}^{e(\alpha^{k+1})} \circ (\varphi_k^-)^{c_k} \circ \cdots \circ (\varphi_1^-)^{c_1}(z_0). \tag{6.11}$$

(ii) By (6.6), (6.9), for $j < k$, $1 \leq s < 2^{n_{j+1}}$, we have

$$F_\Phi^{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)+s\cdot m_j}(\alpha, z_0) = (\underline{s}\tau(\alpha^{k+1}\alpha^{k+2}\ldots), z_{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)+s\cdot m_j}), \tag{6.12}$$

where $z_{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)+s\cdot m_j} = \varphi_{j+1}^{s-1} \circ \varphi_{j+1}^{s-2} \circ \cdots \circ \varphi_{j+1}^1 \circ \varphi_{j+1}^0(z_{m_k-e(\alpha^1\alpha^2\ldots\alpha^k)})$.

6.5. *Uniform rigidity.* A discrete flow (X, F) is *rigid* if there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $F^{n_i}x$ converges to x as i goes to infinity for every $x \in X$ (that is, F^{n_i} converges pointwisely to the identity map). A flow (X, F) is *uniformly rigid* if there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $\sup_{x\in X} d(x, F^{n_i}x) \to 0$ as i goes to infinity (that is, F^{n_i} converges uniformly to the identity map). Refer to [**18**] for more information about topological rigidity.

LEMMA 6.7. *Let $\mathcal{F}$ be a finite subset of $H(X, X)$, where (X, ρ_X) is a compact metric space. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any continuous map $h : X \to X$ with $D_X(h, \mathrm{id}_X) < \delta$, we have*

$$D_X(\psi \circ h \circ \psi^{-1}, \mathrm{id}_X) < \epsilon \quad \textit{for all } \psi \in \mathcal{F}.$$

Proof. For any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\rho_X(x_1, x_2) < \delta$,

$$\rho_X(\psi^{-1}(x_1), \quad \psi^{-1}(x_2)) < \epsilon \quad \text{for all } \psi \in \mathcal{F}.$$

Then for continuous map $h : X \to X$ with $D(h, \mathrm{id}_X) < \delta$, we have that

$$\rho_X(h(\psi(x)), \quad \psi(x)) < \delta \quad \text{for all } x \in X, \text{ for all } \psi \in \mathcal{F}.$$

It follows that

$$\rho_X(\psi^{-1} \circ h \circ \psi(x), x) < \epsilon \quad \text{for all } x \in X, \text{ for all } \psi \in \mathcal{F}.$$

That is,

$$D_X(\psi \circ h \circ \psi^{-1}, \mathrm{id}_X) < \epsilon \quad \text{for all } \psi \in \mathcal{F}.$$

The proof is complete. □

PROPOSITION 6.8. *Suppose that Φ satisfies (C1), (C2), and (C3). Then the skew product flow $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is uniformly rigid.*

Proof. By (6.10), for any $(\alpha, z_0) \in X = Z(2) \times \mathbb{S}^1$, we get

$$\begin{aligned} &F_\Phi^{m_k}(\alpha, z_0) \\ &\quad = (\alpha^1\alpha^2 \ldots \alpha^k \tau(\alpha^{k+1}\alpha^{k+2} \ldots), \psi(\alpha^1\alpha^2 \ldots \alpha^k)\varphi_{k+1}^{e(\alpha^{k+1})}\psi^{-1}(\alpha^1\alpha^2 \ldots \alpha^k)(z_0)), \end{aligned}$$

where $\psi(\alpha^1\alpha^2 \ldots \alpha^k) = \psi_1^+ \circ \psi_2^+ \circ \cdots \circ \psi_k^+$. By the assumption (C1) that

$$\lim_{k\to\infty} \max_{0 \le j \le 2^{n_k}-2} D_{\mathbb{S}^1}(\varphi_k^j, \mathrm{id}_{\mathbb{S}^1}) = 0,$$

and Lemma 6.7, we have that

$$\lim_{k\to\infty} D_X(F_\Phi^{m_k}, \mathrm{id}_X) = 0,$$

that is, $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is uniformly rigid. □

6.6. *Minimality.* Now we specify the homeomorphisms $\{\varphi_{2k-1}^j : 1 \le j \le 2^{n_{2k-1}} - 2\}_{k=1}^\infty$ to make the flow $(Z(2) \times \mathbb{S}^1, F_\Phi)$ minimal. Let

$$\varphi_{2k-1}^j(z) = \begin{cases} \exp\,(i\theta_k) \cdot z, & 1 \le j \le 2^{n_{2k-1}-1} - 1; \\ \exp\,(-i\theta_k) \cdot z, & 2^{n_{2k-1}-1} - 1 < j \le 2^{n_{2k-1}} - 2, \end{cases} \tag{C4}$$

where $\theta_k = 2\pi/(2^{n_{2k-1}-1} - 1)$. That is, φ_{2k-1}^j is a rotation of $\mathbb{S}^1$ with angle θ_k in the anti-clockwise direction if $1 \le j \le 2^{n_{2k-1}-1} - 1$ and in the opposite direction otherwise. We remark that this construction satisfies the assumption (C3), that is, $\psi_{2k-1} = \mathrm{id}$.

PROPOSITION 6.9. *Suppose that Φ satisfies (C1), (C2), (C3), and (C4). Then the skew product flow $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is uniformly rigid and minimal.*

Proof. Let $\alpha \in Z(2)$ and $\mathbb{S}_\alpha^1 = \alpha \times \mathbb{S}^1$. First we show that for any $y_0 \in \mathbb{S}^1$,

$$\overline{\mathcal{O}}((\underline{0}, y_0), F_\Phi) = Z(2) \times \mathbb{S}^1.$$

By (6.6), for $1 \le s < 2^{n_{2k-1}}$, one has that

$$F_\Phi^{s \cdot m_{2k-2}}(\underline{0}, y_0) = (0^{n_1+n_2+\cdots n_{2k-2}}\underline{s}0^\infty, y_{s \cdot m_{2k-2}}),$$

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where $y_{s\cdot m_{2k-2}} = \varphi_{2k-1}^{s-1} \circ \varphi_{2k-1}^{s-2} \circ \cdots \circ \varphi_{2k-1}^{1} \circ \varphi_{2k-1}^{0}(y_0)$. As a result, $\{y_{s\cdot m_{2k-2}} : 1 \leq s \leq 2^{n_{2k-1}-1} - 1\}$ is $\theta_k = 2\pi/(2^{n_{2k-1}-1} - 1)-$dense in $\mathbb{S}^1$. (Recall the definition of ϵ-dense subsets: if S is a subset of a metric space, then B in S is ϵ-dense in S for given $\epsilon > 0$ if for any $s \in S$, there is $b \in B$ such that the distance between s and b is $< \epsilon$.) Moreover, for any $y \in \mathbb{S}^1$, there exists $1 \leq s_k \leq 2^{n_{2k-1}-1} - 1$ such that

$$\lim_{k\to\infty} y_{s_k\cdot m_{2k-2}} = y.$$

Thus,

$$F_\Phi^{s_k\cdot m_{2k-2}}(\underline{0}, y_0) \to (\underline{0}, y), k \to \infty.$$

Thus, $\mathbb{S}^1_{\underline{0}} \subseteq \overline{\mathcal{O}}((\underline{0}, y_0), F_\Phi)$. Since $F_\Phi^n(\mathbb{S}^1_{\underline{0}}) = \mathbb{S}^1_{\underline{n}}$, we have

$$Z(2) \times \mathbb{S}^1 = \overline{\bigcup_{n\in\mathbb{Z}_+} \mathbb{S}^1_{\underline{n}}} \subseteq \overline{\mathcal{O}}((\underline{0}, y_0), F_\Phi).$$

Thus,

$$\overline{\mathcal{O}}((\underline{0}, y_0), F_\Phi) = Z(2) \times \mathbb{S}^1 \text{ for all } y_0 \in \mathbb{S}^1.$$

Now we show that for any $(\alpha, z_0) \in Z(2) \times \mathbb{S}^1$, one has

$$\overline{\mathcal{O}}((\alpha, z_0), F_\Phi) = Z(2) \times \mathbb{S}^1.$$

As $(Z(2), \tau)$ is minimal, there exists some $\{p_k\}$ such that

$$\lim_{k\to\infty} \tau^{p_k}(\alpha) = \underline{0}.$$

Without loss of generality, we may assume that

$$\lim_{k\to\infty} F_\Phi^{p_k}(\alpha, z_0) = (\underline{0}, y_0)$$

for some $y_0 \in \mathbb{S}^1$. Thus, $Z(2) \times \mathbb{S}^1 = \overline{\mathcal{O}}((\underline{0}, y_0), F_\Phi) \subseteq \overline{\mathcal{O}}((\alpha, z_0), F_\Phi)$. Thus,

$$\mathcal{O}((\alpha, z_0), F_\Phi) = Z(2) \times \mathbb{S}^1.$$

Hence, $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is minimal. Moreover, by Proposition 6.8, it is uniformly rigid. The proof is complete. □

6.7. *Proximality.* In this subsection, we specify the homeomorphisms $\{\varphi_{2k}^j : 1 \leq j \leq 2^{n_{2k}} - 2\}_{k=1}^\infty$ to make the extension $\pi : (Z(2) \times \mathbb{S}^1, F_\Phi) \to (Z(2), \tau)$, $(\alpha, z_0) \mapsto \alpha$ to be proximal. Recall that $\mathbb{S}^1 = \{\exp(i2\pi x) : 0 \leq x < 1\}$. Let $\varphi_{2k}^j : \mathbb{S}^1 \to \mathbb{S}^1$ be defined as follows:

$$\varphi_{2k}^j(\exp(i2\pi x)) = \begin{cases} \exp(i2\pi x^{t_k}), & 1 < j \leq 2^{n_{2k}-1} - 1; \\ \exp(i2\pi x^{1/t_k}), & 2^{n_{2k}-1} - 1 < j \leq 2^{n_{2k}} - 2, \end{cases} \tag{C5}$$

where $\{t_k\}_{k\in\mathbb{N}}$ is a decreasing sequence such that $t_k > 1$, $\lim_{k\to\infty} t_k = 1$ and $\lim_{k\to\infty} t_k^{2^{n_{2k}}} = +\infty$. For example, $\{t_k = 3^{2^{-n_{2k}/2}}\}_{k\in\mathbb{N}}$ satisfies the condition. Notice that the construction of φ_{2k}^j also satisfies the convention (C3), that is, $\psi_{2k} = \text{id}$.

To verify that π is proximal, we begin with the following lemma.

LEMMA 6.10. *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. If there exists $y_0 \in Y$ such that for any $x_1, x_2 \in \pi^{-1}(y_0)$, $(x_1, x_2) \in P(X, T)$, then π is proximal.*

Proof. Let $(x_1, x_2) \in R_\pi$ and $\pi(x_1) = \pi(x_2) = y_1$. Since (Y, T) is minimal, we can find $p \in E(X, T)$ such that

$$py_1 = y_0.$$

So $px_1, px_2 \in \pi^{-1}(y_0)$. According to the assumption, there exists $q \in E(X, T)$ such that

$$qpx_1 = qpx_2.$$

It follows that

$$(x_1, x_2) \in P(X, T).$$

Thus, π is proximal. □

PROPOSITION 6.11. *Suppose that Φ satisfies (C2), (C4), and (C5). Then the extension $\pi : (Z(2) \times \mathbb{S}^1, F_\Phi) \to (Z(2), \tau)$, $(\alpha, z) \mapsto \alpha$ is a proximal extension of uniformly rigid minimal flows.*

Proof. First notice that (C4) and (C5) imply (C1) and (C3). It follows that $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is uniformly rigid and minimal by Proposition 6.9.

By Lemma 6.10, it suffices to show that for arbitrary $z_0^{(1)}, z_0^{(2)} \in \mathbb{S}^1$, one has

$$((\underline{0}, z_0^{(1)}), (\underline{0}, z_0^{(2)})) \in P(Z(2) \times \mathbb{S}^1, F_\Phi).$$

Let $z_0^{(1)} = \exp(i2\pi x_1)$, $z_0^{(2)} = \exp(i2\pi x_2)$, where $0 \le x_1, x_2 < 1$. Let $d_{\mathbb{S}^1}$ be the metric of $\mathbb{S}^1$. Let $F_\Phi^m((\underline{0}, z_0^{(j)})) = (\underline{0}, z_m^{(j)})$ for $j = 1, 2$ and $m \in \mathbb{N}$. Let $m_k = 2^{n_1+n_2+\cdots+n_k}$ for all $k \in \mathbb{N}$. Then by Lemma 6.4,

$$\lim_{k\to\infty} d_{\mathbb{S}^1}(z^{(1)}_{2^{n_{2k}-1}\cdot m_{2k-1}}, z^{(2)}_{2^{n_{2k}-1}\cdot m_{2k-1}}) = \lim_{k\to\infty} d_{\mathbb{S}^1}(\exp(i2\pi x_1^{t_k^{(2^{n_{2k}-1}-1)}}), \exp(i2\pi x_2^{t_k^{(2^{n_{2k}-1}-1)}})).$$

Since $\lim_{k\to\infty} t_k^{2^{n_{2k}}} = +\infty$ and $0 \le x_j < 1$, $j = 1, 2$, we have that $\lim_{k\to\infty} x_j^{t_k^{(2^{n_{2k}-1}-1)}} = 0$ for $j = 1, 2$. Thus,

$$\begin{aligned}
&\lim_{k\to\infty} d_{\mathbb{S}^1}(z^{(1)}_{2^{n_{2k}-1}\cdot m_{2k-1}}, z^{(2)}_{2^{n_{2k}-1}\cdot m_{2k-1}}) \\
&\quad = \lim_{k\to\infty} d_{\mathbb{S}^1}(\exp(i2\pi x_1^{t_k^{(2^{n_{2k}-1}-1)}}), \exp(i2\pi x_2^{t_k^{(2^{n_{2k}-1}-1)}})) \\
&\quad = d_{\mathbb{S}^1}(1, 1) = 0.
\end{aligned}$$

In particular,

$$\lim_{k\to\infty} \rho_X((F_\Phi \times F_\Phi)^{2^{n_{2k}-1}\cdot m_{2k-1}}((\underline{0}, z_0^{(1)}), (\underline{0}, z_0^{(2)}))) = 0.$$

It follows that $((\underline{0}, z_0^{(1)}), (\underline{0}, z_0^{(2)})) \in P(Z(2) \times \mathbb{S}^1, F_\Phi)$. □

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6.8. *Proximal extensions and n-weakly mixing extensions.* In §4.1, we mentioned that the fact that $\pi : (X, T) \to (Y, T)$ is weakly mixing does not imply that π is totally weakly mixing. In this subsection, we give such examples. The main result of this subsection is the following theorem.

THEOREM 6.12

(1) *There are proximal extensions of discrete minimal flows which are weakly mixing but not* 3*-weakly mixing.*
(2) *There are proximal extensions of discrete minimal flows which are totally weakly mixing.*

Proof of (1) of Theorem 6.12. Suppose that Φ satisfies (C2), (C4), and (C5). We show that the extension

$$\pi : (Z(2) \times \mathbb{S}^1, F_\Phi) \to (Z(2), \tau), \ (\alpha, z) \mapsto \alpha$$

is weakly mixing but not 3-weakly mixing. Since π is open proximal by Proposition 6.11, it is weakly mixing by Theorem 4.6. Now we show that π is not 3-weakly mixing, that is, $(R_\pi^3, F_\Phi^{(3)})$ is not transitive.

We may regard R_π^3 as $Z(2) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Suppose $(R_\pi^3, F_\Phi^{(3)})$ is transitive, and let $(\alpha, x_1, x_2, x_3) \in Z(2) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be a transitive point. Let y_1, y_2, y_3 be distinct points of $\mathbb{S}^1$ and $\beta \in Z(2)$. Then (β, y_1, y_2, y_3) is in the orbit closure of the transitive point (α, x_1, x_2, x_3). By the construction of Φ ((C4) and (C5)), F_Φ preserves orientation, that is, for all $\alpha \in Z(2)$, F_Φ maps $\{\alpha\} \times \mathbb{S}^1$ to $\{\tau\alpha\} \times \mathbb{S}^1$ such that it sends the unit circle with the anti-clockwise orientation into the unit circle with the anti-clockwise orientation. It follows that (β, y_1, y_3, y_2) can not be in the orbit closure of the transitive point (α, x_1, x_2, x_3). Thus, $(R_\pi^3, F_\Phi^{(3)})$ is not transitive. □

For the proof of (2) of Theorem 6.12, we need some preparations. We use the notation in §6.1.

THEOREM 6.13. [19, Theorem 4] *Let $\mathcal{G}$ be a pathwise connected subgroup of $H(Z, Z)$ such that $(Z, \mathcal{G})$ is a weakly mixing flow. Then for a residual subset $\mathcal{R} \subseteq \overline{\mathcal{S}_{\mathcal{G}}(f)}$, (X, F) is a weakly mixing extension of (Y, f) for every $F \in \mathcal{R}$.*

By the same proof of [19, Theorem 4], we can show the following: if $\mathcal{G}$ is a pathwise connected subgroup of $H(Z, Z)$ such that $(Z^n, \mathcal{G})$ is a transitive flow, then for a residual subset $\mathcal{R}_n \subseteq \overline{\mathcal{S}_{\mathcal{G}}(f)}$, (X, F) is an n-weakly mixing extension of (Y, f) for every $F \in \mathcal{R}_n$. Let $\mathcal{R} = \bigcap_{n=2}^{\infty} \mathcal{R}_n$. Then we have the following result which slightly generalizes Theorem 6.13.

THEOREM 6.14. *Let $\mathcal{G}$ be a pathwise connected subgroup of $H(Z, Z)$ such that $(Z, \mathcal{G})$ is a flow such that $(Z^n, \mathcal{G})$ is transitive for all $n \geq 2$. Then for a residual subset $\mathcal{R} \subseteq \overline{\mathcal{S}_{\mathcal{G}}(f)}$, (X, F) is a totally weakly mixing extension of (Y, f) for every $F \in \mathcal{R}$.*

According to [19], if we choose $Z = \mathbf{P}^n$, $n \geq 2$ to be the projective n-space, and let $\mathcal{G}$ be a pathwise connected component of id_Z in $H(Z, Z)$, then $\mathcal{G}$ satisfies the conditions of

Theorems 6.1, 6.2, and 6.14. Thus, for an arbitrary minimal infinite flow (Y, f), there are many minimal homeomorphisms of $Y \times Z$ which are proximal and totally weakly mixing extensions of (Y, f). And we have proved (2) of Theorem 6.12.

Remark 6.15. Note that the examples in Theorem 6.12(1) are uniformly rigid by Proposition 6.11. By the proof of Proposition 6.5 of [**18**], one may also require that the examples in Theorem 6.12(2) are uniformly rigid.

6.9. *Almost proximal.* In this subsection, we modify the construction of homeomorphisms $\{\varphi_{2k}^j : 1 \le j \le 2^{n_{2k}} - 2\}_{k=1}^{\infty}$ to make the extension

$$\pi : (Z(2) \times \mathbb{S}^1, F_\Phi) \to (Z(2), \tau), \quad (\alpha, z_0) \mapsto \alpha$$

almost proximal.

Let $n \in \mathbb{N}$ be a fixed number. Let $\omega = \exp(i(2\pi/n))$. We always write intervals on $\mathbb{S}^1$ anti-clockwise, so $[z_1, z_2]$ denotes the anti-clockwise closed interval beginning at z_1 and ending at z_2:

$$S_q = [\omega^{q-1}, \omega^q] = \bigg[\exp\bigg(i\frac{2\pi}{n}(q-1)\bigg), \exp\bigg(i\frac{2\pi}{n}q\bigg)\bigg], \quad q \in \{1, 2, \ldots, n\}.$$

Then

$$\mathbb{S}^1 = S_1 \cup S_2 \cup \cdots \cup S_n.$$

Thus, $\mathbb{S}^1$ is divided into n closed intervals equally. For each $q \in \{1, 2, \ldots, n\}$, we may regard S_q as $[0, 1]$ (via the map $S_q = [\exp(i(2\pi/n)(q-1)), \exp(i(2\pi/n)q)] \to [0, 1], \exp(i2\pi x) \mapsto nx - (q-1)$, where $((q-1)/n) \le x \le q/n$) and let the map $\varphi_{2k}^j|_{S_q} : S_q \to S_q$ be isomorphic to $g_k(x) = x^{t_k} : [0, 1] \to [0, 1]$ when $1 \le j \le 2^{n_{2k}-1} - 1$; and $g_k^{-1}(x) = x^{1/t_k} : [0, 1] \to [0, 1]$ when $2^{n_{2k}-1} - 1 < j \le 2^{n_{2k}} - 2$, where $\{t_k\}_{k\in\mathbb{N}}$ is a decreasing sequence tending to 1 and $\lim_{k\to\infty} t_k^{2^{n_{2k}}} = +\infty$.

To be precise, for each $q \in \{1, 2, \ldots, n\}$, when $z = \exp(i2\pi x) \in S_q$, $(q-1)/n \le x \le q/n$, $\varphi_{2k}^j : S_q \to S_q$ is defined as follows:

$$\begin{aligned}\varphi_{2k}^j(z) &= \varphi_{2k}^j(\exp(i2\pi x))\\ &= \begin{cases} \exp\bigg(i2\pi\dfrac{(nx-(q-1))^{t_k} + (q-1)}{n}\bigg), & 1 \le j \le 2^{n_{2k}-1} - 1;\\ \exp\bigg(i2\pi\dfrac{(nx-(q-1))^{1/t_k} + (q-1)}{n}\bigg), & 2^{n_{2k}-1} - 1 < j \le 2^{n_{2k}} - 2,\end{cases}\end{aligned} \tag{C6}$$

where $\{t_k\}$ is a decreasing sequence tending to 1 and $\lim_{k\to\infty} t_k^{2^{n_{2k}}} = +\infty$. Thus, for each $q \in \{1, 2, \ldots, n\}$, $\varphi_{2k}^j|_{S_q} : S_q \to S_q$ has exactly two fixed points. Notice that the construction of φ_{2k}^j also satisfies the convention (C3), that is, $\psi_{2k} = \mathrm{id}$.

Note that when $n = 1$, S_1 is exactly $\mathbb{S}^1$ and (C6) coincides with (C5).

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PROPOSITION 6.16. *Suppose that Φ satisfies (C2), (C4), and (C6). Then the extension*

$$\pi : (Z(2) \times \mathbb{S}^1, F_\Phi) \to (Z(2), \tau), \quad (\alpha, z) \mapsto \alpha$$

is an almost proximal extension of uniformly rigid minimal flows, and the cardinality of maximal almost periodic set in each fiber is n.

Proof. First notice that (C4) and (C6) imply (C1) and (C3). It follows that $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is uniformly rigid and minimal by Proposition 6.9.

Let $\alpha \in Z(2)$ and $z_0 \in \mathbb{S}^1$. Let

$$x_j = \left(\alpha, z_0 \exp\left(i\frac{2\pi}{n}(j-1)\right)\right) = (\alpha, z_0\omega^{j-1}) \in \pi^{-1}\alpha), \quad 1 \le j \le n.$$

First we show that $\{x_1, x_2, \ldots, x_n\}$ is an almost periodic set. To attain that aim, we show $(x_1, x_2, \ldots, x_n)$ is a minimal point of $(X^n, F_\Phi^{(n)})$, where $F_\Phi^{(n)} = F_\Phi \times \cdots \times F_\Phi$ (n times).

By the conditions (C4) and (C6), we have

$$\rho_X(F_\Phi(x_j), F_\Phi(x_{j+1})) = \rho_X(x_j, x_{j+1}) = \frac{2\pi}{n}, \quad 1 \le j \le n-1. \tag{6.13}$$

Let $(y_1, y_2, \ldots, y_n)$ be a minimal point of $\overline{\mathcal{O}}((x_1, x_2, \ldots, x_n), F_\Phi^{(n)})$ with $y_1 = x_1$. (First we choose any minimal point $(y_1', y_2', \ldots, y_n')$ in $\overline{\mathcal{O}}((x_1, x_2, \ldots, x_n), F_\Phi^{(n)})$. Since (X, F_Φ) is minimal, there is some $p \in \mathbf{M}$ such that $py_1' = x_1$. Let $(y_1, y_2, \ldots, y_n) = p(y_1', y_2', \ldots, y_n')$. Then $(y_1, y_2, \ldots, y_n)$ is a minimal point of $\overline{\mathcal{O}}((x_1, x_2, \ldots, x_n), F_\Phi^{(n)})$ with $y_1 = x_1$.) By (6.13),

$$\rho_X(y_j, y_{j+1}) = \rho_X(x_j, x_{j+1}) = \frac{2\pi}{n}, \quad 1 \le j \le n-1.$$

Since F_Φ preserves orientation, we have

$$[x_1, x_2] = [y_1, y_2] = [x_1, y_2],$$

and hence $x_2 = y_2$. By the same reason, we have $x_j = y_j$ for $3 \le j \le n$. Thus, $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ is minimal.

Next we show that for each $\alpha \in Z(2)$, $z_1, z_2 \in \mathbb{S}^1$, if $\rho_X((\alpha, z_1), (\alpha, z_2)) < 2\pi/n$, then (α, z_1), (α, z_2) are proximal. Without loss of generality, we may assume that $\alpha = \underline{0}$ and $z_1, z_2 \in [1, \omega)$. By (C6), for $z = \exp(i2\pi x) \in S_1 = [1, \omega]$, $(0 \le x \le 1/n)$

$$\varphi_{2k}^j(z) = \varphi_{2k}^j(\exp(i2\pi x)) = \begin{cases} \exp\left(i2\pi\dfrac{(nx)^{t_k}}{n}\right), & 1 \le j \le 2^{n_{2k}-1} - 1; \\ \exp\left(i2\pi\dfrac{(nx)^{1/t_k}}{n}\right), & 2^{n_{2k}-1} - 1 < j \le 2^{n_{2k}} - 2. \end{cases}$$

Let m_k and y_m be defined as in Lemma 6.4. According to (6.6), for $1 \le s < 2^{n_{2k}}$,

$$F_\Phi^{s \cdot m_{2k-1}}(\underline{0}, y_0) = (0^{n_1+n_2+\cdots n_{2k-1}}\underline{s}0^\infty, y_{s\cdot m_{2k-1}}),$$

where $y_{s\cdot m_{2k-1}} = \varphi_{2k}^{s-1} \circ \varphi_{2k}^{s-2} \circ \cdots \circ \varphi_{2k}^1 \circ \varphi_{2k}^0(y_0)$. Let $z_1 = \exp(i2\pi a_1)$, $z_2 = \exp(i2\pi a_2)$ and $s = 2^{n_{2k}-1}$, where $0 \le a_1, a_2 < 1/n$. Then for $j = 1, 2$, by conditions on

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$\{t_k\}_{k\in\mathbb{N}}$ and $na_j < 1$, we have that $\lim_{k\to\infty}(na_j)^{t_k^{2^{n_{2k-1}}}} = 0$, and

$$F_\Phi^{sm_{2k-1}}(\underline{0}, z_j) = \left(0^{n_1+n_2+\cdots n_{2k-1}}\underline{s}0^\infty, \exp\left(i2\pi\frac{(na_j)^{t_k^{2^{n_{2k-1}}}}}{n}\right)\right) \to (\underline{0}, 1), \quad k \to \infty.$$

Thus,

$$((\underline{0}, z_1), (\underline{0}, z_2)) \in P(Z(2) \times \mathbb{S}^1, F_\Phi).$$

To sum up, we have showed that for $\alpha \in Z(2)$, $A \subseteq \pi^{-1}(\alpha)$ is an almost periodic set with maximal cardinality if and only if $A = \{(\alpha, z_0 \exp(i(2\pi/n)(j-1))) = (\alpha, z_0\omega^{j-1}) : 1 \le j \le n\}$ for some $z_0 \in \mathbb{S}^1$. The proof is complete. □

6.10. *Some remarks.* The method to construct the flow $(Z(2) \times \mathbb{S}^1, F_\Phi)$ is modified from the examples in [7]. This kind of construction originally was from the study of a triangular map of the unit square $[0, 1]^2$, which is a continuous map $F : [0, 1]^2 \to [0, 1]^2$ of the form $F(x, y) = (f(x), g_x(y))$. For a short survey of triangular maps, see [25].

We may replace $\mathbb{S}^1$ by $\mathbb{T}^n$, $\mathbf{P}^n$, etc., to get similar minimal flows. Since $\mathbb{S}^1$ is enough for our purpose, we do not use them in this paper. However, for different purposes, using manifolds with higher dimension may be useful.

7. *Further discussion*

In this section, we give some questions.

7.1. *Proximality and chaos.* First we restate Problem 5.23. in [2].

Question 1. If a minimal flow is not point distal (that is, for any point $x \in X$, there is $x' \neq x$ such that (x, x') is proximal), is it chaotic in the sense of Li–Yorke?

In this paper, we show that if a minimal flow (X, T) is an almost proximal but not almost finite-to-one extension of some flow (Y, T), then (X, T) is not point distal and it is Li–Yorke chaotic. See [2, §5] for another special case about Question 1.

7.2. *Proximality and weak mixing.* As mentioned before, any minimal proximal flow is weakly mixing. In fact, for minimal proximal flows, one can say more.

THEOREM 7.1. *Let (X, T) be a proximal minimal flow. Then for every $x \in X$, the set*

$$\{y \in X : \overline{\mathcal{O}}((x, y), T) = X \times X\}$$

is residual in X.

Proof. First we show that for any non-empty open subsets V, U', V' of X, there is some $t \in T$ such that

$$(\{tx\} \times tV) \cap U' \times V' \neq \emptyset. \tag{7.1}$$

Since (X, T) is minimal, there is a finite subset $\{t_1, t_2, \ldots, t_n\}$ of T such that $X = \bigcup_{i=1}^n t_iV$. As (X, T) is proximal, $(t_1x, t_2x, \ldots, t_nx) \in P^{(n)}$. And there is some $t_0 \in T$ such that

$$t_0t_1x, t_0t_2x, \ldots, t_0t_nx \in U'.$$

Note that $t_0^{-1}V' \cap \bigcup_{i=1}^n t_iV = t_0^{-1}V' \cap X \neq \emptyset$, there is some $j \in \{1, 2, \ldots, n\}$ such that $t_0^{-1}V' \cap t_jV \neq \emptyset$. Thus,

$$(\{t_0t_jx\} \times t_0t_jV) \cap U' \times V' \neq \emptyset.$$

So we have (7.1).

Let $\{U_i\}_{i=1}^\infty$ be a base for $X \times X$. By (7.1), the set

$$A_i = \{y \in X : \text{there is some } t \in T \text{ such that } t(x, y) \in U_i\}$$

is a dense open subset of X. It follows that

$$\{y \in X : \overline{\mathcal{O}}((x, y), T) = X \times X\} = \bigcap_{i=1}^{\infty} A_i$$

is residual in X. □

Our question is does the relative version of the theorem above hold?

Question 2. Let $\pi : (X, T) \to (Y, T)$ be a non-trivial open proximal extension of minimal flows. For each $y \in Y$ and each $x \in \pi^{-1}(y)$, is the subset

$$\{x' \in \pi^{-1}(y) : \overline{\mathcal{O}}((x, x'), T) = R_\pi\}$$

residual in $\pi^{-1}(y)$?

7.3. *Openness and perfectness.* It is a well-known fact that for a homomorphism of ergodic measure-preserving systems, either almost all fibers have constant, finite cardinality or almost all fibers have the cardinality of the continuum. In Theorems 4.2 and 4.7, almost all fibers are perfect. In fact, in the topological case, we have the following general result.

THEOREM 7.2. [1, Theorem 6.31] *Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. Then one of the following holds:*

(1) *π is almost finite to one;*
(2) *every fiber $\pi^{-1}(y)$ is infinite and $\{y \in Y : \pi^{-1}(y)$ is perfect $\}$ is a residual subset of Y.*

However, the proof of theorem above does not imply that in (2), every fiber is perfect even for open extensions. Thus we have the following question.

Question 3. Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal flows. If π is open and is not finite-to-one, is every fiber $\pi^{-1}(y)$ perfect?

7.3.1. *Some special cases.* For some special cases, we have positive answer for Question 3. First, we have that for some special weakly mixing extensions, each fiber is perfect.

THEOREM 7.3. [23] *Let $\pi : (X, T) \to (Y, T)$ be a non-trivial weakly mixing RIC extension of minimal flows. Then each fiber is perfect.*

In the rest of the section, we show that if a distal extension is not finite-to-one, each fiber is perfect. To prove this result, we need the Furstenberg–Ellis structure theorem.

Furstenberg's structure theorem for distal flows [13] says that any distal minimal flow can be constructed by equicontinuous extensions. We state the result in its relative version of Ellis [10]. Let $\pi : (X, T) \to (Y, T)$ be a distal extension of minimal flows. Then there is an ordinal η (which is countable when X is metrizable) and a family of flows $\{(Z_n, T)\}_{n\leq\eta}$ such that:

(i) $Z_0 = Y$;

(ii) for every $n < \eta$ there exists a homomorphism $\rho_{n+1} : Z_{n+1} \to Z_n$ which is equicontinuous;

(iii) for a limit ordinal $\nu \leq \eta$ the flow Z_ν is the inverse limit of the flows $\{Z_\iota\}_{\iota<\nu}$;

(iv) $Z_\eta = X$.

$$Y = Z_0 \xleftarrow{\rho_1} Z_1 \xleftarrow{\rho_2} \cdots \xleftarrow{\rho_n} Z_n \xleftarrow{\rho_{n+1}} Z_{n+1} \longleftarrow \cdots \xleftarrow{\rho_\eta} Z_\eta = X. \tag{7.2}$$

When $Y = \{pt\}$ is the trivial flow, we have the structure theorem for a distal minimal flow.

THEOREM 7.4. *Let $\pi : (X, T) \to (Y, T)$ be a distal extension of minimal flows. If π is not finite-to-one, then each fiber is perfect.*

Proof. We need an equivalent characterization of an equicontinuous extension. Let M be a homogeneous compact metric space. By this, we mean a compact metric space such that for any two points $x, y \in M$, there is an isometry of M taking x into y. The isometries of M form a compact group H, M may be identified with a coset space H/H_0, where H_0 is the subgroup of H leaving a given point of M fixed.

Let $\pi : X \to Y$ be an extension of flows. Then π is equicontinuous if and only if there exists a continuous map $\rho : R_\pi \to \mathbb{R}$ such that for each $y \in Y$, ρ defines a metric on the fiber $X_y = \pi^{-1}(y)$ under which X_y is isometric to M, and $\rho(tx_1, tx_2) = \rho(x_1, x_2)$ for all $t \in T$ and $(x_1, x_2) \in R_\pi$ [13]. Thus, if π is equicontinuous, then each fiber is isometric to $M = H/H_0$. Hence, if π is not finite-to-one, then each fiber is perfect.

To deal with distal extensions, we use the Furstenberg–Ellis structure theorem as stated above. Let $\{(Z_n, T)\}_{n\leq\eta}$ be the factors. Since π is not finite-to-one, either η is not finite ordinal and each ρ_n is finite-to-one, or there is some $n \leq \eta$ such that ρ_n is not finite-to-one. In the first case, each fiber is an inverse limit of finite sets and it is a Cantor set; in the second case, ρ_n is an infinite-to-one equicontinuous extension and each fiber of ρ_n is perfect and by this, we claim that each fiber of π is also perfect. To prove the second case, we need the following claim: if $\pi_1 : (X_1, T) \to (X_2, T)$, $\pi_2 : (X_2, T) \to (X_3, T)$ are open extensions such that each fiber of π_1 or π_2 is perfect, then each fiber of $\pi_2 \circ \pi_1$ is perfect. First, by definition, it is easy to see that when each fiber of π_1 is perfect, we have each fiber of $\pi_2 \circ \pi_1$ is perfect. Next, we show the other case. Suppose that each fiber of π_2 is perfect. We show that for each $z \in X_3$, $(\pi_2 \circ \pi_1)^{-1}(z)$ is perfect. Let $x \in (\pi_2 \circ \pi_1)^{-1}(z)$ and $y = \pi_1(x)$. Clearly, $y \in \pi_2^{-1}(z)$, and by perfectness of $\pi_2^{-1}(z)$, one can find $y_n \in$

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$\pi_2^{-1}(z)$ such that $y_n \to y$ as $n \to \infty$. Since π_1 is open, there exists $x_n \in X$ such that $\pi_1(x_n) = y_n$ and $x_n \to x$ as $n \to \infty$. To sum up, there are $x_n \in (\pi_2 \circ \pi_1)^{-1}(z)$ such that $x_n \to x, n \to \infty$. Thus, each point of $(\pi_2 \circ \pi_1)^{-1}(z)$ is not isolated and $(\pi_2 \circ \pi_1)^{-1}(z)$ is perfect. Thus, we have the claim. By this claim and the Furstenberg–Ellis structure theorem (7.2), one can show that each fiber of π is perfect. □

7.4. *Open proximality and entropy.* Our last question is about entropy. Let $\pi : (X, \mathbb{Z}) \to (Y, \mathbb{Z})$ be an extension of discrete flows. If $h_{\mathrm{top}}(X) > h_{\mathrm{top}}(Y)$, then lots of fibers will have very complex properties (see [**29**] for example). We are not sure that open proximal extensions can reach those kinds of complexity. Thus, we have the following question.

Question 4. Let $\pi : (X, \mathbb{Z}) \to (Y, \mathbb{Z})$ be an open proximal extension of discrete minimal flows. Is it true that $h_{\mathrm{top}}(X) = h_{\mathrm{top}}(Y)$?

Note that in Question 4, openness is a necessary condition, since there are many minimal flows which are almost one-to-one extensions of their maximal equicontinuous factors, and they have positive entropy by [**14**, Theorem 1].

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