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ON q-EXPONENTIAL FUNCTIONS FOR $|q| = 1$

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ABSTRACT. We discuss the q-exponential functions e_q, E_q for q on the unit circle, especially their continuity in q, and analogues of the limit relation $\lim_{q\to 1} e_q((1-q)z) = e^z$.

1. **Introduction and results.** In recent years, there has been increasing interest in q-series for $|q| = 1$. The case where q is not a root of unity has been useful in investigating various phenomena in Padé approximation [3–7], [15] and the case where q is a root of unity has been useful in the former and in various applications in theoretical physics [17], [19–21]. While investigating Ramanujan's continued fraction for $|q| = 1$, the author was led to consider continuity properties of q-exponentials for q on the unit circle. Recall that the q-exponential functions are [10, p. 9]

$$e_q(z) := \sum_{j=0}^{\infty} z^j \big/ (q;q)_j; \tag{1.1}$$

$$E_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} z^j \big/ (q;q)_j; \tag{1.2}$$

where $(a;q)_0 = 1$ and for $1 \le n \le \infty$,

$$(a;q)_n := \prod_{j=1}^{n} (1 - aq^{j-1}). \tag{1.3}$$

If $|q| < 1$, then e_q and E_q admit the product representations [1], [2], [10]

$$e_q(z) = 1\big/(z;q)_\infty; \quad E_q(z) = (-z;q)_\infty \tag{1.4}$$

and hence

$$e_q(z)E_q(-z) = 1. \tag{1.5}$$

Their connection with the exponential function is the last functional equation, and the limit

$$\lim_{q\to 1} e_q\big((1-q)z\big) = e^z = \lim_{q\to 1} E_q\big((1-q)z\big) \tag{1.6}$$

Here the limit is taken with q restricted to $0 < q < 1$. See [1], [2], [8–10]. McIntosh [16] has studied the asymptotic behaviour of series that include the functions $e_q(z), E_q(z)$ as $q \longrightarrow 1$ with $0 < q < 1$ and z restricted to be real, but without scaling the variable z.

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In this paper, we consider e_q, E_q for q on the unit circle. Obviously e_q, E_q are not defined for q a root of unity, but at least their Maclaurin series coefficients are defined for q not a root of unity. The radius of convergence of both e_q and E_q is

$$R(q) := \liminf_{n\to\infty} \left|\prod_{j=1}^{n}(1-q^j)\right|^{1/n}. \tag{1.7}$$

It follows from a well known identity (and we shall indicate the proof in Section 2) that

$$R(q) = \liminf_{n\to\infty} |1-q^n|^{1/n}. \tag{1.8}$$

The latter is readily formulated in terms of diophantine approximation: If $q = e^{i\theta}$, and $\beta := \theta/(2\pi)$, and $\{x\}$ denotes the distance from $x \in \mathbb{R}$ to its nearest integer, it is easy to see that

$$R(q) = \liminf_{n\to\infty} |\{n\beta\}|^{1/n}. \tag{1.9}$$

It is then clear that $R(q) = 1$ for "most" q. Indeed, if

$$G := \{q : |q| = 1, R(q) < 1\} \tag{1.10}$$

then G is an F_σ set that has linear measure 0, Hausdorff dimension 0, and even logarithmic dimension 2. This is a consequence of the Jarnik-Besicovitch theorem, see *e.g.* [14]. It is by no means obvious that $R(q)$ may assume any value in $[0, 1]$; this follows from a lemma of G. Petruska [18, Lemma 2].

We note that (1.5) persists for $|q| = 1$; this is easily verified from the Maclaurin series. In fact, in view of the simple identity

$$E_q(z) = e_{\bar{q}}(-\bar{q}z) \tag{1.11}$$

it takes the more attractive form

$$e_q(z)e_{\bar{q}}(\bar{q}z) = 1. \tag{1.12}$$

Moreover, as a consequence of the functional equations

$$e_q(qz) = e_q(z)(1-z); \quad E_q(qz) = E_q(z)/(1+z) \tag{1.13}$$

which are easily verified from the Maclaurin series (the infinite products in (1.4) no longer have meaning), we note the following simple:

PROPOSITION 1.1. *Let $q = e^{i\theta}$, $\theta/(2\pi)$ irrational. Then e_q and E_q have natural boundaries on the circle $|z| = R(q)$.*

We include the proof of this, although it is a special case of more general results in [3]. Our goal is to study two questions that arise in analysing q-series for $|q| = 1$:

(I) To what extent are e_q, E_q continuous in q?

(II) For q a root of unity, what is the correct analogue of (1.6)?

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The latter would suggest how to define a q-exponential function for q a root of unity other than 1. We feel that (I) has intrinsic interest in that it indicates how solutions of the functional equation (1.13) vary as q varies on the unit circle. However, our main motivation arose in analyzing Ramanujan's continued fraction for $|q| = 1$, where the continuity properties of e_q and similar functions give insight into Padé convergence theory (we shall present this elsewhere). We feel that (I), (II) provide a model for many of the problems that arise in treating q-series for $|q| = 1$.

Obviously since e_q, E_q are not defined for q a root of unity we must be careful not to stray too close to roots of unity in limiting processes. Accordingly, we define the (closed and, as we shall see, perfect) set

$$S(\rho, L) := \{q : \forall n \geq L, |1 - q^n|^{1/n} \geq \rho\}. \tag{1.14}$$

We prove:

THEOREM 1.2. *Let* $|q_0| = 1$ *and* $R(q_0) > 0$. *Let* $|q_k| = 1\ \forall k$, *and assume that*

$$\lim_{k\to\infty} q_k = q_0. \tag{1.15}$$

The following are equivalent:

(I) For each $\rho \in (0, 1)$ *there exist* L *and* k_0 *such that* $\{q_k\}_{k=k_0}^{\infty} \subset S\big(\rho R(q_0), L\big)$.

(II) Uniformly in compact subsets of $|z| < R(q_0)$,

$$\lim_{k\to\infty} e_{q_k}(z) = e_{q_0}(z), \tag{1.16}$$

and

$$\lim_{k\to\infty} E_{q_k} = E_{q_0}(z).$$

It is noteworthy that the (simpler) radial limit

$$\lim_{r\to 1-} e_{rq_0}(z) = e_{q_0}(z), \quad |z| < R(q_0)$$

was established by Hardy and Littlewood [11]. We remark that more generally, if

$$f_q(z) := \sum_{j=0}^{\infty} \frac{h_j(q)}{(q;q)_j} z^j \tag{1.17}$$

where each $h_j(q)$ is continuous in q and

$$\lim_{j\to\infty} \Big(\sup_{|q|=1} |h_j(q)|\Big)^{1/j} = 1; \tag{1.18}$$

then the above proof shows that if (I) holds, then locally uniformly in $|z| < R(q_0)$,

$$\lim_{k\to\infty} f_{q_k} = f_{q_0}(z). \tag{1.19}$$

Next, we turn to analogues of (1.6). As a polynomial in q, $(q;q)_n$ has a zero of multiplicity n at $q = 1$, but only a zero of multiplicity $[n/2]$ at $q = -1$. (Here and in the sequel, $[x]$ denotes the greatest integer $\leq x$). More generally at a primitive l-th root of unity, $(q;q)_n$ has a zero of multiplicity only $[n/l]$. As a consequence, the scaling of the variable z in (1.6) gives for q_0 a primitive l-th root of unity, $l \geq 2$,

$$\lim_{q \to q_0} e_q\big((1-q)z\big) = 1 \tag{1.20}$$

under the conditions of the following theorem. So a more meaningful scaling of the variable z must be sought, and intuitively, it seems that it should involve $(1 - \frac{q}{q_0})^{1/l}$. This is the situation in the following theorem. The scaling also allows us to let $q \longrightarrow q_0$ through a wider class than in Theorem 1.2. Accordingly we define for $\sigma > 0$

$$\mathcal{T}(\sigma, l, L) := \{q : \forall n \geq L, |1-q^n|^{1/n} \geq \sigma |1-q^l|^{1/l} > 0\}. \tag{1.21}$$

Then we can state:

THEOREM 1.3. *Let q_0 be a primitive l-th root of unity. Let $|q_k| = 1\ \forall k$, and assume that (1.15) holds. The following are equivalent:*

(I) For each $\sigma > 0$, there exist L and k_0 such that $\{q_k\}_{k=k_0}^{\infty} \subset \mathcal{T}(\sigma, l, L)$.

(II) Uniformly in compact subsets of $\mathbb{C}$,

$$\lim_{k \to \infty} e_{q_k}\left(\left(1 - \frac{q_k}{q_0}\right)^{1/l} z\right) = e^{z^l/l^2}, \tag{1.22}$$

The same limit holds if we replace e_q by E_q.

Note that one may use any of the l values of $(1 - \frac{q_k}{q_0})^{1/l}$ in (1.22). Alternative formulations of (1.22) include

$$\lim_{k \to \infty} e_{q_k}\left(\left[\prod_{j=1}^{l}(1-q_k^j)\right]^{1/l} z\right) = e^{z^l} \tag{1.23}$$

or

$$\lim_{k \to \infty} e_{q_k}\big([l(1-q_k^l)]^{1/l} z\big) = e^{z^l}.$$

So it seems that e^{z^l/l^2} is the proper q-exponential function when q is a primitive l-th root of unity.

We shall also consider limits as $q \longrightarrow q_0$ from inside the unit circle, where we can allow somewhat more than a non-tangential limit: Define for given $q_0, \alpha > 0$,

$$\Omega(q_0, \alpha) := \left\{q : |q| < 1 \text{ and } 1 - |q| \geq |1 - \frac{q}{q_0}|^{\alpha}\right\}.$$

The case $\alpha = 1$ corresponds essentially to non-tangential limits, that is a cone with vertex at q_0; the region when restricted to $|1 - \frac{q}{q_0}| \leq 1$ increases as α increases. We prove:

THEOREM 1.4. *Let q_0 be a primitive l-th root of unity and $\alpha > 0$. Then*

$$\lim_{\substack{q \longrightarrow q_0 \\ q \in \Omega(q_0,\alpha)}} e_q\left(\left(1 - \frac{q}{q_0}\right)^{1/l} z\right) = e^{z^l/l^2}$$

uniformly in compact subsets of $\mathbb{C}$.

An obvious question is whether there exist for a given q_0 sequences $\{q_k\}$ fulfilling the hypotheses of Theorems 1.2 or 1.3, so that the specified convergence can take place! We shall prove this using elementary continued fraction theory; we shall also reformulate the conditions of Theorem 1.2. For a given $q = e^{i\theta}$, $\theta \in [0, 2\pi)$, set

$$\beta := \beta(\theta) := \frac{\theta}{2\pi} \tag{1.24}$$

and let β have continued fraction expansion

$$\beta(\theta) = \frac{\theta}{2\pi} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \tag{1.25}$$

with convergents

$$\frac{\pi_j(\theta)}{\chi_j(\theta)} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \frac{1|}{|a_j}. \tag{1.26}$$

(So all a_j, π_j, χ_j are non-negative integers). Small values of $|1 - q^n|$ correspond to large denominators χ_j in the convergents of β. Accordingly, we define for $0 < \sigma < 1$,

$$U(\sigma, L) := \left\{\theta \in [0, 2\pi) : \frac{\log \chi_{n+1}(\theta)}{\chi_n(\theta)} \leq \log \frac{1}{\sigma}, n \geq L\right\}. \tag{1.27}$$

We may now reformulate the condition of Theorem 1.2:

PROPOSITION 1.5. *Let $0 < \tau \leq 1$. Let $q_k = e^{i\theta_k}$, $k \geq 0$ and assume that (1.15) holds. The following are equivalent:*

(I) $\forall \rho \in (0, 1)$, $\exists L$ and k_0 such that $\{q_k\}_{k=k_0}^{\infty} \subset S(\rho\tau, L)$.

(II) $\forall \rho \in (0, 1)$, $\exists L$ and k_0 such that $\{\theta_k\}_{k=k_0}^{\infty} \subset U(\rho\tau, L)$.

As a consequence, we can construct sequences fulfilling the hypotheses of Theorem 1.2:

THEOREM 1.6. *Let $q_0 = e^{i\theta_0}$ where $\theta_0/(2\pi)$ is irrational. Let $0 < \tau \leq 1$. The following are equivalent:*

(I) $\exists \{q_k\}$ with $q_k \neq q_0$; $R(q_k) \geq \tau$, $k \geq 1$; with $q_k \longrightarrow q_0$, $k \longrightarrow \infty$ and such that for each $\rho \in (0, 1)$ there exist L and k_0 such that $\{q_k\}_{k=k_0}^{\infty} \subset S(\rho\tau, L)$.

(II) $R(q_0) \geq \tau$.

In particular, choosing $\tau = R(q_0)$ gives a sequence satisfying the requirements of Theorem 1.2. We note that the proof shows that each $S(\rho, L)$ is perfect, that is has no isolated points. Our proof of (II) $\Rightarrow$ (I) is constructive as is the proof of the following result:

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THEOREM 1.7. *For each* q_0 *that is a primitive l-th root of unity, there exists a sequence* $\{q_k\}$ *with* $q_k \neq q_0$; $R(q_k) = 1$, $k \geq 1$, *with* $q_k \longrightarrow q_0$, $k \longrightarrow \infty$ *and satisfying the hypothesis (I) of Theorem 1.3.*

Our proof of Theorems 1.6, 1.7 is easily modified to give infinitely many non-trivially distinct sequences with the desired property. It does not however indicate "what proportion" of the sequences that approach q_0 have the desired property. Intuitively, the restriction that every q_k with $k \geq k_0$ should lie in $S(\rho, L)$ severely restricts the sequence.

We present the proofs in Section 2.

2. **Proofs.** We shall make use of the following identity:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \exp\Bigl(\sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}\Bigr) =: \exp\bigl(\Phi_q(z)\bigr). \tag{2.1}$$

Hardy and Littlewood [11] proved that even for q on the unit circle this identity holds inside the radius of convergence of either series, and hence Φ_q and e_q have the same radius of convergence. Then (1.8) follows. We can now give the

PROOF OF PROPOSITION 1.1. We remarked that the functional equations (1.13) may easily be verified from the Maclaurin series for e_q and E_q even for q on the unit circle. Thus if z_0 is a point of analyticity of e_q on $|z| = R(q)$, then so is $q^{-1}z_0$ and hence so are $\{q^{-j}z_0\}_{j=0}^{\infty}$ and as the latter is dense, we obtain that e_q is analytic on the circle of convergence of its power series, a contradiction. Similarly for E_q; Alternatively one may use either (1.5) or the simple identity

$$E_q(z) = e_{\bar{q}}(-\bar{q}z)$$

and note that $R(\bar{q}) = R(q)$. ■

We turn to the

PROOF OF THEOREM 1.2. Note first that (1.12) shows that e_{q_0} has no zeros in $B := \{z : |z| < R(q_0)\}$. Thus the convergence of e_{q_k} to e_{q_0} is equivalent to that of $e_{q_k}^{\pm 1}$ to $e_{q_0}^{\pm 1}$. Our hypotheses on $\{q_k\}$ ensure that for each fixed L,

$$\lim_{k\longrightarrow\infty} \sum_{j=0}^{L} \frac{z^j}{(q_k;q_k)_j} = \sum_{j=0}^{L} \frac{z^j}{(q_0;q_0)_j}$$

uniformly in compact subsets of B. Thus the locally uniform convergence in Theorem 1.2 is equivalent to the uniform boundedness of $\{e_{q_k}^{\pm 1}\}$ in compact subsets of B. The identity (2.1) shows that this is equivalent to uniform boundedness of $\{|\operatorname{Re}\Phi_{q_k}|\}$ in compact subsets of B, and consequently of $\{\Phi_{q_k}\}$ in compact subsets of B [13, p. 193]. Since a fixed number of terms of the Maclaurin series of Φ_{q_k} converge as $k \longrightarrow \infty$ to the corresponding terms in the Maclaurin series of Φ_{q_0}, and since L_2 norms on a circle centre 0 may be used to bound above L_∞ norms on a smaller concentric circle, we see

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that the convergence in Theorem 1.2 is equivalent to the following assertion: For each $0 < r < R(q_0)$, there exists L and k_0 such that

$$\sum_{n=L}^{\infty} \frac{r^{2n}}{|n(1-q_k^n)|^2} \leq C, \quad k \geq k_0. \tag{2.2}$$

Clearly if $\{q_k\}_{k=k_0}^{\infty} \subset S(r, L)$ for some k_0, then the series in (2.2) is bounded above by $\sum_{n=1}^{\infty} n^{-2}$, independently of $k \geq k_0$. Conversely, if (2.2) holds for a given r and k_0, then for any $s < r$ we claim there exists k_1 and L_1 such that $\{q_k\}_{k=k_1}^{\infty} \subset S(s, L_1)$. If not, we choose infinitely many k and $n = n(k)$ for which $n(k) \longrightarrow \infty$, $k \longrightarrow \infty$ and

$$|1 - q_k^n|^{1/n} < s$$

and hence

$$\frac{r^{2n}}{|n(1-q_k^n)|^2} > n^{-2}\Big(\frac{r}{s}\Big)^{2n} \longrightarrow \infty, \quad n \longrightarrow \infty$$

contradicting (2.2). Since r and hence s may be made arbitrarily close to $R(q_0)$, we have the converse assertion of Theorem 1.2. ∎

We turn to the

PROOF OF THEOREM 1.3. We first note the following limit, which concerns individual terms of the Maclaurin series of $e_q\big((1-\frac{q}{q_0})^{1/l}z\big)$:

$$\lim_{q \to q_0} \frac{(1-\frac{q}{q_0})^{n/l}}{(q;q)_n} = \begin{cases} l^{-2k}/k! & \text{if } n = kl \\ 0 & \text{otherwise} \end{cases}. \tag{2.3}$$

Recall that $[x]$ denotes the greatest integer $\leq x$. The factor $(1-\frac{q}{q_0})$ occurs precisely $[n/l]$ times in $(q;q)_n$ as a polynomial in q, since q_0 is a zero of $1-q^j$ iff j is a multiple of l. It follows that for n not a multiple of l, we have $n/l > [n/l]$ and so we have the desired limit. Now let us suppose that $n = kl$. Then as $q \longrightarrow q_0$,

$$\frac{(1-\frac{q}{q_0})^{n/l}}{(q;q)_n} = \prod_{r=0}^{k-1} \frac{1-\frac{q}{q_0}}{\prod_{j=1}^{l}(1-q^{rl+j})} \longrightarrow \prod_{r=0}^{k-1} \frac{1}{\big(\prod_{j=1}^{l-1}(1-q_0^j)\big)(rl+l)}$$

But if $Q(z) := (z^l - 1)/(z-1)$, then Q has zeros at the l-th roots of unity other than 1, that is at q_0^j, $1 \leq j \leq l-1$, so

$$Q(z) = \prod_{j=1}^{l-1}(z - q_0^j)$$

and hence

$$\prod_{j=1}^{l-1}(1-q_0^j) = Q(1) = l.$$

So we have (2.3). By much the same reasoning as in the proof of Theorem 1.2, we obtain the locally uniform convergence in Theorem 1.3, iff for each $r > 0$, there exist k_0 and L such that

$$\sum_{n=L}^{\infty} \frac{r^{2n}|1-\frac{q_k}{q_0}|^{2n/l}}{|n(1-q_k^n)|^2} \leq C, \quad k \geq k_0.$$

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In turn, as

$$|1-q_k^l|/|1-\frac{q_k}{q_0}| \longrightarrow l, \quad k \longrightarrow \infty$$

this is equivalent to the following: For each $s > 0$, there exist k_1 and L_1 such that

$$\sum_{n=L_1}^{\infty} \frac{s^{2n}|1-q_k^l|^{2n/l}}{l^{2n/l}|n(1-q_k^n)|^2} \le C, \quad k \ge k_1. \tag{2.4}$$

We see that the $2n$-th root of the term with index n in the series is

$$\frac{s|1-q_k^l|^{1/l}}{l^{1/l}n^{1/n}|1-q_k^n|^{1/n}}.$$

Because of the freedom of choice in s, we see much as in the previous proof that (2.4) holds with a corresponding value of k_1, L_1 for each s, iff for each $\sigma > 0$, there exist k_2 and L_2 such that

$$|1-q_k^n|^{1/n} \ge \sigma|1-q_k^l|^{1/l}, \quad n \ge L_2, \ k \ge k_2,$$

that is $\{q_k\}_{k=k_2}^{\infty} \subset T(\sigma, l, L_2)$. ∎

PROOF OF THEOREM 1.4. We note from (2.3) that individual terms of the Maclaurin series of $e_q\big((1-\frac{q}{q_0})^{1/l}z\big)$ converge to the corresponding terms of e^{z^l/l^2} as $q \longrightarrow q_0$. Thus it suffices to establish boundedness independent of $q \in \Omega(q_0, \alpha)$. As before, using (2.1), this boils down to estimation, for each fixed s, of

$$\Delta := \sum_{n=L_1}^{\infty} \frac{s^{2n}|1-q^l|^{2n/l}}{l^{2n/l}|n(1-q^n)|^2}.$$

But for $q \in \Omega(q_0, \alpha)$ and close enough to q_0

$$\begin{aligned} |1-q^n| &\ge 1-|q|^n \ge 1-|q| \\ &\ge \left|1-\frac{q}{q_0}\right|^{\alpha} \ge C\left|1-\left(\frac{q}{q_0}\right)^l\right|^{\alpha} = C|1-q^l|^{\alpha}. \end{aligned}$$

Thus

$$\Delta \le C^{-2} \sum_{n=L_1}^{\infty} \frac{s^{2n}}{l^{2n/l}n^2}|1-q^l|^{2n/l-2\alpha}.$$

For a suitably large L_1 and q close to q_0 this is clearly bounded above independent of q. ∎

Before turning to the proof of Proposition 1.5, we recall some elementary properties of continued fractions [12]. Our notation is as in (1.24) to (1.26). Firstly

$$\frac{1}{2\chi_{j+1}(\theta)} \le |\chi_j(\theta)\beta(\theta) - \pi_j(\theta)| \le \frac{1}{\chi_{j+1}(\theta)}. \tag{2.5}$$

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Moreover, if π, χ are coprime and π/χ is not a convergent, then

$$|\chi\beta(\theta) - \pi| \geq \frac{1}{2\chi}. \tag{2.6}$$

The recurrence relation for the denominators of the convergents is

$$\chi_j(\theta) = a_j\chi_{j-1}(\theta) + \chi_{j-2}(\theta), \quad j \geq 2. \tag{2.7}$$

Finally, since $\{x\} \in [-\frac{1}{2}, \frac{1}{2}]$ for real x, we have for $q = e^{i\theta}, \beta(\theta) = \frac{\theta}{2\pi}$,

$$|1 - q^n| = 2|\sin \pi\{n\beta\}| \begin{cases} \leq 2\pi|\{n\beta(\theta)\}| \\ \geq 4|\{n\beta(\theta)\}| \end{cases}. \tag{2.8}$$

We turn to the

PROOF OF PROPOSITION 1.5. In view of the last inequality, we see that (I) of Proposition 1.5 is equivalent to the following: For each $\rho \in (0, 1)$, $\exists L$ and k_0 such that

$$|\{n\beta(\theta_k)\}|^{1/n} \geq \rho\tau, \quad n \geq L,\ k \geq k_0.$$

Now if n is not a denominator of a convergent of the continued fraction for $\beta(\theta_k)$, then (2.6) shows that

$$|\{n\beta(\theta_k)\}|^{1/n} \geq (2n)^{-1/n}.$$

If n is a denominator, say, $n = \chi_j(\theta_k)$, then (2.5) shows that

$$\big(2\chi_{j+1}(\theta_k)\big)^{-1/\chi_j(\theta_k)} \leq |\{n\beta(\theta_k)\}|^{1/n} \leq \chi_{j+1}(\theta_k)^{-1/\chi_j(\theta_k)}.$$

It follows that (I) of Proposition 1.5 is equivalent to the following: For each $\rho \in (0, 1)$, $\exists L$ and k_0 such that

$$\chi_{j+1}(\theta_k)^{-1/\chi_j(\theta_k)} \geq \rho\tau \quad \text{for } \chi_j(\theta_k) \geq L,\ k \geq k_0.$$

This is almost (II) of Proposition 1.5, the only difference being that instead of $\chi_j(\theta_k) \geq L$, we want $j \geq L_1$. This follows in view of the fact that for fixed j, we have for large enough k that $\chi_j(\theta_k) = \chi_j(\theta_0)$. ∎

PROOF OF THEOREM 1.6. We first show that (II) implies (I) Let us assume that $q_0 = e^{i\theta_0}$ and $R(q_0) \geq \tau$. Write

$$\beta(\theta_0) = \frac{\theta_0}{2\pi} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots$$

(note that $\beta(\theta_0)$ is irrational, so the c.f. does not terminate). We define

$$\beta(\theta_k) = \frac{\theta_k}{2\pi} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots \frac{1|}{|a_{k-1}} + \frac{1|}{|a_k + 1} + \frac{1|}{|a_{k+1}} + \cdots.$$

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Thus the c.f. for $\beta(\theta_k)$ is obtained from that for $\beta(\theta_0)$ by adding 1 to its k-th c.f. coefficient. Then the c.f. of both $\beta(\theta_0)$ and $\beta(\theta_k)$ have the same first $k-1$ convergents and then (2.5) shows that $\beta(\theta_k) \longrightarrow \beta(\theta_0)$, $k \longrightarrow \infty$, and hence $q_k \longrightarrow q_0$, $k \longrightarrow \infty$. Next, in our situation the recurrence relations for the denominators of the convergents become:

$$\chi_j(\theta_k) = a_j\chi_{j-1}(\theta_k) + \chi_{j-2}(\theta_k), \quad j \neq k$$

and

$$\chi_j(\theta_k) = (a_j + 1)\chi_{j-1}(\theta_k) + \chi_{j-2}(\theta_k), \quad j = k.$$

Then as $\chi_j(\theta_k) = \chi_j(\theta_0)$, $j \leq k$,

$$\chi_k(\theta_k) = \chi_k(\theta_0) + \chi_{k-1}(\theta_0)$$

so for $j = k$

$$\chi_j(\theta_0) \leq \chi_j(\theta_k) \leq 2\chi_j(\theta_0).$$

This inequality also holds trivially for $j \leq k$, and an easy induction on the recurrence relation shows that it holds for all $j \geq 1$. Then for all j, k

$$\frac{\log \chi_{j+1}(\theta_k)}{\chi_j(\theta_k)} \leq \frac{\log 2}{\chi_j(\theta_0)} + \frac{\log \chi_{j+1}(\theta_0)}{\chi_j(\theta_0)}.$$

But since $R(q_0) \geq \tau$, for each $0 < \rho < \rho' < \rho'' < 1$, we have

$$|1 - q_0^n|^{1/n} \geq \rho''\tau$$

for n large enough, and as in the previous proof, we deduce that

$$\frac{\log \chi_{j+1}(\theta_0)}{\chi_j(\theta_0)} \leq \log \frac{1}{\rho'\tau}$$

for j large enough. Thus we can find L such that

$$\frac{\log \chi_{j+1}(\theta_k)}{\chi_j(\theta_k)} \leq \log \frac{1}{\rho\tau}, \quad j \geq L, \ k \geq 1.$$

So $\{\theta_k\}_{k=1}^{\infty} \subset U(\rho\tau, L)$ and hence Proposition 1.5 shows that $\{q_k\}$ has the required properties. Moreover, the last inequality for each $\rho < 1$ shows that for each k, we have $R(q_k) \geq \tau$.

We turn to the proof that (I) implies (II). Note that

$$S(\rho\tau, L) = \bigcap_{n=L}^{\infty} \{q : |q| = 1 \quad \text{and} \quad |1 - q^n|^{1/n} \geq \rho\tau\}$$

so is closed. Then if q_0 is the limit of $\{q_k\}_{k=k_0}^{\infty} \subset S(\rho\tau, L)$ it follows that $q_0 \in S(\rho\tau, L)$ and hence $R(q_0) \geq \rho\tau$. Since this is true for each $\rho < 1$, we have $R(q_0) \geq \tau$. ∎

Next, we give the

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PROOF OF THEOREM 1.7. Write (for some r, m depending on $\beta(\theta_0)$)

$$\beta(\theta_0) = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots \frac{1|}{|a_m} = \frac{r}{l}$$

and set

$$\beta(\theta_k) = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots \frac{1|}{|a_m} + \frac{1|}{|k} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots .$$

Then as $\beta(\theta_k)$ is a quadratic irrational, $R(q_k) = 1$, and the estimate (2.5) shows that $\beta(\theta_k) \longrightarrow \beta(\theta_0)$, $k \longrightarrow \infty$. The c.f.'s of $\beta(\theta_0)$ and $\beta(\theta_k)$ have the same first m convergents. Moreover,

$$\chi_m(\theta_k) = l;$$
$$\chi_{m+1}(\theta_k) = k\chi_m(\theta_k) + \chi_{m-1}(\theta_k) \begin{cases} \leq (k+1)l \\ \geq kl \end{cases}.$$

For $j \geq m+1$,

$$\chi_{j+1}(\theta_k) = \chi_j(\theta_k) + \chi_{j-1}(\theta_k)$$

and so

$$\chi_{j+1}(\theta_k) \leq 2\chi_j(\theta_k), \quad j \geq m+1.$$

Then

$$\chi_{j+1}(\theta_k)^{-1/\chi_j(\theta_k)} = \exp\left(-\frac{\log \chi_{j+1}(\theta_k)}{\chi_j(\theta_k)}\right) \geq \exp\left(-\frac{\log 2 + \log \chi_j(\theta_k)}{\chi_j(\theta_k)}\right)$$
$$\geq \exp\left(-\frac{\log 2}{kl} + \frac{\log(kl)}{kl}\right), \quad j \geq m+1.$$

Next, as $l = \chi_m(\theta_k)$, (2.8) and then (2.5) show that

$$|1 - q_k^l| \leq 2\pi |\{l\beta(\theta_k)\}| \leq \frac{2\pi}{kl}.$$

Then if n is not a denominator of a convergent of the c.f. of $\beta(\theta_k)$, we obtain (recall (2.6))

$$\frac{|1-q_k^n|^{1/n}}{|1-q_k^l|^{1/l}} \geq (2n)^{-1/n}\left(\frac{kl}{2\pi}\right)^{1/l}$$

and if n is a denominator, say $n = \chi_j(\theta_k)$, with $j \geq m+1$, then (recall (2.5))

$$\frac{|1-q_k^n|^{1/n}}{|1-q_k^l|^{1/l}} \geq \left(\frac{kl}{2\pi}\right)^{1/l} \left(2\chi_{j+1}(\theta_k)\right)^{-1/\chi_j(\theta_k)}$$

$$\geq 2^{-1/n}\left(\frac{kl}{2\pi}\right)^{1/l} \exp\left(-\frac{\log 2}{kl} + \frac{\log(kl)}{kl}\right).$$

Clearly, given $\sigma > 0$, we can find k_0 and L such that this last terms exceeds σ for $n \geq L$, $k \geq k_0$. So $\{q_k\}$ satisfies the hypothesis (I) of Theorem 1.3. ∎

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