2

CATEGORIES AND EXACT SEQENCES

The definition of an abstract category imposes few conditions on the objects and morphisms of a category, with the consequence that a great diversity of mathematical entities can be regarded as categories. In contrast, a category of modules has a richer structure than the minimum required for the definition of a category. The set of homomorphisms $\mathrm{Hom}(M, N)$ between two modules is always an abelian group, and we can recognise whether or not a homomorphism is an injection or surjection. Furthermore, the category may or may not contain such things as direct sums, kernels, projective modules, etc., depending on the nature of the modules that belong to it.

Our aim in this chapter is to analyse the extra structure that is available to module categories, and thereby to formulate sets of conditions which ensure that an abstract category behaves as a module category. We start by investigating the functorial properties of the abelian groups $\mathrm{Hom}(M, N)$ and we see how to reformulate the basic properties of modules and homomorphisms in terms of these functorial properties. We then use these interpretations to define a series of types of abstract category – preadditive, additive, abelian – which increasingly resemble module categories. Ultimately, we arrive at the notion of an exact category, which is a category that has enough structure for the purposes of algebraic K-theory.

2.1 THE HOMOMORPHISM FUNCTORS

In this section, we take a break from the theory of categories in general and return to the consideration of module categories. Our aim is to investigate the functorial properties of the group of homomorphisms $\mathrm{Hom}(M_R, N_R)$ as M and N vary through the category $\mathcal{M}\mathit{od}_R$ of right R-modules, where R is a ring.

We find that certain properties, such as the injectivity or surjectivity of

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an R-module homomorphism, or the injectivity or projectivity of a module, can be reinterpreted in terms of the behaviour of the functors $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$. Such results are useful not only for their applications in module theory, but also because they suggest a method whereby concepts such as injectivity and projectivity can be extended to abstract categories, provided the morphism sets in these categories share enough of the properties of $\mathrm{Hom}(-, -)$. This theme will be developed in the next section. The Hom functors are basic to the study of homology, and their introduction in [Eilenberg & Mac Lane 1942] predates category theory.

As usual, we work with categories of right modules, except in a few places where we are obliged to consider left modules. The transcription of the main part of our discussion to left modules is not quite immediate, although straightforward. The complication arises since we usually view $\mathrm{Hom}(M, N)$ as a member of the right category $\mathcal{A}\mathcal{B}$, regardless of the chirality of the modules M and N. There are also some awkward but essentially trivial variations in notation which arise when we need to regard $\mathrm{Hom}(M, N)$ as a left module over some ring.

2.1.1 Basic properties

We start with a discussion of the functorial properties of the sets of homomorphisms between modules.

Let R be a ring and let L_R and X_R be right R-modules. Since $\mathrm{Hom}(L_R, X_R)$ is simply the set $\mathrm{Mor}_{\mathcal{C}}(L, X)$ where the category $\mathcal{C}$ is the category $\mathcal{M}\mathit{od}_R$ of right R-modules, the discussion in (1.2.7) shows that Hom defines a pair of functors $\mathrm{Hom}(-, X)$ and $\mathrm{Hom}(L, -)$, which are respectively contravariant and covariant.

Explicitly, given a right R-module homomorphism $\lambda : L \to M$, the map

$$\lambda^* : \mathrm{Hom}(M_R, X_R) \longrightarrow \mathrm{Hom}(L_R, X_R)$$

is given by $\lambda^*(\xi) = \xi\lambda$. If $\mu : M \to N$ is another right R-module homomorphism, then

$$(\mu\lambda)^* = (\lambda)^*(\mu)^* : \mathrm{Hom}(N_R, X_R) \longrightarrow \mathrm{Hom}(L_R, X_R).$$

On the other hand, for any R-module homomorphism $\xi : X \to Y$, the map

$$\xi_* : \mathrm{Hom}(L_R, X_R) \longrightarrow \mathrm{Hom}(L_R, Y_R)$$

is given by $\xi_*(\alpha) = \xi\alpha$, and $(\eta\xi)_* = \eta_*\xi_*$ for an R-module homomorphism $\eta : Y \to Z$.

Each set $\mathrm{Hom}(L_R, X_R)$ is an abelian group, with addition given by the rule

$$(\alpha + \alpha')\ell = \alpha\ell + \alpha'\ell \text{ for } \ell \in L \text{ and } \alpha, \alpha' \in \mathrm{Hom}(L_R, X_R),$$

and it is easy to verify that the maps λ^* and ξ_* are homomorphisms of abelian groups, that is,

$$\lambda^*(\alpha + \alpha') = \lambda^*\alpha + \lambda^*\alpha' \text{ and } \xi_*(\alpha + \alpha') = \xi_*\alpha + \xi_*\alpha'$$

for $\alpha, \alpha' \in \mathrm{Hom}(L_R, X_R)$.

Thus $\mathrm{Hom}(-, X)$ and $\mathrm{Hom}(L, -)$ are functors with values in the category $\mathcal{A}\mathcal{B}$ of abelian groups rather than the category $\mathcal{S}\mathcal{ET}$ of sets. Furthermore, the equalities

$$(\lambda + \lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi + \xi')_* = \xi_* + \xi'_*$$

hold for pairs of homomorphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$ and, as in (1.3.2)(i), each pair of homomorphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(M_R, Y_R) \\ \downarrow \lambda^* & & \downarrow \lambda^* \\ \mathrm{Hom}(L_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(L_R, Y_R) \end{array}$$

which can be interpreted as showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Hom}(M_R, -)$ to the functor $\mathrm{Hom}(L_R, -)$, or, equally, that the homomorphisms ξ_* give a natural transformation from $\mathrm{Hom}(-, X_R)$ to $\mathrm{Hom}(-, Y_R)$.

2.1.2 Exact sequences

We now interpret the exactness of sequences of modules in terms of the Hom functors. By definition, a sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

of right R-modules and right R-module homomorphisms is *exact* if

$$\mathrm{Ker}(\mu) = \mathrm{Im}(\lambda).$$

More generally, a sequence of right R-modules

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \longrightarrow \cdots$$

is called an exact sequence if it is exact at each of its *terms* M_i. Such a sequence may have finite length, or it may be infinite. (A more detailed treatment of exact sequences can be found in [BK: IRM], particularly section 2.4.) Thus the sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M$$

is exact precisely when the homomorphism λ is injective, while

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact precisely when μ is surjective.

2.1.3 Proposition

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y$$

is exact in $\mathcal{M}\textsc{od}_R$ *if and only if*

$$0 \longrightarrow \operatorname{Hom}(L, X) \xrightarrow{\xi_*} \operatorname{Hom}(L, Y)$$

is an exact sequence of abelian groups for any right R-module L.

(i)$^{\text{op}}$ *The sequence*

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact in $\mathcal{M}\textsc{od}_R$ *if and only if*

$$0 \longrightarrow \operatorname{Hom}(N, X) \xrightarrow{\mu^*} \operatorname{Hom}(M, X)$$

is an exact sequence of abelian groups for any right R-module X.

Proof

We prove (i) only, (i)$^{\text{op}}$ being similar. Suppose that ξ is injective and that $\xi_*\alpha = 0, \alpha : L \to X$. By definition, this means that $\xi\alpha = 0$, and so $\alpha = 0$.

Conversely, suppose that ξ_* is injective. Choose any element $x \in \operatorname{Ker}\xi$, put $L = xR$ and let $\alpha : L \to X$ be the inclusion map. Then $\xi_*\alpha = 0$, hence $\alpha = 0$, and so $xR = 0$, which means that $x = 0$. □

Next we examine what happens when we extend the exact sequences by one term. The assertion that

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

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is an exact sequence of R-modules tells us that X is isomorphic via ξ to the kernel of η, while the exactness of

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

is equivalent to the fact that N is isomorphic via μ to the cokernel of λ.

2.1.4 Proposition

(i) *A sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z)$$

is exact for any right R-module L.

(i)$^{\mathrm{op}}$ *A sequence*

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X)$$

is exact for any right R-module X.

Proof

We prove (i) only. Suppose that $0 \to X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$ is exact. By the previous result, ξ_* is injective, so it is enough to check exactness at $\mathrm{Hom}(L, Y)$. For any $\alpha \in \mathrm{Hom}(L, X)$, we have $\eta_*\xi_*\alpha = \eta\xi\alpha = 0$ since $\eta\xi = 0$. Suppose that $\beta \in \mathrm{Hom}(L, Y)$ and $\eta_*\beta = 0$. Then $\eta\beta = 0$, and so there is a homomorphism $\alpha' : L \to \mathrm{Ker}\,\eta$ such that $\kappa\alpha' = \beta$, where κ is the canonical injection of $\mathrm{Ker}\,\eta$ into Y. But, as we noted above, $\mathrm{Ker}\,\eta$ is isomorphic via ξ to X, so we can construct a homomorphism $\alpha : L \to X$ with $\beta = \xi\alpha = \xi_*\alpha$.

Conversely, suppose that the Hom sequence is exact. Again, the previous result tells us that ξ is injective. Substituting X for L, we see that $0 = \eta_*\xi_*(id_X) = \eta\xi$, that is, $\xi X \subseteq \mathrm{Ker}\,\eta$.

To obtain equality, take an element $y \in \mathrm{Ker}\,\eta$ and put $L = yR$. Let $\beta : L \to Y$ be the inclusion map. Since $\eta_*\beta = 0$, $\beta = \xi_*\alpha = \xi\alpha$ for some $\alpha : L \to X$, which shows that $y \in \xi X$. □

2.1.5 Short exact sequences

A *short exact sequence* is a sequence

$$0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

which is exact at each of the terms M', M and M''. There is an evident extension of the statement of the preceding result in which all the sequences are taken to be short exact sequences, but the result then fails, as can be seen from the following example.

Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } \mathbb{Z} \xrightarrow{\ \pi\ } \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in which ι is left multiplication by 2: $z \mapsto 2z$, and π is the canonical surjection. This induces a Hom sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\ \pi^*\ } \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\ \iota^*\ } \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0.$$

Now $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ itself under the map which assigns to an endomorphism α of $\mathbb{Z}$ its value $\alpha(1)$ at 1, and it is not hard to see that the endomorphism of $\mathbb{Z}$ induced by ι^* is ι again and hence non-surjective. Thus the Hom sequence is not exact on the right.

For the dual example, we look at

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\ \iota_*\ } \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\ \pi_*\ } \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

The terms $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ are evidently 0, while $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So again the sequence fails to be exact on the right.

There are several interesting circumstances in which the Hom sequences are guaranteed to be short exact sequences. The first is the most immediate extension of the last proposition. To state the result, we need the following definition. A short exact sequence of R-modules (or of abelian groups)

$$0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

is said to be a *split exact sequence* if we can find homomorphisms

$$\gamma : M \longrightarrow M' \ \text{ and } \ \delta : M'' \longrightarrow M$$

such that

$$\beta\delta = id_{M''}, \ \ \gamma\alpha = id_{M'} \ \text{ and } \ \alpha\gamma + \delta\beta = id_M.$$

It can be shown ([BK: IRM] Theorem 2.4.5) that it is enough to know the existence of only one of the maps γ or δ, the other then being constructible, and that when the sequence is split, we have $M \cong M' \oplus M''$.

Conversely, if we are given a direct sum decomposition of M, then we can reconstruct a split exact sequence as above.

2.1.6 Proposition

(a) *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\ \xi\ } Y \xrightarrow{\ \eta\ } Z \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L,X) \xrightarrow{\ \xi_*\ } \mathrm{Hom}(L,Y) \xrightarrow{\ \eta_*\ } \mathrm{Hom}(L,Z) \longrightarrow 0$$

is split exact for any right R-module L.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L,X) \xrightarrow{\ \xi_*\ } \mathrm{Hom}(L,Y) \xrightarrow{\ \eta_*\ } \mathrm{Hom}(L,Z) \longrightarrow 0$$

is exact for any right R-module L.

(a)$^{\mathrm{op}}$ *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow L \xrightarrow{\ \lambda\ } M \xrightarrow{\ \mu\ } N \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{\ \mu^*\ } \mathrm{Hom}(M,X) \xrightarrow{\ \lambda^*\ } \mathrm{Hom}(L,X) \longrightarrow 0$$

is split exact for any right R-module X.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{\ \mu^*\ } \mathrm{Hom}(M,X) \xrightarrow{\ \lambda^*\ } \mathrm{Hom}(L,X) \longrightarrow 0$$

is exact for any right R-module X.

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Proof

We prove (a) only. Suppose that (i) holds, that is,

$$0 \longrightarrow X \xrightarrow{\ \xi\ } Y \xrightarrow{\ \eta\ } Z \longrightarrow 0$$

is split exact. Then there are R-module homomorphisms $\vartheta : Y \to X$ and $\zeta : Z \to Y$ such that

$$\vartheta\xi = id_X,\ \eta\zeta = id_Z \text{ and } \xi\vartheta + \zeta\eta = id_Y.$$

Since the identity map id_A of a module induces the identity map $(id_A)_*$ on a homomorphism group $\mathrm{Hom}(L, A)$, we have

$$\vartheta_*\xi_* = id,\ \eta_*\zeta_* = id \text{ and } \xi_*\vartheta_* + \zeta_*\eta_* = id,$$

with the identity maps appropriately interpreted. This shows that (ii) holds.

Trivially (ii) gives (iii). Assume that (iii) holds. Taking the special case $L = Z$, we see that there is some homomorphism $\zeta : Z \to Y$ with $\xi_*(\zeta) = id_Y$, that is, $\xi\zeta = id_Y$, which establishes that the module sequence is split exact. □

2.1.7 Projective and injective modules

An alternative procedure for ensuring that the Hom sequences are short exact is to impose conditions on the 'fixed' term rather than the exact sequence of 'variables'. This leads to a characterization of projective modules and their duals, the injective modules.

2.1.8 Proposition

(a) *Let L be a right R-module. Then the following are equivalent.*

(i) *L is projective.*

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\ \xi_*\ } \mathrm{Hom}(L, Y) \xrightarrow{\ \eta_*\ } \mathrm{Hom}(L, Z) \longrightarrow 0$$

is exact for every exact sequence

$$0 \longrightarrow X \xrightarrow{\ \xi\ } Y \xrightarrow{\ \eta\ } Z \longrightarrow 0$$

of right R-modules.

(a)$^{\mathrm{op}}$ *Let X be a right R-module. Then the following are equivalent.*

(i) *X is injective.*

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(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{\mu^*} \mathrm{Hom}(M,X) \xrightarrow{\lambda^*} \mathrm{Hom}(L,X) \longrightarrow 0$$

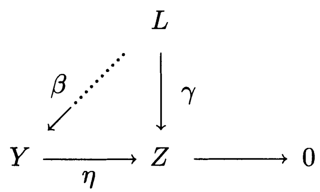
is exact for every exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules.

Proof

For (a), we note that the surjectivity of η_* is the same as the assertion that, for any homomorphism $\gamma : L \to Z$, there is a homomorphism $\beta : L \to Y$ which makes the following diagram commute:



But this is a standard criterion for L to be projective (see [BK: IRM] Theorem 2.5.4).

For (a)$^{\mathrm{op}}$, use the dual characterization of injective modules (Exercise 2.5.7 of [BK: IRM]). □

2.1.9 Homomorphism functors arising from bimodules

So far, we have regarded $\mathrm{Hom}(M_R, X_R)$ simply as an abelian group, so that both the functors $\mathrm{Hom}(-, X_R)$ and $\mathrm{Hom}(M_R, -)$ take values in the right category $\mathcal{A}$B. However, it sometimes happens that we wish to take account of a further module structure on $\mathrm{Hom}(M_R, X_R)$ under which it may be most naturally viewed as a left module, and then our functors will take values in a left category. We therefore introduce a separate notation to alert the reader to our changed point of view.

Suppose that ${}_SX_R$ is an S-R-bimodule for some rings R and S. Then $\mathrm{Hom}(M_R, X_R)$ has a natural left S-module structure given by

$$(s\alpha)(m) = s(\alpha m), \quad s \in S, \ m \in M, \ \alpha : M \to X.$$

Thus $\mathrm{Hom}(-, X_R)$ is a contravariant functor from $\mathcal{M}\mathrm{OD}_R$ to ${}_S\mathcal{M}\mathrm{OD}$, which we write as $H_X(-)$ to distinguish it from its more common manifestation as a functor with values in $\mathcal{A}$B.

If N is another right R-module and $\mu : M \to N$ is an R-module homomorphism, we define

$$H_X(\mu) : H_X(N) \longrightarrow H_X(M) \text{ by } \alpha H_X(\mu) = \alpha\mu.$$

In the language of (1.2.6), H_X is a contravariant and contrachiral functor from $\mathcal{M}od_R$ to ${}_S\mathcal{M}od$, with $H_X(\mu\nu) = H_X(\mu)H_X(\nu)$ for composable μ and ν.

Now assume that M is a T-R-bimodule for some ring T. Then $\mathrm{Hom}(M_R, X_R)$ becomes a right T-module for any right R-module X, under the rule $(\alpha t)m = \alpha(tm)$ for $\alpha \in \mathrm{Hom}(M_R, X_R), t \in T, m \in M$.

We write $H^M(-) = \mathrm{Hom}(M_R, -)$. The action of this functor on morphisms is given by $H^M(\xi)\alpha = \xi\alpha$, where $\alpha \in \mathrm{Hom}(M_R, X_R)$ and $\xi \in \mathrm{Hom}(X_R, Y_R)$, and Y is a right R-module. We see that $H^M(\eta)H^M(\xi) = H^M(\eta\xi)$ whenever the composition is defined, and that $H^M(-)$ is a covariant, cochiral functor from $\mathcal{M}od_R$ to $\mathcal{M}od_T$.

Note that these constructions include the case that X or M is simply an R-module, since any right R-module can be viewed as a $\mathbb{Z}$-R-bimodule. The functor H^M is then the same as the functor $\mathrm{Hom}(M_R, -)$, both having values in the right category $\mathcal{A}b$, which is $\mathcal{M}od_{\mathbb{Z}}$ under another name. However, the functor H_X takes values in the mirror category ${}_{\mathbb{Z}}\mathcal{M}od$ of $\mathcal{A}b$, so that $\mathrm{Mor}(-, X_R) = \mathrm{Mir} \circ H_X(-)$: compare with (1.1.5) and (1.2.6).

We can also regard both M and X simultaneously as being variables for Hom, which is then a bifunctor from $\mathcal{M}od_R \times \mathcal{M}od_R$ to $\mathcal{A}b$. More generally, Hom gives rise to bifunctors from $\mathcal{M}od_R \times {}_S\mathcal{M}od_R$ to ${}_S\mathcal{M}od$ and from ${}_T\mathcal{M}od_R \times \mathcal{M}od_R$ to $\mathcal{M}od_T$. These will have mixed variance and chirality.

2.1.10 The extension functors

As we saw in (2.1.4), an exact sequence

$$0 \longrightarrow X \xrightarrow{\ \xi\ } Y \xrightarrow{\ \eta\ } Z \longrightarrow 0$$

of right R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\ \xi_*\ } \mathrm{Hom}(L, Y) \xrightarrow{\ \eta_*\ } \mathrm{Hom}(L, Z)$$

for any R-module L. By the example in (2.1.5), the Hom sequence cannot always be extended on the right to form a short exact sequence.

This fault can be repaired by introducing the *extension functors* $\mathrm{Ext}^n_R(-,-)$, $n \geq 1$, which, like $\mathrm{Hom}_R(-,-)$, are bifunctors from $\mathcal{M}od_R \times \mathcal{M}od_R$ to $\mathcal{A}b$.

There are then long exact sequences of the form

$$\begin{array}{lllllll} 0 & \longrightarrow & \mathrm{Hom}(L,X) & \longrightarrow & \mathrm{Hom}(L,Y) & \longrightarrow & \mathrm{Hom}(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^1(L,X) & \longrightarrow & \mathrm{Ext}^1(L,Y) & \longrightarrow & \mathrm{Ext}^1(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^2(L,X) & \longrightarrow & \cdots & & & \end{array}$$

which may continue for ever to the right, depending on the nature of the ring R and the particular modules involved. There are analogous long exact sequences in which the roles of the variables are exchanged.

The term 'extension functor' is used since the elements of $\mathrm{Ext}^1_R(L,X)$ describe all the possible extensions of L by X, that is, all short exact sequences

$$0 \longrightarrow X \longrightarrow M \longrightarrow L \longrightarrow 0$$

with M a right R-module, under a suitable equivalence relation. The higher extension groups $\mathrm{Ext}^n_R(L,X)$ have a similar interpretation.

The construction of these functors is outlined in Exercise 2.1.7 below.

Exercises

2.1.1 Let $_RM$ and $_RX$ be left R-modules. Following (2.1.1), define functors $\mathrm{Hom}(_RM,-) : {_R\mathcal{M}od} \to \mathcal{A}b$ and $\mathrm{Hom}(-,{_RX}) : {_R\mathcal{M}od} \to \mathcal{A}b$, and determine their chiralities.

Following (2.1.9), show that a bimodule $_RM_S$ defines a functor $H^M(-) : {_R\mathcal{M}od} \to {_S\mathcal{M}od}$ and that a bimodule $_RX_T$ defines a functor $H_X(-) : {_R\mathcal{M}od} \to \mathcal{M}od_T$. Give the chiralities of these functors.

Thus $\mathrm{Hom}(-,-)$ affords an example of a bifunctor of mixed variance and chirality.

2.1.2 Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of right R-modules, where the index set Λ may be infinite. Show that, for any right R-modules L and X, there are isomorphisms of abelian groups

$$\mathrm{Hom}(\textstyle\bigoplus_\Lambda M_\lambda, X) \cong \prod_\Lambda \mathrm{Hom}(M_\lambda, X)$$

and

$$\mathrm{Hom}(L, \textstyle\prod_\Lambda M_\lambda) \cong \prod_\Lambda \mathrm{Hom}(L, M_\lambda)$$

Show that the inverse of the second of the above isomorphisms restricts to an injective homomorphism

$$\textstyle\bigoplus_\Lambda \mathrm{Hom}(L, M_\lambda) \longrightarrow \mathrm{Hom}(L, \bigoplus_\Lambda M_\lambda),$$

which is in fact an isomorphism when L is finitely generated.

Warning. The description of the fourth term $\mathrm{Hom}(\prod_\Lambda M_\lambda, X)$ is

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far from straightforward, since it requires some notion of convergence. A related question is treated in Exercise 2.3.1 of [BK: IRM].

2.1.3 Regard the ring R as an R-R-bimodule, and let M be a right R-module. Define a map $\epsilon : M_R \to \mathrm{Hom}(R_R, M_R)$ by $\epsilon(m)(r) = mr$. Show that ϵ is an isomorphism of right R-modules, and deduce that the functor $\mathrm{Hom}(R_R, -) = H^R(-)$ is naturally isomorphic to the identity functor on $\mathcal{M}od_R$.

Verify the left-handed version of this result.

(The map ϵ is sometimes called *left multiplication by* m; its inverse is *evaluation at* 1.)

2.1.4 Show that an arbitrary direct product of injective R-modules is again injective.

2.1.5 **Injectives exist**

This longish exercise outlines the proof that, for any ring R and right R-module M, there is an injective right R-module N and an injective R-module homomorphism from M to N.

(i) Let Q be any divisible $\mathbb{Z}$-module. Then it is known that Q is an injective $\mathbb{Z}$-module (see Exercise 2.5.11 of [BK: IRM]). Put $\widehat{R} = \mathrm{Hom}(R_{\mathbb{Z}}, Q_{\mathbb{Z}})$. Verify that $\widehat{R}$ is a right R-module by $(\rho r)(x) = \rho(rx)$, for ρ in $\widehat{R}$, r, x in R.

(ii) Let $\mathfrak{a}$ be a right ideal of R, let $\iota : \mathfrak{a} \to R$ be the inclusion map, and let $\varphi : \mathfrak{a} \to \widehat{R}$ be an R-module homomorphism. It is a fact that $\widehat{R}$ is injective if for any such φ there is a homomorphism $\psi : R \to \widehat{R}$ with $\psi\iota = \varphi$ – see Exercise 2.5.8 in [BK: IRM].
Let $a \in \mathfrak{a}$. Verify that $\gamma : \mathfrak{a} \to Q$, $\gamma(a) = (\varphi a)(1_R)$, is a homomorphism of $\mathbb{Z}$-modules. Choose $\delta : R_{\mathbb{Z}} \to Q_{\mathbb{Z}}$ such that $\delta\iota = \gamma$ and put $(\psi r)(x) = \delta(rx)$. Verify that $\psi r \in \widehat{R}$ and that ψ is as required.

(iii) Let M be any right R-module. Using the fact that there is an isomorphism of $\mathbb{Z}$-modules $M \cong \mathbb{Z}^{\Lambda}/A$ for some free abelian group $\mathbb{Z}^{\Lambda}$ and subgroup A of $\mathbb{Z}^{\Lambda}$, and the divisibility of the group $\mathbb{Q}$ of additive rationals, show that M is a subgroup of a divisible abelian group Q.

(iv) Confirm that there is a chain of injective R-module homomorphisms

$$M \rightarrowtail \mathrm{Hom}(R, M) \rightarrowtail \mathrm{Hom}(R, Q) = \widehat{R}.$$

Remark. It can be shown that an R-module M can be embedded in a 'minimal' injective R-module $E(M)$; this means that there is an injective R-module homomorphism $\epsilon : M \to E(M)$ such that for any other injective R-module homomorphism $\eta : M \to I$ with I an injective module, there is an injective homomorphism $\rho : E(M) \to I$ with $\eta = \rho\epsilon$. As a universal object, the module $E(M)$ is unique up to isomorphism and is known as the *injective hull* of M. The construction can be found in most texts on ring theory, for instance [Rotman 1979], §3.

2.1.6 Let Q be a divisible abelian group. For a left R-module N, let $\Delta(N) = \mathrm{Hom}(N_{\mathbb{Z}}, Q_{\mathbb{Z}})$. Verify that $\Delta(N)$ is a right R-module, with $\Delta(R) = \widehat{R}$ as in the previous exercise.

Confirm that Δ is a contravariant functor from ${}_R\mathcal{M}\textsc{od}$ to $\mathcal{M}\textsc{od}_R$ and that Δ restricts to a functor from ${}_R\mathcal{P}\textsc{roj}$ to $\mathcal{I}\textsc{nj}_R$, the category of injective right R-modules.

2.1.7 **The group** Ext^1_R

This substantial exercise gives one construction, using the Baer sum, of the groups $\mathrm{Ext}^1_R(M'', M')$. Our treatment is based on Chapter III, §2, of [Mac Lane 1975]. Alternative definitions, and further details, can be found in most texts on homological algebra (see, for example, [Weibel 1995]).

Our arguments depend on some basic methods of manipulating short exact sequences that are described in more detail in section 2.4 of [BK: IRM]. Let

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

be a short exact sequence and let $\theta : L'' \to M''$ be a homomorphism. Then the pull-back of M and L'' over M'' gives an exact sequence

$$\theta^*\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\ \mu\ } M \times_{M''} L'' \xrightarrow{\ \overline{\beta}\ } L'' \longrightarrow 0,$$

the *pull-back* of $\mathbf{E}$ along θ. (An alternative expression is that $\theta^*\mathbf{E}$ is obtained from $\mathbf{E}$ by *base change.*)

Similarly, given a homomorphism $\phi : M' \to N'$, the push-out of N' and M over M' gives a short exact sequence

$$\phi_*\mathbf{E} \qquad 0 \longrightarrow N' \xrightarrow{\ \overline{\alpha}\ } N' \oplus_{M'} M \xrightarrow{\ \nu\ } M'' \longrightarrow 0,$$

the *push-out* of $\mathbf{F}$ along ϕ. (Alternatively, $\phi_*\mathbf{E}$ is said to be obtained from $\mathbf{E}$ by *cobase change.*)

Two short exact sequences

$$\mathbf{E}_1 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} M'' \longrightarrow 0$$

and

$$\mathbf{E}_2 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_2} M_2 \xrightarrow{\beta_2} M'' \longrightarrow 0$$

are *equivalent* if there is an isomorphism $\chi : M_1 \to M_2$ with $\chi\alpha_1 = \alpha_2$ and $\beta_2\chi = \beta_1$. The notation is $\mathbf{E}_1 \equiv \mathbf{E}_2$.

For fixed M' and M'', equivalence of short exact sequences is an eqivalence relation on the set of all short exact sequences $\mathbf{E}$, and it is compatible with pull-backs and push-outs.

We write $[\mathbf{E}]$ for the class of $\mathbf{E}$ and we define $\mathrm{Ext}^1_R(M'', M')$ to be the set of all such equivalence classes.

The direct sum $\mathbf{E}_1 \oplus \mathbf{E}_2$ is defined to be the short exact sequence

$$0 \longrightarrow M' \oplus M' \xrightarrow{\alpha_1 \oplus \alpha_2} M_1 \oplus M_2 \xrightarrow{\beta_1 \oplus \beta_2} M'' \oplus M'' \longrightarrow 0.$$

Define an operation, the *Baer sum*, on $\mathrm{Ext}^1_R(M'', M')$, by

$$[\mathbf{E}_1] + [\mathbf{E}_2] = [(\Sigma')_*(\Delta'')^*(\mathbf{E}_1 \oplus \mathbf{E}_2)],$$

where the diagonal homomorphism

$$\Delta'' : M'' \longrightarrow M'' \oplus M'' \text{ has } \Delta'' m'' = (m'', m'')$$

and the codiagonal homomorphism

$$\Sigma' : M' \oplus M' \longrightarrow M' \text{ has } \Sigma'(m', n') = m' + n'.$$

(i) Verify that the Baer sum depends only on the equivalence classes of $\mathbf{E}_1$ and $\mathbf{E}_2$, so that it is a well-defined law of composition on $\mathrm{Ext}^1_R(M'', M')$.

(ii) Confirm that the middle term of $\mathbf{E}_1 + \mathbf{E}_2$ is

$$(M_1 \times_{M''} M_2)/\{(\alpha_1 m', -\alpha_2 m') \mid m' \in M'\}.$$

(iii) Verify that the composition is associative and commutative.

(iv) Deduce that $\mathrm{Ext}^1_R(M'', M')$ is an abelian group, with zero element corresponding to the standard split exact sequence, and

$$-[\mathbf{E}] = [(-id)_*\mathbf{E}]$$

where $-id$ is the negative of the identity map on M'.

(v) Let $\phi, \psi : L'' \to M''$. Verify the following chain of equivalences:

$$\begin{aligned}\phi^*\mathbf{E} + \psi^*\mathbf{E} &\equiv (\Sigma')_*(\Delta'')^*(\phi^*\mathbf{E} \oplus \psi^*\mathbf{E}) \\ &\equiv (\Sigma')_*(\Delta'')^*(\phi \oplus \psi)^*(\mathbf{E} \oplus \mathbf{E}) \\ &\equiv (\Sigma''(\phi \oplus \psi)\Delta'')^*\mathbf{E} \\ &\equiv (\phi + \psi)^*\mathbf{E}.\end{aligned}$$

Deduce that $\operatorname{Ext}^1_R(-, M')$ is a contravariant functor. Show also that $\operatorname{Ext}^1_R(M'', -)$ is a covariant functor and that $\operatorname{Ext}^1_R(-,-)$ is a bifunctor.

(vi) Show that $\operatorname{Ext}^1_R(P, X) = 0$ for all R-modules X if and only if P is a projective R-module.

(vii) Let

$$\mathbf{Y} \qquad 0 \longrightarrow X \xrightarrow{\ \xi\ } Y \xrightarrow{\ \eta\ } Z \longrightarrow 0$$

be an exact sequence of right R-modules. For $\lambda \in \operatorname{Hom}(L, Z)$, define

$$\partial(\lambda) = [\lambda^*\mathbf{Y}] \in \operatorname{Ext}^1_R(Z, X),$$

and, anticipating the Snake Lemma (Exercise 2.3.13), verify that

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(L,X) &\longrightarrow \operatorname{Hom}_R(L,Y) \longrightarrow \operatorname{Hom}_R(L,Z) \\ \xrightarrow{\ \partial\ } \operatorname{Ext}^1_R(L,X) &\longrightarrow \operatorname{Ext}^1_R(L,Y) \longrightarrow \operatorname{Ext}^1_R(L,Z)\end{aligned}$$

is an exact sequence of abelian groups.

(viii) Derive the corresponding exact sequence with fixed second variable, and show that $\operatorname{Ext}^1_R(L, J) = 0$ for all R-modules L if and only if J is an injective R-module.

(ix) Here is an inductive definition of $\operatorname{Ext}^n_R(L, X)$ for $n \geq 2$. Choose a short exact sequence

$$0 \longrightarrow S \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P projective (such a sequence exists; we may take P to be free by Lemma 2.5.7 of [BK: IRM]), and put

$$\operatorname{Ext}^n_R(L, X) = \operatorname{Ext}^{n-1}_R(S, L) \ \text{ for } \ n \geq 2.$$

It can be shown that the right-hand side is independent of the choice of the short exact sequence above.

It is also possible to make a definition in terms of the second variable by taking a short exact sequence

$$0 \longrightarrow X \longrightarrow J \longrightarrow C \longrightarrow 0$$

with J injective (see Exercise 2.1.5 above). Much hard work reveals that the two definitions coincide, or, more properly, that the functors obtained are naturally isomorphic. Full details are given in [Mac Lane 1975] and [Rotman 1979].

2.2 ADDITIVE CATEGORIES

When [Eilenberg & Mac Lane 1945] had given the abstract definition of a category, a natural next step was to seek the conditions on the morphism sets of an abstract category which enable the definitions and constructions of module theory to be developed in that category. The exploration led to the definitions of additive, exact (in the non-K-theory sense) [Buchsbaum 1955] and abelian categories [Grothendieck 1957]. These abstractions were motivated by the desire to apply homological methods in categories that arise in algebraic geometry, such as categories of sheaves and of vector bundles (which we do not encounter in this text), as well as intellectual curiosity. The widest class of categories that share some properties of modules comprises the preadditive categories. In such a category $\mathcal{C}$, each morphism set $\mathrm{Mor}_{\mathcal{C}}(L, X)$ is an abelian group and the functors arising from morphisms must respect addition. These requirements suffice to permit the recognition of direct sums, exact sequences, kernels and cokernels, and projective and injective objects in $\mathcal{C}$. However, there need not be any nontrivial examples of such things within the category.

The next step is the definition of an additive category, which is required to contain a direct sum of any two of its objects. Many interesting module categories are additive categories, for example, the categories $\mathcal{M}_R$ of finitely generated modules and $\mathcal{P}_R$ of finitely generated projectives. Also, many of the categories of interest in K-theory are additive categories.

There is some variation in the literature as to the precise definition of a preadditive category, as there is with the definitions of (pre)additive subcategories of (pre)additive categories. Indeed, some authorities do not make formal definitions. Even where terms like additive and abelian categories have standard meanings, there is a complex web of interacting axioms woven through the literature. Our own choice of axioms has been guided by intuitive appeal rather than minimality.

Throughout this section, we work with right categories, it being clear that

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there is a corresponding discussion for left categories. Modules are right modules over an arbitrary ring R, unless otherwise specified.

To make the comparison and transcription of results easier, we often use the same symbols – $L, M, \dots, X, Y, \dots$ – for objects of an abstract category $\mathcal{C}$ as we used for modules in the preceding section.

2.2.1 Preadditive categories

A category $\mathcal{C}$ is called *preadditive* if the following conditions are satisfied.

Ad 1. For each pair of objects M and N of $\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(M, N)$ is an abelian group.

Ad 2. If $\lambda, \lambda' : L \to M$ and $\mu, \mu' : M \to N$ are morphisms in $\mathcal{C}$, then

$$\mu(\lambda + \lambda') = \mu\lambda + \mu\lambda' \text{ and } (\mu + \mu')\lambda = \mu\lambda + \mu'\lambda.$$

Ad 3. $\mathcal{C}$ contains a zero object 0 such that $\mathrm{Mor}_{\mathcal{C}}(0, N) = 0$ and $\mathrm{Mor}_{\mathcal{C}}(M, 0) = 0$, the trivial group, for all objects M and N.

Here, a zero object is as defined in (1.4.14). There is (usually) more than one zero object, but any two zero objects are isomorphic by a unique isomorphism, so we often refer to 'the' zero object of $\mathcal{C}$.

Note that the set $\mathrm{End}_{\mathcal{C}}(M)$ of endomorphisms of any object M in $\mathcal{C}$ must be a ring.

Of course, $\mathcal{A}\mathcal{B}$ is a preadditive category, and so is the category $\mathcal{R}\mathcal{NG}$ of nonunital rings. However, the category $\mathcal{R}\mathcal{ING}$ of rings is *not* preadditive because the zero mapping is not in general a ring homomorphism. It is clear that $\mathcal{M}\mathcal{OD}_R$ is preadditive for any ring R, and it is easy to see that the basic functorial properties of $\mathrm{Hom}_R(-,-)$, given in (2.1.1), transcribe to $\mathrm{Mor}_{\mathcal{C}}(-,-)$. We list them for convenience.

A morphism $\lambda : L \to M$ in $\mathcal{C}$ defines a map

$$\lambda^* : \mathrm{Mor}_{\mathcal{C}}(M, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, X) \text{ by } \lambda^*(\xi) = \xi\lambda.$$

From Ad 2, λ^* is a homomorphism of abelian groups. For another morphism $\mu : M \to N$,

$$(\mu\lambda)^* = \lambda^*\mu^* : \mathrm{Mor}_{\mathcal{C}}(N, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, X).$$

In the other variable, a morphism $\xi : X \to Y$ gives a homomorphism of abelian groups

$$\xi_* : \mathrm{Mor}_{\mathcal{C}}(L, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, Y) \text{ by } \xi_*(\alpha) = \xi\alpha,$$

and $(\eta\xi)_* = \eta_*\xi_*$ for a morphism $\eta : Y \to Z$.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,X)$ and $\mathrm{Mor}_{\mathcal{C}}(L,-)$ are functors on $\mathcal{C}$, contravariant and covariant respectively, with values in the category $\mathcal{A}\mathcal{B}$ of abelian groups. As with the Hom functor, in a preadditive category we have equalities

$$(\lambda+\lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi+\xi')_* = \xi_* + \xi'_*$$

for pairs of morphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$. Each pair of morphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}(M,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(M,Y) \\ \downarrow \lambda^* & & \downarrow \lambda^* \\ \mathrm{Mor}(L,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(L,Y) \end{array}$$

showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Mor}(M,-)$ to the functor $\mathrm{Mor}(L,-)$ and that the homomorphisms ξ_* give a natural transformation from $\mathrm{Mor}(-,X)$ to $\mathrm{Mor}(-,Y)$.

2.2.2 Preadditive subcategories

Next we define a preadditive subcategory of a preadditive category. There is no general agreement as to the proper definition, so we have chosen one suited to our immediate purposes.

A subcategory $\mathcal{D}$ of a preadditive category $\mathcal{C}$ is said to be a *preadditive subcategory* if

PAS 1. $\mathrm{Mor}_{\mathcal{D}}(L,X)$ is a subgroup of $\mathrm{Mor}_{\mathcal{C}}(L,X)$ for any pair of objects L and X of $\mathcal{D}$, and

PAS 2. $\mathcal{D}$ contains a zero object of $\mathcal{C}$.

Evidently, such a subcategory $\mathcal{D}$ is itself a preadditive category.

The zero object of a preadditive category can be viewed as defining a preadditive subcategory with one object. Less trivially, a nonzero object X of $\mathcal{C}$ gives a minimal full preadditive subcategory $\mathcal{P}\mathrm{READD}(X)$ of $\mathcal{C}$; the set of objects of $\mathcal{P}\mathrm{READD}(X)$ is $\{0,X\}$ and the set of morphisms is $\mathrm{End}_{\mathcal{C}}(X)$ together with the requisite zero morphisms.

2.2.3 Monomorphisms and epimorphisms

We now use the Mor functors to transcribe definitions and results from module categories $\mathcal{M}\mathrm{OD}_R$ to arbitrary preadditive categories. The fact that the Mor

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functors on a preadditive category take values in $\mathcal{A}\mathcal{B}$ (rather than $\mathcal{S}\mathcal{E}\mathcal{T}$) is crucial in what follows.

We start with monomorphisms and epimorphisms, which are the analogues of injective and surjective homomorphisms – the reason for the change in terminology will be explained shortly.

By analogy with (2.1.3), we make the following definitions in a preadditive category $\mathcal{C}$.

A morphism $\mu : X \to Y$ in $\mathcal{C}$ is said to be a *monomorphism* if, for any object L of $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, X) \xrightarrow{\mu_*} \mathrm{Mor}_{\mathcal{C}}(L, Y)$$

is an exact sequence of abelian groups.

A morphism $\epsilon : M \to N$ in $\mathcal{C}$ is said to be an *epimorphism* if, for any object X of $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(N, X) \xrightarrow{\epsilon^*} \mathrm{Mor}_{\mathcal{C}}(M, X)$$

is an exact sequence of abelian groups.

Important examples of monomorphisms and epimorphisms are provided by the following result.

2.2.4 Lemma

If $\mu : X \to Y$ has a left inverse $\rho : Y \to X$ (that is, $\rho\mu = id_X$), then μ is a monomorphism.

If $\epsilon : M \to N$ has a right inverse $\sigma : N \to M$ (that is, $\epsilon\sigma = id_N$), then ϵ is an epimorphism.

Proof

By functoriality of $\mathrm{Mor}_{\mathcal{C}}(L, -)$, $\rho_*\mu_*$ is just the identity homomorphism on the group $\mathrm{Mor}_{\mathcal{C}}(L, X)$. So μ_* is injective. Similarly, ϵ_* is surjective. □

A morphism with a left inverse is often called a split monomorphism, while a morphism with a right inverse is known as a split epimorphism.

Warning. These descriptions are valid only for right categories! The reader may wish to formulate their left category counterparts. (The definitions of epimorphism and monomorphism can be extended to arbitrary categories; see Exercise 2.2.4.)

In a module category $\mathcal{M}\mathit{od}_R$, the basic result (2.1.3) shows that a monomorphism is the same thing as an injective morphism while an epimorphism is just a surjective morphism. However, these coincidences need not happen in an abstract category, which explains the need for new terminology.

The most blatant examples in which a monomorphism cannot be an injection are provided by categories whose objects are not sets. More seriously, it may happen that the objects of a preadditive category are sets, but that a monomorphism need not be an injection or an epimorphism need not be a surjection. Such pathologies are somewhat tricky to find.

We give examples of both phenomena.

2.2.5 Example: the opposite category

Recall from (1.1.6) that the opposite category $\mathcal{C}^{\text{op}}$ of a category $\mathcal{C}$ has objects C^{op} in bijective correspondence with the objects C of $\mathcal{C}$, and that for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\text{op}}$, there is a bijection between the morphisms α in $\text{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\text{Mor}_{\mathcal{C}^{\text{op}}}(D^{\text{op}}, C^{\text{op}})$.

The composition in $\mathcal{C}^{\text{op}}$ is given by the rule

$$(\alpha^{\text{op}})(\beta^{\text{op}}) = (\beta\alpha)^{\text{op}}.$$

If $\mathcal{C}$ is preadditive, then so also is $\mathcal{C}^{\text{op}}$, under the natural definition

$$\alpha^{\text{op}} + \beta^{\text{op}} = (\alpha + \beta)^{\text{op}}.$$

A morphism $\xi : X \to Y$ in $\mathcal{C}$ induces

$$\xi_* : \text{Mor}_{\mathcal{C}}(L, X) \longrightarrow \text{Mor}_{\mathcal{C}}(L, Y),$$

while its opposite counterpart $\xi^{\text{op}} : Y^{\text{op}} \to X^{\text{op}}$ gives

$$(\xi^{\text{op}})^* : \text{Mor}_{\mathcal{C}^{\text{op}}}(X^{\text{op}}, L^{\text{op}}) \longrightarrow \text{Mor}_{\mathcal{C}^{\text{op}}}(Y^{\text{op}}, L^{\text{op}}).$$

For any $\lambda : L \to X$, we have $(\xi^{\text{op}})^*(\lambda^{\text{op}}) = (\xi_*(\lambda))^{\text{op}}$, so that $(\xi^{\text{op}})^*$ and ξ_* are either both injective or both not injective.

Thus ξ^{op} is an epimorphism exactly when ξ is a monomorphism. Similarly, ξ^{op} is a monomorphism precisely when ξ is an epimorphism.

Thus, if $\mathcal{C}$ contains both monomorphisms and epimorphisms, so does $\mathcal{C}^{\text{op}}$. However, the objects of $\mathcal{C}^{\text{op}}$ are not sets, so its monomorphisms cannot be injective mappings, nor can its epimorphisms be surjections.

2.2.6 Example: topological abelian groups

This example relies on some elementary topology which we do not develop in this text. (The definitions of terms below can be found in [Willard 1970] §13.) As well as providing an epimorphism which is not a surjection, it shows that an epimorphism in a subcategory may cease to be an epimorphism in a larger category.

Let $\mathcal{HA}b$ be the category of Hausdorff topological abelian groups. Thus an object A of $\mathcal{HA}b$ is a Hausdorff topological space which is also an abelian group in such a way that the group operations are continuous maps, and a morphism from A to B in $\mathcal{HA}b$ is a continuous map which is also a group homomorphism. We exploit the following result *ibid,* (13.14): if $f, g : X \to Y$ are continuous maps of topological spaces with Y Hausdorff, and f and g agree on a dense subset of X, then $f = g$.

Now the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are both objects of $\mathcal{HA}b$ with their customary additions and topologies, and moreover $\mathbb{Q}$ is dense in $\mathbb{R}$. The result quoted above shows that the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathcal{HA}b$, although not a surjection. Note that ι is also a monomorphism and, of course, an injection.

In particular, we have an example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

The quotient group $\mathbb{R}/\mathbb{Q}$ can be given a topology (indiscrete) so that it becomes the cokernel of ι in the category $\mathcal{T}op\mathcal{A}b$ of all topological abelian groups, but it is not Hausdorff (see [Higgins 1974] Chapter II, §3 and Proposition 5).

We also note that $\mathcal{HA}b$ is a preadditive subcategory of $\mathcal{T}op\mathcal{A}b$ but that ι is no longer an epimorphism in $\mathcal{T}op\mathcal{A}b$.

2.2.7 Kernel and cokernel

Before we define short exact sequences in a preadditive category, we record the next, easily verified lemma, which shows that the concepts of kernel and cokernel in a preadditive category correspond to their counterparts in $\mathcal{A}b$ via the morphism functors.

2.2.8 Lemma

Let $\alpha : A \to B$ be a morphism in a preadditive category $\mathcal{C}$. Then the following assertions hold.

(i) *$\kappa : K \to A$ is a kernel for α if and only if, for any L in $\mathcal{C}$,*

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, K) \xrightarrow{\kappa_*} \mathrm{Mor}_{\mathcal{C}}(L, A) \xrightarrow{\alpha_*} \mathrm{Mor}_{\mathcal{C}}(L, B)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Ker}\,\alpha$,

$$\mathrm{Mor}_{\mathcal{C}}(L, \mathrm{Ker}\,\alpha) = \mathrm{Ker}\,\mathrm{Mor}_{\mathcal{C}}(L, \alpha).$$

(*Note that α_* is an alternative notation for* $\mathrm{Mor}_{\mathcal{C}}(L, \alpha)$.)

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(ii) *$\gamma : B \to C$ is a cokernel for α if and only if, for any X in $\mathcal{C}$,*

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, X) \xrightarrow{\gamma^*} \mathrm{Mor}_{\mathcal{C}}(B, X) \xrightarrow{\alpha^*} \mathrm{Mor}_{\mathcal{C}}(A, X)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Cok}\,\alpha$,

$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{Cok}\,\alpha, X) = \mathrm{Ker}\,\mathrm{Mor}_{\mathcal{C}}(\alpha, X),$$

(*where α^* is an alternative notation for* $\mathrm{Mor}_{\mathcal{C}}(\alpha, X)$). □

An immediate consequence is that a morphism is a monomorphism precisely when its kernel is the zero object, and an epimorphism precisely when its cokernel is the zero object.

2.2.9 Short exact sequences

A *short exact sequence* in a preadditive category is a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in which α is a kernel for β and β is a cokernel for α. Thus, in particular, α is a monomorphism and β is an epimorphism.

An easy exercise is that β is an isomorphism if and only if A is the zero object, while α is an isomorphism just when $C = 0$.

The distinction between an epimorphism and a surjection means that some caution must be exercised when working with short exact sequences in a general preadditive category. The unexpected phenomena that can arise are illustrated by the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$.

As we noted in (2.2.6), ι is both an epimorphism and monomorphism in the category $\mathcal{HA}_{\mathcal{B}}$ of Hausdorff abelian topological groups, but it is not an isomorphism. Thus the sequence

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\iota} \mathbb{R} \xrightarrow{0} 0 \longrightarrow 0$$

is not short exact. In fact, $\mathbb{Q}$ cannot be the kernel of $\mathbb{R} \to 0$ because the identity map on $\mathbb{R}$ cannot be factored through $\mathbb{Q}$.

2.2.10 Projective and injective objects

There are three equivalent definitions for a projective object in an arbitrary preadditive category $\mathcal{C}$, which generalize well-known characterizations of projective modules.

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Proj 1. The generalization of our basic definition for modules (1.1.7): an object P of $\mathcal{C}$ is *projective* (in $\mathcal{C}$) if every short exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$$

in $\mathcal{C}$ *splits* at P, meaning that there exists $\sigma : P \to M$ in $\mathcal{C}$ with $\pi\sigma = id_P$.

Proj 2. P is projective if and only if, whenever the row is exact, the following diagram can be completed (so as to provide a commuting triangle):

$$\begin{array}{ccccc} & & P & & \\ & \overset{\beta}{\swarrow} & \downarrow \gamma & & \\ Y & \xrightarrow[\eta]{} & Z & \longrightarrow & 0 \end{array}$$

(The equivalence of Proj 1 and Proj 2 for modules is shown in Theorem 2.5.4 of [BK: IRM].)

Proj 3. P is projective if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Mor}\,(P, X) \xrightarrow{\xi_*} \mathrm{Mor}\,(P, Y) \xrightarrow{\eta_*} \mathrm{Mor}\,(P, Z) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

in $\mathcal{C}$ (see (2.1.8) for the module version).

The proof of the fact that these definitions are mutually equivalent is much the same as that for modules, and is left to the reader. Likewise, the reader should have no problem in formulating and verifying the equivalence of three definitions for an injective object.

However, there is no reason why a preadditive category should contain any nontrivial projective or injective object. For example, the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups (which can also be described as the category of finitely generated torsion $\mathbb{Z}$-modules or the category of Artinian $\mathbb{Z}$-modules) has no nonzero projectives – this can be seen as in Exercise 2.2.10 below.

More seriously, it may happen that a projective object in a subcategory is not projective in a larger category. For an illustration, consider the category $\mathcal{ASS}_{\mathbb{Z}}$ of Artinian semisimple $\mathbb{Z}$-modules. Such a module is a finite sum of irreducible $\mathbb{Z}$-modules, that is, abelian groups $\mathbb{Z}/p\mathbb{Z}$ of prime order for various primes p (see (2.3.20 - F) below). A fundamental result, the Artinian Splitting

Theorem† ([BK: IRM] (4.1.17)) tells us that, for any module M in $\mathcal{ASS}_{\mathbb{Z}}$, any submodule or quotient module of M must be a direct summand, and so every module in $\mathcal{ASS}_{\mathbb{Z}}$ is both projective and injective as an object of that category. In contrast, no such (nonzero) module can be projective or injective in the category $\mathcal{FA}$B.

Although the concept of exactness was introduced in [Hurewicz 1941], in the context of homology groups of topological spaces, the term 'exact sequence' did not appear until [Kelley & Pitcher 1947]. The crucial role of exact sequences in contemporary module theory and homology theory is evident from almost any modern text on these subjects.

2.2.11 Additive categories

As we remarked in (1.4.10), the direct sum $M' \oplus M''$ of two modules M' and M'' is characterized up to isomorphism by the existence of homomorphisms

$$\sigma' : M' \longrightarrow M, \ \sigma'' : M'' \longrightarrow M, \ \pi' : M \longrightarrow M' \text{ and } \pi'' : M \longrightarrow M''$$

satisfying the relations

$$\pi'\sigma' = id_{M'}, \ \pi''\sigma'' = id_{M''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_M.$$

These relations make sense in any preadditive category $\mathcal{C}$, and so we may define the *direct sum* of objects M' and M'' of $\mathcal{C}$ to be an object M of $\mathcal{C}$ which satisfies these relations. However, a preadditive category need not contain the direct sum of a pair of objects; an easy example is given by $\mathcal{P}$READD$(\mathbb{Z})$ in $\mathcal{A}$B. These observations lead to the following definition.

A category $\mathcal{C}$ is said to be *additive* if it is a preadditive category satisfying the next condition.

Ad 4. Any two objects of $\mathcal{C}$ have a direct sum in $\mathcal{C}$.

As in the case of modules (the same arguments apply), such a direct sum is both a product and a coproduct and so it is unique to within unique isomorphism. The notation $M' \oplus M''$ is used to indicate any convenient choice of the direct sum – in most contexts, there is an obvious candidate for this choice.

It can be shown (Exercise 2.2.6) that a product in a preadditive category must be a direct sum, as must be a coproduct. Conversely, when products and coproducts exist and they are the same as direct sums in a category $\mathcal{C}$ satisfying Ad 3, then Ad 1 and Ad 2 hold, so that $\mathcal{C}$ is additive. (See [Gabriel 1962] for details.)

† The Artinian Splitting Theorem holds over any ring R.

We also note in passing that when the direct sum $M' \oplus M''$ exists, so also does $M'' \oplus M'$, which is isomorphic to $M' \oplus M''$ via a unique isomorphism. Thus one may speak of the direct sum of an unordered pair of objects.

Note also that it is possible to characterize direct sums in terms of *split exact sequences*, again mimicking the result for modules.

A short exact sequence

$$0 \longrightarrow M' \xrightarrow{\ \sigma'\ } M \xrightarrow{\ \pi'\ } M'' \longrightarrow 0$$

in a preadditive category is *split* if there are morphisms σ'' and π'' as above. As with modules, it is enough to know that one of the splitting morphisms σ'' or π'' exists, since the other can then be constructed, and we then find that $M \cong M' \oplus M''$.

2.2.12 Additive subcategories

A subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$ is an *additive subcategory* if, as well as being a preadditive subcategory, $\mathcal{D}$ is additive.

Let M', M'' be a pair of objects in $\mathcal{D}$. Any direct sum of M' and M'' in $\mathcal{D}$ is necessarily their direct sum in $\mathcal{C}$ also, but the favoured choice $M' \oplus M''$ of the direct sum in $\mathcal{C}$ need not itself be an object of $\mathcal{D}$. However, there is to be some choice, necessarily isomorphic to this, which, together with its associated inclusions and projections $\sigma', \sigma'', \pi', \pi''$, does lie in $\mathcal{D}$.

As with preadditive categories, we can define a minimal additive subcategory $\mathcal{A}\mathrm{dd}(X)$ of an additive category $\mathcal{C}$ in terms of a nonzero object X of $\mathcal{C}$ by taking $\mathcal{A}\mathrm{dd}(X)$ to be the full subcategory of $\mathcal{C}$ with objects the finite direct sums X^n, $n \geq 1$, together with the zero object.

2.2.13 Examples

Before developing the theory of additive categories, we give some more examples.

All the module categories introduced in (1.1.9), namely, $\mathcal{P}\mathrm{roj}_R$, $\mathcal{F}\mathrm{ree}_R$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$, are additive subcategories of $\mathcal{M}\mathrm{od}_R$; the verifications rely on simple facts about direct sums which are mostly immediate consequences of the definitions. For projective modules, we use the fact that $M' \oplus M''$ is projective if and only if both summands are projective ([BK: IRM] Corollary 2.5.6); and for free modules, we note that the zero module is free on the empty set, and that if M and N are free on sets X and Y respectively, then $M \oplus N$ is isomorphic to the free module on the disjoint union $X \sqcup Y$ ([BK: IRM] Exercise

2.1.8). Also, by Lemma 2.5.7 of [BK: IRM], $M' \oplus M''$ is finitely generated if and only if its summands are finitely generated, which equivalence is easy to check directly.

We record the results as follows.

2.2.14 Lemma

The categories $\mathcal{P}\text{ROJ}_R$, $\mathcal{F}\text{REE}_R$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$ are additive subcategories of $\mathcal{M}\text{OD}_R$. If S is a subring of R, then $\mathcal{M}\text{OD}_R$ is an additive subcategory of $\mathcal{M}\text{OD}_S$. □

Next, we consider product categories.

2.2.15 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Then $\mathcal{C} \times \mathcal{D}$ is an additive category.

Proof

The sum of morphisms (α, β) and (α', β') from (C, D) to (C', D') in $\mathcal{C} \times \mathcal{D}$ is given componentwise:

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$$

In effect, there is an identity

$$\mathrm{Mor}_{\mathcal{C}}(C, C') \times \mathrm{Mor}_{\mathcal{D}}(D, D') = \mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D'))$$

which gives us the abelian group structure that we require in $\mathcal{C} \times \mathcal{D}$. Note that the zero is the pair $(0, 0)$ and the direct sum is

$$(C, D) \oplus (C', D') = (C \oplus C', D \oplus D').$$ □

2.2.16 Morphism categories

Equally straightforward is the extension of the above result to morphism categories. Let $\mathcal{C}$ be an additive category. If (α, β) and (α', β') are both morphisms from (C, D, γ) to (C', D', γ') in $\mathcal{M}\text{OR}_{\mathcal{C}}$, then so is their sum. Composition of morphisms is distributive over addition because this is already true in the product category $\mathcal{C} \times \mathcal{C}$, and the zero is $(0, 0, id_0)$.

The direct sum in $\mathcal{M}\text{OR}_{\mathcal{C}}$ is given by

$$(C, D, \gamma) \oplus (C', D', \gamma') = (C \oplus C', D \oplus D', \gamma \oplus \gamma'),$$

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where

$$\gamma \oplus \gamma' = \sigma_D \gamma \pi_C + \sigma_{D'} \gamma' \pi_{C'} : C \oplus C' \to D \oplus D',$$

π_C and $\pi_{C'}$ being the split epimorphisms associated with $C \oplus C'$, and σ_D and $\sigma_{D'}$ being the split monomorphisms associated with $D \oplus D'$. In a module category, this becomes the familiar formula

$$(\gamma \oplus \gamma')(c, c') = (\gamma c, \gamma' c') \text{ for all } c \in C, c' \in C'.$$

We record the consequences of our discussion.

2.2.17 Lemma

Suppose that $\mathcal{C}$ is an additive category. Then the categories $\mathcal{M}\textsc{or}_{\mathcal{C}}$, $\mathcal{E}\textsc{nd}_{\mathcal{C}}$, $\mathcal{I}\textsc{so}_{\mathcal{C}}$ and $\mathcal{A}\textsc{ut}_{\mathcal{C}}$ are additive categories. □

2.2.18 Additive functors

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. For each pair of objects C', C in $\mathcal{C}$, the set $\mathrm{Mor}_{\mathcal{C}}(C', C)$ is an additive group, as is $\mathrm{Mor}_{\mathcal{D}}(D', D)$ for each pair D', D in $\mathcal{D}$. In this circumstance, it is natural to consider functors between $\mathcal{C}$ and $\mathcal{D}$ that respect the additive structures on the sets of morphisms.

A covariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is said to be *additive* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is a group homomorphism for all objects C' and C in $\mathcal{C}$:

$$F(\alpha + \beta) = F\alpha + F\beta \quad \text{always.}$$

Additive contravariant functors are defined similarly, as are additive multifunctors on products of additive categories.

First examples are the morphism functors themselves. For a fixed object C of an additive category, $\mathrm{Mor}_{\mathcal{C}}(C, -)$ is a covariant additive functor from $\mathcal{C}$ to $\mathcal{A}\textsc{b}$ and $\mathrm{Mor}_{\mathcal{C}}(-, C)$ is a contravariant additive functor from $\mathcal{C}$ to $\mathcal{A}\textsc{b}$.

It is clear that if $\mathcal{C}'$ is an additive subcategory of the additive category $\mathcal{C}$, then the inclusion functor is additive. If F is an additive functor from $\mathcal{C}$ to $\mathcal{D}$, then the restriction of F to $\mathcal{C}'$ will also be additive.

An important property of an additive functor $F : \mathcal{C} \to \mathcal{D}$ is that it preserves zero morphisms, zero objects and direct sums.

2.2.19 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive

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functor. If 0 is a zero morphism or object in $\mathcal{C}$, then $F0$ is a zero morphism or object in $\mathcal{D}$.

Proof

Temporarily, we indicate a zero object in $\mathcal{C}$ by Z to distinguish it from the various zero maps that we use.

If $0 : C' \to C$ is any zero map in $\mathcal{C}$, then $F0$ must be the zero map from FC' to FC in $\mathcal{D}$, since F is additive. As $id_Z = 0$ on Z itself, $id_{FZ} = F(id_Z) = 0$ on FZ. Hence for a zero object Z' in $\mathcal{D}$, the unique maps $FZ \to Z'$ and $Z' \to FZ$ are mutually inverse isomorphisms. □

2.2.20 Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Then for any two objects C' and C'' of $\mathcal{C}$,

$$F(C' \oplus C'') \cong F(C') \oplus F(C'') \text{ in } \mathcal{D}.$$

Proof

Recall from (2.2.11) that the direct sum of the pair of objects C' and C'' in $\mathcal{C}$ can be characterized as an object C of $\mathcal{C}$ for which there are morphisms

$$\sigma' : C' \longrightarrow C,\ \sigma'' : C'' \longrightarrow C,\ \pi' : C \longrightarrow C' \text{ and } \pi'' : C \longrightarrow C''$$

satisfying the relations

$$\pi'\sigma' = id_{C'},\ \pi''\sigma'' = id_{C''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_C.$$

Since F preserves sums and products, and converts identities to identities and zeroes to zeroes, the maps $F\sigma'$, $F\sigma''$, $F\pi'$ and $F\pi''$ satisfy the equations that show FC to be a direct sum also. □

2.2.21 Functor categories

Recall from (1.3.9) that, given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ has as objects the covariant functors from $\mathcal{C}$ to $\mathcal{D}$, the morphisms from F to G in $[\mathcal{C}, \mathcal{D}]$ comprising the set $\text{Nat}(F, G)$ of all natural transformations from F to G.

Our aim now is to show that properties of $\mathcal{D}$ can be carried over to $[\mathcal{C}, \mathcal{D}]$ by making *pointwise* definitions; the category $\mathcal{C}$ can be completely arbitrary.

The essential observation is that a set $\text{Nat}(F, G)$ of morphisms in $[\mathcal{C}, \mathcal{D}]$ is in fact a disjoint union of sets $\text{Mor}_{\mathcal{D}}(F(C), G(C))$, one for each object C of $\mathcal{C}$,

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so that any property or condition on $\mathrm{Nat}(F,G)$ can be verified, or imposed, separately in each $\mathrm{Mor}_{\mathcal{D}}(F(C),G(C))$.

Suppose that $\mathcal{D}$ is additive. If η and ζ are in $\mathrm{Nat}(F,G)$, the pointwise definition of their sum $\eta+\zeta$ in $\mathrm{Nat}(F,G)$ is

$$(\eta+\zeta)_C = \eta_C + \zeta_C \text{ for all } C \in \mathcal{C},$$

and the zero natural transformation 0 in $\mathrm{Nat}(F,G)$ is given by

$$0_C = 0 : FC \longrightarrow GC.$$

Thus $\mathrm{Nat}(F,G)$ is an abelian group for any pair of functors.

Since $\mathcal{D}$ has a zero object, we can define the *zero functor*

$$0 : \mathcal{C} \longrightarrow \mathcal{D}$$

by

$$0(C) = 0 \text{ for all } C \text{ in } \mathcal{C},$$

which is a zero object in $[\mathcal{C},\mathcal{D}]$.

We can also define the *direct sum* of two functors F and G in $[\mathcal{C},\mathcal{D}]$ in the obvious way:

$$(F\oplus G)(C) = F(C)\oplus G(C) \text{ for all } C \text{ in } \mathcal{C}.$$

To verify that this actually gives a direct sum, we again use the characterization of direct sums in a preadditive category which we gave in (2.2.11). For each object C, there are morphisms

$$\alpha_C : FC \longrightarrow FC\oplus GC, \quad \beta_C : FC\oplus GC \longrightarrow GC,$$

$$\gamma_C : FC\oplus GC \longrightarrow FC \text{ and } \delta_C : GC \longrightarrow FC\oplus GC$$

with

$$\beta_C\alpha_C = 0,\ \gamma_C\delta_C = 0,\ \beta_C\delta_C = id_{GC},\ \gamma_C\alpha_C = id_{FC}$$

and

$$\alpha_C\gamma_C + \delta_C\beta_C = id_{FC\oplus GC},$$

which gives a set $\alpha,\beta,\gamma,\delta$ of natural transformations that exhibit $F\oplus G$ as a direct sum in $[\mathcal{C},\mathcal{D}]$.

Thus we have the following result.

2.2.22 Proposition

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an additive category. Then the functor category $[\mathcal{C},\mathcal{D}]$ is an additive category. □

Exercises

2.2.1 Let $\mathcal{D}$ be a preadditive subcategory of the preadditive subcategory $\mathcal{C}$. Show that $\mathrm{End}_{\mathcal{D}}(X)$ is a subring of $\mathrm{End}_{\mathcal{C}}(X)$ for every object X in $\mathcal{D}$.

Given an object X of $\mathcal{C}$ and a subring D of $\mathrm{End}_{\mathcal{C}}(X)$, define a minimal preadditive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

2.2.2 Given a nonzero object X of an additive category $\mathcal{C}$ and a subring D of $\mathrm{Mor}_{\mathcal{C}}(X, X)$, define a minimal additive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

Show that the morphisms in $\mathcal{D}$ can be represented as sets $M_{m,n}(D)$ of $m \times n$ matrices over D.

Hint. Exercise 2.1.6 of [BK: IRM] helps.

2.2.3 There seems to be a curious asymmetry in the exact sequences of abelian groups obtained in (2.2.3) above, in that they begin, but need not end, with the zero group. Show that this is inevitable for the following reason.

In a preadditive (right) category, the following statements are equivalent.

(i) $\varphi : X \to Y$ has a right inverse;
(ii) for all $L \in \mathcal{C}$, $\mathrm{Mor}(L, X) \xrightarrow{\varphi_*} \mathrm{Mor}(L, Y) \to 0$ is exact;
(iii) for all $L \in \mathcal{C}$, $\varphi_* : \mathrm{Mor}(L, X) \to \mathrm{Mor}(L, Y)$ has a right inverse.

There is an obvious dualization of the above.

Deduce that the following are equivalent.

(i) φ is an isomorphism;
(ii) $\mathrm{Mor}(-, \varphi)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$;
(iii) $\mathrm{Mor}(\varphi, -)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$.

2.2.4 **Epimorphisms and monomorphisms in arbitrary categories**

Let $\mathcal{C}$ be a (right) category. A morphism $\gamma : C \to D$ in $\mathcal{C}$ is defined to be a *monomorphism* if the following holds.

If β, $\beta' : B \to C$ *are two morphisms with* $\gamma\beta = \gamma\beta'$, *then* $\beta = \beta'$.

Verify that a monomorphism in a preadditive category is again a monomorphism in this sense. Observe too that if γ has a left inverse, then γ is a monomorphism.

Give a definition of an epimorphism by duality, that is, γ is an epimorphism in $\mathcal{C}$ if and only if γ^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$.

Verify directly that the two definitions of epimorphism coincide in a preadditive category, and that any morphism with a right inverse is an epimorphism.

Show that the inclusion map $\iota : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the (nonadditive) category $\mathcal{R}$ING of rings with identity.

2.2.5 Show that $C \oplus 0 \cong C$ for any object C of a preadditive category $\mathcal{C}$.

2.2.6 Let C, C' and C'' be objects of a preadditive category $\mathcal{C}$. Verify that the following statements are equivalent, as asserted in (2.2.11).

(i) C is the product of C' and C''.
(ii) C is the coproduct of C' and C''.
(iii) C is the direct sum of C' and C''.

Generalize this result to an arbitrary finite set of objects in $\mathcal{C}$.

2.2.7 Let $\mathcal{C}$ be a preadditive category, with $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : C \to D$ composable morphisms in $\mathcal{C}$.

Show that if β and γ are epimorphisms, then $\gamma\beta$ is an epimorphism, and if $\gamma\beta$ is an epimorphism, then γ is an epimorphism.

Prove also that if α and β are monomorphisms, so is $\beta\alpha$, and if $\beta\alpha$ is a monomorphism, so also is α.

Give counterexamples to the claims that if $\gamma\beta$ is an epimorphism, then β is an epimorphism, and if $\beta\alpha$ is a monomorphism, so also is β.

2.2.8 Let $A_1, \dots, A_k$ be objects of an additive category $\mathcal{A}$. Show that $A_1 \oplus \cdots \oplus A_k$ is a projective object in $\mathcal{A}$ if and only $A_1, \dots, A_k$ are each projective objects in $\mathcal{A}$. (This generalizes a well-known result for modules (Theorem 2.5.5 of [BK: IRM]).)

2.2.9 **Projectives depend on the category**

Generalize (2.2.10) to show that, for any ring R, every module in $\mathcal{ASS}_R$ is projective as an object in that category.

Suppose that R is right Artinian but not semisimple. Find a module in $\mathcal{ASS}_R$ which is not projective in $\mathcal{M}$OD$_R$.

Hint. [BK: IRM], section 4.3 is relevant, particularly Exercise 4.3.7.

2.2.10 **A category without projectives**

Let $\mathcal{FA}$B be the category of finite abelian groups. It is well known that any finite abelian group is the direct sum of finite cyclic groups (see Theorem 6.3.24 of [BK: IRM], for example). So if $\mathcal{FA}$B contains a nonzero projective, there must be a nontrivial finite cyclic projective, by Exercise 2.2.8 above. However, this is easily seen to be impossible – consider the natural surjection of $\mathbb{Z}/a^2\mathbb{Z}$ to $\mathbb{Z}/a\mathbb{Z}$ for any positive integer a.

2.2.11 **The opposite category**

Show that if $\mathcal{C}$ is an additive category, so likewise is its opposite category $\mathcal{C}^{\mathrm{op}}$, where addition of morphisms is given by $\alpha^{\mathrm{op}} + \beta^{\mathrm{op}} = (\alpha + \beta)^{\mathrm{op}}$ as in (2.2.5).

Show that C^{op} is a projective object in $\mathcal{C}^{\mathrm{op}}$ if and only if C is an injective object in $\mathcal{C}$, and vice-versa.

Give an example of an additive category which contains nontrivial injective objects but only the zero projective object.

2.2.12 Just as a monoid can be regarded as the same thing as a category with one object (Exercise 1.1.4), a ring may be taken to be the same as a preadditive category with one nonzero object. Viewing a ring R in this way, show that right R-modules correspond to contravariant, cochiral additive functors from R to $\mathcal{M}\textsc{od}_{\mathbb{Z}}$, while left R-modules correspond to covariant, contrachiral additive functors from R to ${}_{\mathbb{Z}}\mathcal{M}\textsc{od}$. Show also that R-module homomorphisms correspond to natural transformations.

2.3 ABELIAN CATEGORIES

For the next stage in our examination of categories whose objects in some way resemble modules, we consider abelian categories. Such categories must contain the kernel and cokernel of each of their morphisms, and most constructions that can be carried out in the full module categories $\mathcal{M}\textsc{od}_R$ are reproducible in an abelian category. Indeed, there are Embedding Theorems, which show that an arbitrary abelian category can be realized as a subcategory of $\mathcal{M}\textsc{od}_R$ for some ring R. However, an abelian category may lack nontrivial examples of projective or injective objects.

We show that the various morphism categories associated to abelian categories are again abelian, as are the functor and product categories. We also introduce the idea of a direct sum of categories, which leads to an interesting categorical interpretation of the module theory of a Dedekind domain.

2.3.1 The definition

Although we have defined kernels and cokernels in an arbitrary category with zero object, and thus in an additive category, a morphism in an additive category need not have a kernel or cokernel in that category. Indeed, the category $\mathcal{P}_{A[\epsilon]}$ of finitely generated projective modules over a ring of dual numbers $A[\epsilon]$ (where $\epsilon^2 = 0$) fails to contain either the kernel or the cokernel of the multiplication homomorphism $\epsilon : A[\epsilon] \to A[\epsilon]$, since each is isomorphic to A, which is not projective over $A[\epsilon]$.

The requirement that a category should contain kernels and cokernels leads to the next definition.

Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is an *abelian category* if the following conditions are satisfied.

Ab 1. If $\lambda : L \to M$ is a morphism in $\mathcal{C}$, then $\mathcal{C}$ contains a kernel $(\operatorname{Ker}\lambda, \kappa)$ and a cokernel $(\operatorname{Cok}\lambda, \chi)$ of λ. (This means not only that the objects $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ are in $\mathcal{C}$, but also that the morphisms

$$\kappa : \operatorname{Ker}\lambda \longrightarrow L \quad \text{and} \quad \chi : M \longrightarrow \operatorname{Cok}\lambda$$

belong to $\mathcal{C}$.)

Ab 2. If $\kappa : K \to L$ is a monomorphism in $\mathcal{C}$, then (K, κ) is a kernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Ab 2^{op}. If $\chi : M \to C$ is an epimorphism in $\mathcal{C}$, then (C, χ) is a cokernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Statement Ab 2 is often phrased as 'every monomorphism is normal'. Alternative axiom sets abound in the literature. For example, see [Freyd 1964] or [Herrlich & Strecker 1979] for the following.

2.3.2 Theorem

Suppose that $\mathcal{C}$ is an additive category satisfying Ab 1. Then the following statements are equivalent.

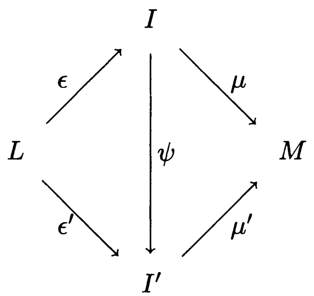
(i) *Ab 2 and Ab 2^{op} hold in $\mathcal{C}$.*

(ii) *For any morphism $\lambda : L \to M$ with kernel $(\operatorname{Ker}\lambda, \kappa)$ and cokernel $(\operatorname{Cok}\lambda, \chi)$, the morphism*

$$\psi : \operatorname{Cok}\kappa \longrightarrow \operatorname{Ker}\chi$$

(which exists by the universal properties of kernels and cokernels) is an isomorphism.

(iii) *Every morphism $\lambda : L \to M$ has a unique (epi, mono) factorization; that is, there exists an epimorphism $\epsilon : L \to I$ and monomorphism $\mu : I \to M$ with $\mu\epsilon = \lambda$ such that, if $\epsilon' : L \to I'$, $\mu' : I' \to M$ is another (epi, mono) factorization of λ, then there is a unique morphism $\psi : I \to I'$ making*



commute; moreover, ψ is an isomorphism. ○

The isomorphic objects $\operatorname{Ker}\chi$ and I of this theorem are commonly referred to as the image $\operatorname{Im}\lambda$ of λ.

2.3.3 Module-like behaviour

In essence, the above result tells us that the Induced Mapping Theorem ([BK: IRM] (1.2.11)) must hold in an abelian category. It is then possible to translate many general results about modules into the context of an abstract abelian category. For example, the Isomorphism Theorems ([BK: IRM] Exercise 1.2.2) can be reproduced. However, the proofs are more delicate since it is not possible to work with elements. Instead, the universal properties of kernels, cokernels, etc. must be used. Arguments of this type are developed in [Mac Lane 1975], Chapter IX and [Mac Lane 1971], Chapter VIII.

On the other hand, there are some features of module categories which need not be reproducible in an abelian category. These usually relate to the existence of an object with some specific property. For instance, an abelian category need not contain any projective object, as is the case with the category $\mathcal{FA}_\mathcal{B}$ of finite abelian groups (see Exercise 2.2.10).

2.3.4 Example

For an example of an additive category that satisfies axiom Ab 1 but not Ab 2, consider the full additive subcategory of $\mathcal{A}_\mathcal{B}$ whose objects comprise all finite direct sums of the form $\mathbb{Z}^m \oplus \mathbb{Q}^n$ with $m, n \geq 0$. Here the inclusion monomorphism $\mathbb{Z} \to \mathbb{Q}$ is not a kernel.

We now look for abelian categories of modules. The first result simplifies the checking in many cases of interest. (Its converse, while also true, is less elementary – see Exercise 2.3.8 for the details.)

2.3.5 Theorem

Let R be a ring and let $\mathcal{C}$ be a full additive subcategory of $\mathcal{M}od_R$ such that, for any homomorphism $\lambda : L \to M$ of R-modules in $\mathcal{C}$, the R-modules $\operatorname{Ker}\lambda$ *and* $\operatorname{Cok}\lambda$ *are both in $\mathcal{C}$.*

Then $\mathcal{C}$ is abelian.

Proof

The R-modules $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ evidently satisfy the universal properties needed to make them the kernel and cokernel of λ in $\mathcal{C}$. So Ab 1 holds. To see that Ab 2 holds, consider a monomorphism $\kappa : K \to L$ in $\mathcal{C}$. As we remarked in (2.2.3), κ is an injective mapping. Now $\mathcal{C}$ must also contain $\operatorname{Cok}\kappa = L/\operatorname{Im}\kappa$, and so K is the kernel of the canonical module epimorphism $\lambda : L \to L/\operatorname{Im}\kappa$, which is in $\mathcal{C}$. The proof of Ab 2^{op} is similar. □

For the next result, recall that, given a ring R, a right R-module M is *Noetherian* if every R-submodule of M is finitely generated. The ring R itself is *right Noetherian* if every right ideal of R is finitely generated. An account of Noetherian rings and modules is given in [BK: IRM], §3.1.

2.3.6 Corollary

Let R be a ring. Then the category $\mathcal{M}_R$ of finitely generated R-modules is an additive subcategory of $\mathcal{M}od_R$, and $\mathcal{M}_R$ is abelian precisely when the ring R is right Noetherian.

Proof

The sole obstruction to $\mathcal{M}_R$ being abelian is that it might not contain the kernel of some homomorphism, that is, there may be a finitely generated module M having a submodule which is not finitely generated. By Corollary 3.1.7 of [BK: IRM], this cannot happen if R is right Noetherian. In the reverse direction, if $\mathcal{M}_R$ is abelian, then every ideal of R is in $\mathcal{M}_R$. □

2.3.7 More examples

The category $\mathcal{HA}b$ of Hausdorff abelian groups is an example of a full additive subcategory of $\mathcal{M}od_{\mathbb{Z}}$ which is not abelian. Observe that the cokernel (in $\mathcal{T}op\mathcal{A}b$) of the inclusion monomorphism $\mathbb{Q} \to \mathbb{R}$ does not lie in $\mathcal{HA}b$.

There are some subcategories of $\mathcal{M}od_R$ that are abelian for an arbitrary ring R. Let $\mathcal{N}oeth_R$ denote the full subcategory of $\mathcal{M}od_R$ whose objects are the Noetherian right R-modules, let $\mathcal{A}rt_R$ denote the full subcategory whose objects are the Artinian right R-modules and let $\mathcal{ASS}_R$ be the full subcategory given by the Artinian semisimple modules. Each of these categories is abelian

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since it contains any submodule or quotient module of any of its members – see [BK: IRM], Proposition 3.1.2, Proposition 4.1.5 and the Artinian Splitting Theorem (4.1.17) respectively. We record our assertion formally.

2.3.8 Theorem

The categories $\mathcal{N}\text{OETH}_R$, $\mathcal{A}\text{RT}_R$ and $\mathcal{ASS}_R$ are abelian. □

By [BK: IRM] Corollary 4.1.18, there is an inclusion of categories

$$\mathcal{ASS}_R \subseteq \mathcal{A}\text{RT}_R$$

and, by the same result together with the definition of Noetherian, there are inclusions

$$\mathcal{ASS}_R \subseteq \mathcal{N}\text{OETH}_R \subseteq \mathcal{M}_R.$$

In general, $\mathcal{A}\text{RT}_R$ is not a subcategory of $\mathcal{N}\text{OETH}_R$. An example (due to P. M. Cohn) of a cyclic Artinian, non-Noetherian module is exhibited in Exercise 4.1.7 of [BK: IRM].

Next, we give the conditions for equality between these various categories.

2.3.9 Proposition

(i) $\mathcal{N}\text{OETH}_R = \mathcal{M}_R$ *if and only if R is right Noetherian.*

(ii) $\mathcal{A}\text{RT}_R = \mathcal{M}_R$ *if and only if R is right Artinian.*

(iii) $\mathcal{ASS}_R = \mathcal{M}_R$ *if and only if R is an Artinian semisimple ring (in which case $\mathcal{ASS}_R = \mathcal{P}_R$, the category of finitely generated projective right R-modules).*

(iv) *Suppose that R is right Artinian. Then the ring $R/\operatorname{rad}(R)$ is Artinian semisimple, where $\operatorname{rad}(R)$ is the Jacobson radical, and*

$$\mathcal{ASS}_R = \mathcal{M}_{R/\operatorname{rad}(R)}.$$

Proof

(i) Since R itself is in $\mathcal{M}_R$, the equality of categories makes R right Noetherian. Conversely, if R is right Noetherian then R^k is Noetherian for any natural number k and hence any finitely generated module M is also Noetherian – see [BK: IRM] (3.1.4).

(ii) The argument is similar to the proof of (i); we use the fact that if R is right Artinian, then so is any finitely generated right R-module ([BK: IRM] (4.1.7)).

(iii) Suppose that R is an Artinian semisimple ring. By the Wedderburn-Artin Theorem ([BK: IRM] (4.2.3)), R is a direct product of matrix rings over division rings, and so the categories are equal since every finitely generated

R-module is both semisimple and projective ([BK: IRM] (4.2.6)). Conversely, if the equalities hold, then R is Artinian semisimple by definition.

(iv) By [BK: IRM] (4.3.15), $R/\operatorname{rad}(R)$ is Artinian semisimple, and, by [BK: IRM] (4.3.16), a right R-module M is semisimple if and only if $M \cdot \operatorname{rad}(R) = 0$. Thus a semisimple R-module can be regarded as an $R/\operatorname{rad}(R)$-module by the scalar multiplication

$$m \cdot \bar{r} = m \cdot r \quad \text{for } r \in R,$$

and conversely, this rule can be used to make any $R/\operatorname{rad}(R)$-module into a semisimple R-module. □

2.3.10 Product and morphism categories

Next, we show that product and morphism categories of abelian categories are also abelian. As we already know that such categories are additive, it suffices to verify the axioms Ab 1, Ab 2 and Ab 2^{op}, which will follow at once from explicit descriptions of kernels, cokernels and so forth.

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A morphism in $\mathcal{C} \times \mathcal{D}$ is a pair

$$(\alpha, \beta) : (C, D) \longrightarrow (C', D'),$$

with α a morphism in $\mathcal{C}$ and β a morphism in $\mathcal{D}$. Thus

$$\operatorname{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \operatorname{Mor}_{\mathcal{C}}(C, C') \times \operatorname{Mor}_{\mathcal{D}}(D, D'),$$

and any assertion about a morphism (α, β) in $\mathcal{C} \times \mathcal{D}$ can be verified on its components α, β separately. Thus (α, β) is a monomorphism precisely when α and β are both monomorphisms, and likewise for epimorphisms.

The kernel of (α, β) is $((\operatorname{Ker}\alpha, \operatorname{Ker}\beta), (\kappa, \psi))$, where $(\operatorname{Ker}\alpha, \kappa)$ and $(\operatorname{Ker}\beta, \psi)$ are the respective kernels of α and β, and the cokernel is $((\operatorname{Cok}\alpha, \operatorname{Cok}\beta), (\chi, \zeta))$. In practice, we omit the maps κ and ψ, and write

$$\operatorname{Ker}(\alpha, \alpha') = (\operatorname{Ker}\alpha, \operatorname{Ker}\alpha'),$$

and similarly for the cokernel.

The image is the pair $\operatorname{Im}(\alpha, \beta) = (\operatorname{Im}\alpha, \operatorname{Im}\beta)$, with the obvious implicit morphisms. We summarize as follows.

2.3.11 Proposition

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories. Then the category $\mathcal{C} \times \mathcal{D}$ is also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D') \longrightarrow (C, D) \longrightarrow (C'', D'') \longrightarrow 0$$

in $\mathcal{C} \times \mathcal{D}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences. □

Similar arguments are used for the morphism categories attached to the category $\mathcal{C}$. In the morphism category $\mathcal{M}\text{OR}\,\mathcal{C}$, the kernel of

$$(\alpha, \beta) : (C, D, \gamma) \longrightarrow (C', D', \delta)$$

is $(\operatorname{Ker}\alpha, \operatorname{Ker}\beta, \overline{\gamma})$, where $\overline{\gamma}$ is the restriction of γ to $\operatorname{Ker}\alpha$, and the cokernel is $(\operatorname{Cok}\alpha, \operatorname{Cok}\beta, \overline{\delta})$, where $\overline{\delta}$ is induced by δ. Images in $\mathcal{M}\text{OR}\,\mathcal{C}$ are defined in the expected way. The result is as follows.

2.3.12 Proposition

Let $\mathcal{C}$ be an abelian category. Then the categories $\mathcal{M}\text{OR}\,\mathcal{C}$, $\mathcal{E}\text{ND}\,\mathcal{C}$, $\mathcal{I}\text{SO}\,\mathcal{C}$ and $\mathcal{A}\text{UT}\,\mathcal{C}$ are also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D', \gamma') \longrightarrow (C, D, \gamma) \longrightarrow (C'', D'', \gamma'') \longrightarrow 0$$

in $\mathcal{M}\text{OR}\,\mathcal{C}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences in $\mathcal{C}$.

Note that in $\mathcal{E}\text{ND}\,\mathcal{C}$ and $\mathcal{A}\text{UT}\,\mathcal{C}$, the two component sequences are identical.

□

2.3.13 Module categories

Let R and S be rings. As we saw in (1.3.16), the product category is equivalent to the category $\mathcal{M}\text{OD}_{R\times S}$, since every object (M, N) in the product category can be regarded as a module over $R \times S$ and every morphism is an $R \times S$-module homomorphism (1.1.12). Thus there are two ways in which, for instance, a kernel can be defined in $\mathcal{M}\text{OD}_R \times \mathcal{M}\text{OD}_S$, either by using the abstract machinery or by taking an equivalent object in $\mathcal{M}\text{OD}_{R\times S}$.

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The reader should not have much difficulty in verifying that these approaches give essentially the same objects, to within isomorphisms.

The kernels and cokernels in the (homo)morphism categories $\mathcal{M}\text{OR}(\mathcal{M}\text{OD}_R)$, $\mathcal{E}\text{ND}(\mathcal{M}\text{OD}_R)$, $\mathcal{I}\text{SO}(\mathcal{M}\text{OD}_R)$ and $\mathcal{A}\text{UT}(\mathcal{M}\text{OD}_R)$ are those in $\mathcal{M}\text{OD}_{R\times R}$.

2.3.14 Functor categories

We can also give pointwise descriptions of kernels, cokernels, and so on, in a functor category $[\mathcal{C}, \mathcal{D}]$ provided the appropriate objects exist in $\mathcal{D}$. For simplicity we suppose that $\mathcal{D}$ is abelian.

For example, the kernel $(\operatorname{Ker}\eta, \kappa)$ of a morphism η from F to G, that is, of a natural transformation $\eta \in \operatorname{Nat}(F, G)$, is obtained as follows.

For each object C of $\mathcal{C}$, write $(\operatorname{Ker}\eta_C, \kappa_C)$ for the kernel of $\eta_C : FC \to GC$. Then the object $\operatorname{Ker}\eta$ in $[\mathcal{C}, \mathcal{D}]$ is the functor from $\mathcal{C}$ to $\mathcal{D}$ given by

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}(\eta_C : FC \longrightarrow GC)$$

for each object C of $\mathcal{C}$.

To see that $\operatorname{Ker}\eta$ is a functor, note that a morphism $\alpha : C' \to C$ gives a commutative diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

which, together with the definition of a kernel as a universal object (1.4.15), shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C).$$

There is a natural transformation

$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

given by

$$\kappa_C : (\operatorname{Ker}\eta)(C) \longrightarrow F(C),$$

κ_C as above, and it is straightforward to verify that the pair $(\operatorname{Ker}\eta, \kappa)$ is a kernel for η.

Similarly, cokernels, monomorphisms, epimorphisms, projective and injective objects in $[\mathcal{C}, \mathcal{D}]$ are given pointwise by the corresponding objects or morphisms in $\mathcal{D}$.

We record the result for future reference.

2.3.15 Theorem

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an abelian category. Then the functor category $[\mathcal{C}, \mathcal{D}]$ is an abelian category. □

2.3.16 Direct sums of categories

Since we are able to define the direct sum of two, or more, functors between additive categories, we can interpret the direct product of two additive categories as a *direct sum*, because we can mimic the description of a direct sum of modules in terms of injections and projections (2.2.11).

Throughout the sequel we demand that an *equivalence* between additive categories must be given by an additive functor.

2.3.17 Theorem

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive categories. Then the following statements are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \mathcal{C}_2.$$

(ii) *There are additive functors*

$$I_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}, \quad I_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}, \quad P_1 : \mathcal{C} \longrightarrow \mathcal{C}_1, \quad P_2 : \mathcal{C} \longrightarrow \mathcal{C}_2$$

such that there are natural isomorphisms

$$I_1P_1 \oplus I_2P_2 \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_1I_1 \simeq \mathrm{Id}_{\mathcal{C}_1}, \ P_2I_2 \simeq \mathrm{Id}_{\mathcal{C}_2},$$

and also

$$P_2I_1 = 0, \ \ P_1I_2 = 0.$$

(I and P stand for inclusion and projection.)

Proof

Suppose that (i) holds, and let $F : \mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2$ and $G : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ be the (additive) functors that give the equivalence. If C is an object of $\mathcal{C}$, we have $FC = (C_1, C_2)$ for objects C_1 of $\mathcal{C}_1$ and C_2 of $\mathcal{C}_2$. Similarly, for a morphism α we have $F\alpha = (\alpha_1, \alpha_2)$. Define $P_1C = C_1$ and $P_1\alpha = \alpha_1$, and likewise for P_2.

For an object C_1 and morphism α_1 in $\mathcal{C}_1$, let $I_1C_1 = G(C_1, 0)$ and $I_1\alpha_1 =$

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$G(\alpha_1, 0)$, and similarly for the other variable. The assertion follows, on noting that

$$(C_1, C_2) \cong (C_1, 0) \oplus (0, C_2) \text{ in } \mathcal{C}_1 \times \mathcal{C}_2.$$

Conversely, if (ii) holds, define $FC = (P_1C, P_2C)$ and $G(C_1, C_2) = I_1C_1 \oplus I_2C_2$, and similarly for morphisms. □

For completeness, we record the extension of the above result to a direct product of any finite set of additive categories. The result is analogous to that for modules (see [BK: IRM], (2.1.7)).

2.3.18 Corollary

Let $\mathcal{C}$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be additive categories. Then the following are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \cdots \times \mathcal{C}_n.$$

(ii) *There are additive functors $I_i : \mathcal{C}_i \to \mathcal{C}$ and $P_i : \mathcal{C} \to \mathcal{C}_i$, for $i = 1, \ldots, n$, with natural isomorphisms*

$$I_1P_1 \oplus \cdots \oplus I_nP_n \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_iI_i \simeq \mathrm{Id}_{\mathcal{C}_i} \text{ for all } i,$$

and further

$$P_jI_i = 0 \text{ for } i \neq j.$$ □

2.3.19 Infinite direct sums of categories

Since an additive category has a zero object, it is sensible to speak of the *direct sum* $\bigoplus_\Lambda \mathcal{C}_\lambda$ of an infinite set $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ of categories, where Λ is an ordered set. The objects of the direct sum are all sequences of the form

$$C_\Lambda = (C_\lambda \mid C_\lambda \in \mathcal{C}_\lambda, \text{ almost all } C_\lambda = 0),$$

and a morphism $\alpha_\Lambda : C'_\lambda \to C_\Lambda$ is a sequence $(\alpha_\lambda : C' \to C_\lambda)$ with each α_λ a morphism in $\mathcal{C}_\lambda$.

It is straightforward to verify that $\bigoplus_\Lambda \mathcal{C}_\lambda$ is again an additive category under the usual componentwise definitions. Thus the zero object is $0_\Lambda = (0)$, and the direct sum in $\bigoplus_\Lambda \mathcal{C}_\lambda$ is given by

$$C'_\Lambda \oplus C''_\Lambda = (C'_\lambda \oplus C''_\lambda).$$

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The reader is invited to characterize infinite direct sums in terms of inclusion and projection functors.

2.3.20 Dedekind domains: a review

Some interesting direct sum decompositions of categories arise from the structure theory of modules over a Dedekind domain. As we use this structure theory in several places in this text, we turn aside to review its basic definitions and results, which are developed in detail in Chapters 5 and 6 of [BK: IRM].

We start with some ideal theory. Let $\mathcal{O}$ be an arbitrary commutative domain with field of fractions $\mathcal{K}$. A *fractional ideal* of $\mathcal{O}$ is a finitely generated nonzero $\mathcal{O}$-submodule $\mathfrak{a}$ of $\mathcal{K}$. If $\mathfrak{b}$ is also a fractional ideal of $\mathcal{O}$, so is their *product*

$$\mathfrak{a}\mathfrak{b} = \{a_1 b_1 + \cdots + a_k b_k \mid a_1, \ldots, a_k \in \mathfrak{a},\ b_1, \ldots, b_k \in \mathfrak{b},\ \ k \geq 1\}.$$

A fractional ideal $\mathfrak{a}$ is *invertible* if

$$\mathfrak{a}\mathfrak{b} = \mathcal{O}$$

for some $\mathfrak{b}$.

Our definition of a Dedekind domain ([BK: IRM] (5.1.10)) is that it is a commutative domain $\mathcal{O}$ all of whose fractional ideals are invertible. It can then be shown that $\mathcal{O}$ is a Noetherian ring ([BK: IRM] Exercise 5.1.2). Clearly, the fractional ideals of a Dedekind domain form a group $\mathrm{Frac}(\mathcal{O})$, and the set of nonzero principal ideals $\mathrm{Pr}(\mathcal{O})$ is a subgroup of $\mathrm{Frac}(\mathcal{O})$. The *ideal class group* of $\mathcal{O}$ is the quotient group

$$\mathrm{Cl}(\mathcal{O}) = \mathrm{Frac}(\mathcal{O}) / \mathrm{Pr}(\mathcal{O}).$$

The image of $\mathfrak{a}$ in $\mathrm{Cl}(\mathcal{O})$ is written $\{\mathfrak{a}\}$.

It is obvious that a commutative principal ideal domain is the same thing as a Dedekind domain with trivial ideal class group.

We record [BK: IRM] (5.1.19).

A The Unique Factorization Theorem for Ideals

Let $\mathfrak{a}$ be a fractional ideal of a Dedekind domain $\mathcal{O}$. Then there is a set of distinct nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and a set of nonzero integers $n_1, \ldots, n_k$ such that

$$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}.$$

The prime ideals are uniquely determined apart from the order in which they *are written, and the exponents are uniquely determined once an order has been fixed.* ○

Let $\mathcal{O}$ again be an arbitrary commutative domain, and let M be an $\mathcal{O}$-module. The *torsion submodule* $T(M)$ is

$$T(M) = \{m \in M \mid mx = 0 \text{ for some } x \in \mathcal{O},\ x \neq 0\}.$$

We say that M is a *torsion module* if $M = T(M)$ and that M is *torsion-free* if $T(M) = 0$. (These definitions anticipate the more general definitions that will be made in section 6.2.)

Here are the fundamental results for projective modules over a Dedekind domain.

B Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated $\mathcal{O}$-module. Then M is projective if and only if M is torsion-free (see ([BK: IRM] (6.3.4)).○

C Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated projective $\mathcal{O}$-module. Then

$$M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$$

for some fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of $\mathcal{O}$ ([BK: IRM] (6.1.2)). ○

The next result gives the criterion for two projective modules to be isomorphic ([BK: IRM] (6.1.6)).

D Steinitz' Theorem

Let $P \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$ be a projective module over a Dedekind domain $\mathcal{O}$. Then

$$P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}_1 \cdots \mathfrak{a}_r.$$

If $Q \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ is also a projective $\mathcal{O}$-module, then the following statements are equivalent.

(i) $P \cong Q$;

(ii) $r = s$ *and* $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_s\}$ *in* $\mathrm{Cl}(\mathcal{O})$. ○

The integer r is called the *rank* of P and the ideal class $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\}$ is called the *ideal class* of P. A projective module is said to be in *standard form* if $P = \mathcal{O}^{r-1} \oplus \mathfrak{a}$.

The next result is immediate from the observation that $M/T(M)$ is torsion-free.

E Corollary

Let M be a finitely generated module over a Dedekind domain $\mathcal{O}$. Then

$$M \cong T(M) \oplus M/T(M)$$

with $M/T(M)$ projective. ○

To complete the picture, we describe the structure of torsion modules over a Dedekind domain $\mathcal{O}$. A finitely generated torsion $\mathcal{O}$-module M has *annihilator*

$$\mathrm{Ann}(M) = \{x \in \mathcal{O} \mid mx = 0 \text{ for all } m \in M\},$$

which is a nonzero ideal of $\mathcal{O}$. If $\mathrm{Ann}(M) = \mathfrak{p}^n$ for some nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and natural number n, then M is said to be $\mathfrak{p}$-*primary*.

Since $\mathcal{O}$ is Noetherian, any set of submodules of a finitely generated module M has maximal members, each of which is finitely generated ([BK: IRM], (3.1.6)). However, the sum of any two $\mathfrak{p}$-primary submodules of M is again $\mathfrak{p}$-primary, so that M has a unique maximal $\mathfrak{p}$-primary submodule, $T_{\mathfrak{p}}(M)$. The submodule $T_{\mathfrak{p}}(M)$ is the $\mathfrak{p}$-*primary component* of M; it is again finitely generated.

Then we have the following result ([BK: IRM] (6.3.15), (6.3.20)).

F The Primary Decomposition Theorem

Let M be a finitely generated torsion module over a Dedekind domain $\mathcal{O}$. Then:

(i) *there is a direct sum decomposition*

$$M = \bigoplus\nolimits_{\mathfrak{p}} T_{\mathfrak{p}}(M),$$

where almost all (that is, all except a finite number) of the $\mathfrak{p}$-primary components of M are zero;

(ii) *for each prime ideal $\mathfrak{p}$ of $\mathcal{O}$,*

$$T_{\mathfrak{p}}(M) = \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},1)} \oplus \cdots \oplus \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},\ell_{\mathfrak{p}})}$$

with $\delta(\mathfrak{p},1) \leq \cdots \leq \delta(\mathfrak{p},\ell_{\mathfrak{p}})$ and $\ell_{\mathfrak{p}} \geq 0$;

(iii) *the collection of integers $\delta(\mathfrak{p},1),\ldots,\delta(\mathfrak{p},\ell_{\mathfrak{p}})$, where $\mathfrak{p}$ ranges over all the nonzero prime ideals of $\mathcal{O}$, is uniquely determined by M and in turn determines M to within isomorphism.* ○

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2.3.21 Module categories over Dedekind domains

We can now give the promised direct sum decompositions of categories that arise from the structure theory of modules over a Dedekind domain $\mathcal{O}$.

Since $\mathcal{O}$ is a Noetherian ring, any finitely generated (right) $\mathcal{O}$-module M is also Noetherian, and in particular, any ascending chain of submodules of M has a maximal member, which is again finitely generated ([BK: IRM] (3.1.6)). Thus the category $\mathcal{M}_{\mathcal{O}}$ of finitely generated (right) $\mathcal{O}$-modules is abelian (2.3.6).

Let $\mathcal{T}_{\mathcal{O}}$ be the full subcategory of $\mathcal{M}_{\mathcal{O}}$ whose objects are all the finitely generated torsion $\mathcal{O}$-modules. Since the direct sum of two modules belonging to $\mathcal{T}_{\mathcal{O}}$ is again a module in $\mathcal{T}_{\mathcal{O}}$, we see that $\mathcal{T}_{\mathcal{O}}$ is an additive subcategory of $\mathcal{M}_{\mathcal{O}}$. As a submodule or quotient module of a module in $\mathcal{T}_{\mathcal{O}}$ again belongs to $\mathcal{T}_{\mathcal{O}}$, the kernel and cokernel of a homomorphism in $\mathcal{T}_{\mathcal{O}}$ is also in $\mathcal{T}_{\mathcal{O}}$, so that $\mathcal{T}_{\mathcal{O}}$ is an abelian category (2.3.5).

Write $\mathcal{TF}_{\mathcal{O}}$ for the category of finitely generated torsion-free (right) $\mathcal{O}$-modules. By (2.3.20 - B), $\mathcal{TF}_{\mathcal{O}}$ is the same as $\mathcal{P}_{\mathcal{O}}$, the category of finitely generated projective $\mathcal{O}$-modules.

Let M and N be finitely generated $\mathcal{O}$-modules. Clearly, an $\mathcal{O}$-module homomorphism $\lambda : M \to N$ induces a homomorphism $T(\lambda) : T(M) \to T(N)$. Thus we have a functor

$$T : \mathcal{M}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathcal{O}}.$$

There is also a functor $F : \mathcal{M}_{\mathcal{O}} \to \mathcal{P}_{\mathcal{O}}$, which sends M to $M/T(M)$. Now, any module M has a direct sum decomposition $M = T(M) \oplus M'$ with $M' \cong M/T(M)$ projective, but there is no direct sum decomposition of the category $\mathcal{M}_{\mathcal{O}}$ in terms of $\mathcal{P}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$, essentially because there is no canonical choice of M' – see Exercise 2.3.4.

For each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ let $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ denote the full subcategory of $\mathcal{T}_{\mathcal{O}}$ given by the $\mathfrak{p}$-primary modules. Again, $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ is an abelian category, and we have functors

$$T_{\mathfrak{p}} : \mathcal{T}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

for each $\mathfrak{p}$.

By (2.3.20 - F), there is a direct sum decomposition of

$$\mathcal{T}_{\mathcal{O}} = \bigoplus\nolimits_{\mathfrak{p}\in\mathbf{P}} \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

of the category $\mathcal{T}_{\mathcal{O}}$ into its subcategories $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$.

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2.3.22 The Embedding Theorems

There are several results to the effect that an abstract additive or abelian category can be viewed as embedded as a subcategory in a concrete abelian category, such as $\mathcal{A}b$, or $\mathcal{M}od_R$ for some ring R.

If $\mathcal{C}$ is a small additive category, then $\mathcal{C}$ can be embedded contravariantly as an additive subcategory in the functor category $[\mathcal{C}, \mathcal{A}b]$ by associating with an object C the morphism functor $\mathrm{Mor}_{\mathcal{C}}(C, -)$ ([Mitchell 1965], IV (2.3)). Since the opposite of an additive category is also additive, this result gives a covariant embedding of $\mathcal{C}$ in the additive category $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}b]$.

This result applies also to abelian categories, and extends to show that a small additive or abelian category can be embedded in $\mathcal{A}b$ (*ibid.* IV (2.6)), which result is sometimes known as the Lubkin-Heron-Freyd Representation Theorem, although it is also attributed to Mitchell. (The discoverers of this circle of results are generous with their attributions.)

In [Pareigis 1970] (4.14) Theorem 3, we find the more refined result that, given a small abelian category $\mathcal{C}$, there is a ring R and a covariant full faithful functor (1.3.13) from $\mathcal{C}$ to the category $\mathcal{M}od_R$. Anticipating the next section, we remark that this functor is also 'exact' in that it preserves exact sequences.

The philosophical consequence of these results is that it suffices to verify any sufficiently general statement about abelian categories in the category $\mathcal{A}b$; this point of view is discussed in [Mitchell 1965], IV. However, it is usually more natural to work directly in a given additive or abelian category.

2.3.23 Example: The Famous Five Lemma

This lemma† is one of a number of useful results which hold in an abstract abelian category, but which are proved more readily by a 'diagram chase' in a module category.

Suppose that we have a commuting diagram, with both rows exact, in an additive category:

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 & \xrightarrow{\alpha_4} & M_5 \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\ N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4 & \xrightarrow{\beta_4} & N_5 \end{array}$$

† Possibly so-named because it leads to the Smuggler's Top Theorem, a truly marvellous result which this footnote is unfortunately too small to contain ([Blyton 1950]).

Then, by embedding the category in $\mathcal{A}_{\mathcal{B}}$, we may work with elements in the various objects. For example:

> *Suppose that γ_2 and γ_4 are epimorphisms, and that γ_5 is a monomorphism. Then γ_3 is also an epimorphism.*

To prove this, let $y_3 \in N_3$. Taking $x_4 \in M_4$ with $\gamma_4 x_4 = \beta_3 y_3 \in N_4$, we have

$$\gamma_5 \alpha_4 x_4 = \beta_4 \beta_3 y_3 = 0.$$

Thus $x_4 \in \operatorname{Ker} \alpha_4 = \operatorname{Im} \alpha_3$, and so we can find $x_3 \in M_3$ with $\alpha_3 x_3 = x_4$; then

$$y_3 - \gamma_3 x_3 \in \operatorname{Ker} \beta_3 = \operatorname{Im} \beta_2 = \operatorname{Im} \beta_2 \gamma_2 = \operatorname{Im} \gamma_3 \alpha_2.$$

So $y_3 \in \operatorname{Im} \gamma_3$ as required.

With slightly less rigmarole, one can prove the dual statement for γ_3 to be a monomorphism. In combination, these diagram chases yield the following highly useful result.

> *If γ_1, γ_2, γ_4 and γ_5 are isomorphisms, then so also is γ_3.*

Other, less widely used, diagram chase theorems to be found in the literature have such evocative names as the Snake Lemma, the Horseshoe Lemma, the Windmill Lemma, the 3×3 Lemma, the Nine Lemma, etc.

Exercises

2.3.1 Show that the opposite category of an abelian category is also abelian.

2.3.2 Let $\alpha : L \to M$ and $\beta : M \to N$ be morphisms in an abelian category. Show (for example, by (2.3.2)), that the following statements are equivalent.

(i) $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence;

(ii) α is a monomorphism and $\beta = \operatorname{Cok} \alpha$;

(iii) β is an epimorphism and $\alpha = \operatorname{Ker} \beta$.

Recall that our example $0 \to \mathbb{Q} \to \mathbb{R} \to 0 \to 0$ in $\mathcal{HA}_{\mathcal{B}}$ shows that (ii) need not imply (i) when the category is additive but not abelian.

2.3.3 Show that in an abelian category $\mathcal{C}$ any morphism $\alpha : L \to M$ gives rise to a commuting diagram of the following kind.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\kappa} & L & \xrightarrow{\tau} & I & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\theta} & & \\ 0 & \longleftarrow & N & \xleftarrow{\chi} & M & \xleftarrow{\nu} & I' & \longleftarrow & 0 \end{array}$$

Here the horizontal sequences are both short exact, θ is an isomorphism, and $\kappa = \operatorname{Ker}\alpha$ and $\chi = \operatorname{Cok}\alpha$. Thus $I = \operatorname{Cok}\operatorname{Ker}\alpha$ is the image $\operatorname{Im}\alpha$ of α.

Generalizing (2.1.2) from module categories to arbitrary abelian categories, we say that a sequence

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \longrightarrow \cdots$$

in $\mathcal{C}$ is *exact at* A_i if $\operatorname{Ker}\alpha_i = \operatorname{Im}\alpha_{i-1}$, and *exact* if it is exact at each term A_i in the sequence. Show that a short exact sequence (as defined in (2.2.7)) is an exact five-term sequence which begins and ends in the zero object 0 of $\mathcal{C}$.

2.3.4 Suppose that an additive category $\mathcal{C}$ is the direct sum of its subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$, and that P_1, P_2 are the projection functors as in (2.3.17). Show that for any pair of objects M, N of $\mathcal{C}$, there is an isomorphism of abelian groups

$$\operatorname{Mor}_{\mathcal{C}}(M,N) \cong \operatorname{Mor}_{\mathcal{C}_1}(P_1M, P_1N) \oplus \operatorname{Mor}_{\mathcal{C}_2}(P_2M, P_2N)$$

which is natural in each term M, N.

Show that, for any nonzero integer a,

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/a\mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z}$$

and deduce that $\mathcal{M}_{\mathbb{Z}}$ is *not* the direct sum of $\mathcal{T}_{\mathbb{Z}}$ and $\mathcal{TF}_{\mathbb{Z}}$ – see (2.3.21).

2.3.5 Let $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ be a set of categories, where the index set Λ may be infinite. Show that if each $\mathcal{C}_\lambda$ is additive (or abelian), then the direct product $\prod_\Lambda \mathcal{C}_\lambda$ (1.1.13) is again additive (or abelian).

Describe the projective objects in the direct product $\prod_\Lambda \mathcal{C}_\lambda$ and direct sum $\bigoplus_\Lambda \mathcal{C}_\lambda$ – see (2.3.19).

2.3.6 Let $\mathcal{C}$ be an abelian category. Show that a pull-back in $\mathcal{C}$ can be obtained as the kernel of a suitably defined morphism, and that a push-out can be obtained as a cokernel.

Deduce that $\mathcal{C}$ contains the pull-back of any pull-back diagram of objects and morphisms in $\mathcal{C}$ and likewise for push-outs.

2.3.7 Let $\mathcal{C}$ be a preadditive category. Show that if $\mathcal{C}$ contains all possible pull-backs and push-outs of its objects and morphisms, then $\mathcal{C}$ is abelian.

2.3.8 Here, for a ring T and T-module homomorphism λ, we denote by $\mathrm{Ker}_T \lambda$ and $\mathrm{Cok}_T \lambda$ the kernel and cokernel of λ in the category $\mathcal{M}\text{od}_T$.

By the Embedding Theorems (2.3.22), given an abelian category $\mathcal{C}$, there is a ring S and embedding of $\mathcal{C}$ in $\mathcal{M}\text{od}_S$ such that for any morphism λ in $\mathcal{C}$ both $\mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_S \lambda$ lie in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is also embedded in $\mathcal{M}\text{od}_R$ for another ring R, then $\mathrm{Ker}_R \lambda \cong \mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_R \lambda \cong \mathrm{Cok}_S \lambda$ as abelian groups, and hence $\mathrm{Ker}_R \lambda$ and $\mathrm{Cok}_R \lambda$ lie in $\mathcal{C}$ (at least to within a canonical isomorphism). This proves the converse to (2.3.5).

2.3.9 Let $\mathcal{C}$ be a full subcategory of $\mathcal{M}\text{od}_R$ containing the zero module. Show that the following are equivalent.

(a) $\mathcal{C}$ is abelian.

(b) For any short exact sequence of right R-modules

$$0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

both the following hold:

(i) if $M \in \mathcal{C}$, then both M' and M'' are in $\mathcal{C}$, and

(ii) when the sequence is split, if M' and M'' are in $\mathcal{C}$, then $M \in \mathcal{C}$.

2.3.10 **Extensions depend on the category**

Generalizing (2.3.5), we may define an *abelian subcategory* $\mathcal{C}$ of an abelian category $\mathcal{A}$ to be an additive subcategory with the further property that for any morphism $\gamma : C \to D$ in $\mathcal{C}$, then the kernel and cokernel of γ in $\mathcal{A}$ are (isomorphic to) objects in $\mathcal{C}$. More precisely, if (K, κ) is a kernel for γ in $\mathcal{A}$, then there is an object K' in $\mathcal{C}$, a morphism $\kappa' : K' \to C$ in $\mathcal{C}$ and an isomorphism $\theta : K \to K'$ (in $\mathcal{A}$) such that $\kappa = \kappa'\theta$, and similarly for cokernels.

Verify that an abelian subcategory is indeed abelian.

By the discussion in (2.3.7), for any ring R, the category of Artinian semisimple modules $\mathcal{ASS}_R$ is abelian. Verify that $\mathcal{ASS}_R$ is an abelian subcategory of $\mathcal{M}\text{od}_R$.

Now suppose that L, N lie in $\mathcal{ASS}_R$. By the Artinian Splitting

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Theorem ([BK: IRM] (4.1.17)), any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with term M in $\mathcal{ASS}_R$ must be split. However, this need not be so when L and N are viewed as modules in $\mathcal{M}\text{OD}_R$ and M can lie outside $\mathcal{ASS}_R$.

For an explicit example, take $R = \mathcal{K}[\epsilon]$, the ring of dual numbers over a field $\mathcal{K}$. (This is the polynomial ring whose indeterminate ϵ has $\epsilon^2 = 0$.)

Show that the Jacobson radical of R is $\text{rad}(R) = \epsilon\mathcal{K}[\epsilon]$ and that $\mathcal{ASS}_R = \mathcal{M}_\mathcal{K}$, with ϵ acting as 0 on $\mathcal{K}$-spaces (see (2.3.9)(iv)). Now consider the exact sequence of R-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}[\epsilon] \longrightarrow \mathcal{K} \longrightarrow 0.$$

2.3.11 Let A be an object of an additive category $\mathcal{C}$. Write $\sigma_j : A \to A \oplus A$, $\pi_j : A \oplus A \to A$ for the usual monomorphisms and epimorphisms, $j = 1, 2$. Then the *diagonal map* $\Delta_A : A \to A \oplus A$ is defined by universality to be the unique morphism with $\pi_j \Delta_A = id_A$, $j = 1, 2$. Dually, the *codiagonal map* $\nabla_A : A \oplus A \to A$ is defined to be the unique map such that $\nabla_A \sigma_j = id_A$, $j = 1, 2$.

Show that, when $\mathcal{C}$ is abelian, $\Delta_A : A \to A \oplus A$ is the kernel of $\pi_1 - \pi_2 : A \oplus A \to A$, while $\nabla_A : A \oplus A \to A$ is the cokernel of $\sigma_1 - \sigma_2 : A \to A \oplus A$.

Confirm also that these definitions extend those given for modules in Exercise 2.1.7.

Show too that, for morphisms $\alpha : A \to C$ and $\beta : B \to D$ in $\mathcal{C}$,

$$\alpha \oplus \beta : A \oplus B \longrightarrow C \oplus D,$$

as defined in (2.2.16), is the unique morphism such that

$$(\alpha \oplus \beta)\sigma_A = \sigma_C \alpha \quad \text{and} \quad (\alpha \oplus \beta)\sigma_B = \sigma_D \beta.$$

Further, when $A = B$ and $C = D$, $\alpha + \beta : A \to C$ is uniquely defined by

$$\alpha + \beta = \nabla_C (\alpha \oplus \beta) \Delta_A.$$

[Pareigis 1970] 4.1 develops this line of argument to give an alternative treatment of additive categories.

Finally, prove the converse of (2.2.20): if $F : \mathcal{C} \to \mathcal{D}$ is a functor between additive categories, which preserves finite direct sums of objects, together with the corresponding monomorphisms and epimorphisms, then F is additive.

2.3.12 **The Three Lemma**

Extend the Three Lemma ([BK: IRM] Exercise 2.4.1) from modules to an arbitrary abelian category $\mathcal{A}$: if $\alpha : L \to M$ and $\beta : M \to N$ are any morphisms in $\mathcal{A}$, there is an induced exact sequence

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta\alpha \to \operatorname{Ker}\beta \to \operatorname{Cok}\alpha \to \operatorname{Cok}\beta\alpha \to \operatorname{Cok}\beta \to 0.$$

2.3.13 **The Snake Lemma**

Extend the Snake Lemma ([BK: IRM] Exercise 2.4.2) from modules to an arbitrary abelian category $\mathcal{A}$: given a commuting diagram of objects and morphisms in $\mathcal{A}$ with both rows exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\ & & \downarrow{\alpha'} & & \downarrow{\alpha} & & \downarrow{\alpha''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0 \end{array}$$

construct a *connecting morphism* $\delta : \operatorname{Ker}\alpha'' \to \operatorname{Cok}\alpha'$ which fits into an exact sequence

$$0 \longrightarrow \operatorname{Ker}\alpha' \longrightarrow \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\alpha'' \xrightarrow{\delta} \operatorname{Cok}\alpha' \longrightarrow \operatorname{Cok}\alpha \longrightarrow \operatorname{Cok}\alpha'' \longrightarrow 0$$

2.4 EXACT CATEGORIES

We now introduce a structure that plays a fundamental role in K-theory, namely, that of an exact category. This is an additive category $\mathcal{C}$ together with a specified class of 'admissible' exact sequences chosen from the class of all short exact sequences in $\mathcal{C}$. The definitions, and properties, of the K-groups $K_0(\mathcal{C}), K_1(\mathcal{C}), \ldots$ depend not only on the category $\mathcal{C}$ but also on the class $\operatorname{Ex}(\mathcal{C})$. To illustrate this phenomenon, we discuss a few basic properties of the Grothendieck group $K_0(\mathcal{C})$. We state the properties required of $\operatorname{Ex}(\mathcal{C})$ in two parts, one 'elementary', the other 'advanced'. The elementary properties suffice for many purposes, in particular, the construction of the Grothendieck group. The more refined advanced properties are designed to ensure that Quillen's Q-construction can be carried out, as needed for a definition of higher K-theory. We do not discuss Quillen's construction in this text.

Several authors have formulated definitions of a 'category with admissible exact sequences', sometimes from the point of view of homological algebra, sometimes from that of K-theory. [Mac Lane 1975], Ch. XII §4, introduces *proper* classes of exact sequences in an abelian category. Such a class satisfies conditions GE1–3 of (2.4.1) together with QE1, QE1$^{\text{op}}$, QE2 and QE2$^{\text{op}}$ of (2.4.10). Other variations can be found in [Buchsbaum 1959], [Buchsbaum 1960] and [Heller 1965]. The definition of a 'K-theoretic' exact category that we adopt here appears in [Quillen 1973].

To aid the exposition, we use the term 'G-exact' for a category which satisfies the elementary conditions that allow the construction of the Grothendieck group, and 'Q-exact' for a category satisfying Quillen's conditions. The reason for these neologisms is that the term 'exact category' is already used in category theory with a different meaning ([Mitchell 1965], p. 18), and we feel it useful to preserve a distinction in an expository text. On the other hand, some authors of papers on K-theory use 'exact' to mean 'Q-exact' as in this text.

As usual, we discuss right categories only; it is clear that everything works also for left categories.

2.4.1 *G-exact categories*

A *G-exact category* is a pair $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ consisting of an additive category $\mathcal{C}$ together with a specified class $\mathrm{Ex}(\mathcal{C})$ of short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

all of whose terms M', M and M'' are in $\mathcal{C}$. The class $\mathrm{Ex}(\mathcal{C})$ must satisfy the following requirements.

GE1. Every split exact sequence in $\mathcal{C}$ belongs to $\mathrm{Ex}(\mathcal{C})$.

GE2. $\mathrm{Ex}(\mathcal{C})$ is *closed under isomorphism*: if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0 \end{array}$$

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in which all the vertical arrows are isomorphisms in $\mathcal{C}$ and the top sequence is in $\mathrm{Ex}(\mathcal{C})$, then so also is the bottom sequence.

GE3. $\mathrm{Ex}(\mathcal{C})$ is *closed under direct sums*: if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so also is

$$0 \longrightarrow M' \oplus N' \longrightarrow M \oplus N \longrightarrow M'' \oplus N'' \longrightarrow 0.$$

The *admissible exact sequences* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ are the short exact sequences belonging to $\mathrm{Ex}(\mathcal{C})$. An *admissible monomorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a monomorphism $\alpha : M' \to M$ in $\mathcal{C}$ that occurs in some short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, and an *admissible epimorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is similarly one occurring in an admissible short exact sequence. Thus any admissible monomorphism has a cokernel, which must be an admissible epimorphism, and dually.

In many contexts where there is a standard choice for the class $\mathrm{Ex}(\mathcal{C})$, a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is referred to simply as $\mathcal{C}$.

2.4.2 Split and repletely G-exact categories

Given an additive category $\mathcal{C}$, there are two extreme choices of $\mathrm{Ex}(\mathcal{C})$ that make $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ into a G-exact category.

On the one hand, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all short exact sequences in $\mathcal{C}$. In this case, we say that $\mathcal{C}$ is a *repletely G-exact category.* As we frequently wish to consider an additive category $\mathcal{C}$ to be a repletely G-exact category, we use the same symbol for both and omit $\mathrm{Ex}(\mathcal{C})$ from the notation.

At the other extreme, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all split short exact sequences with terms in $\mathcal{C}$. By condition GE1, this is the smallest possible choice for $\mathrm{Ex}(\mathcal{C})$, and a series of routine verifications establishes that axioms GE2 and GE3 hold. In this case, we say that the category is *split G-exact* and we use the notation $\mathcal{C}^{\oplus}$ to indicate that we view $\mathcal{C}$ as a G-exact category in this way.

In some categories, notably the categories $\mathcal{P}_R$ and $\mathcal{F}_R$, these extremal choices coincide, since any short exact sequence in them must be split.

We note next that many familiar module categories are G-exact.

2.4.3 Theorem

For any ring R, the following are repletely G-exact categories of right R-modules.

(i) $\mathcal{M}\text{OD}_R$ *itself.*
(ii) $\mathcal{P}_R$*, the category of finitely generated projective modules.*
(iii) $\mathcal{F}_R$*, the category of free modules of finite rank.*
(iv) $\mathcal{A}\text{RT}_R$*, the category of Artinian modules.*
(v) $\mathcal{N}\text{OETH}_R$*, the category of Noetherian modules.*
(vi) $\mathcal{M}_R$*, the category of finitely generated modules.*
(vii) $\mathcal{ASS}_R$*, the category of Artinian semisimple modules.*

The categories $\mathcal{P}_R$*,* $\mathcal{F}_R$ *and* $\mathcal{ASS}_R$ *are also split G-exact.*

Proof

By (2.2.14), $\mathcal{P}_R$, $\mathcal{F}_R$ and $\mathcal{M}_R$ are additive categories, and by (2.3.8), $\mathcal{N}\text{OETH}_R$, $\mathcal{A}\text{RT}_R$ and $\mathcal{ASS}_R$ are also additive (in fact, abelian). Assertions (i) to (vii) follow by trivial verifications.

The definition of a projective module shows that any short exact sequence in $\mathcal{P}_R$, and hence in $\mathcal{F}_R$, must be split, while any short exact sequence in $\mathcal{ASS}_R$ is split by the Artinian Splitting Theorem ([BK: IRM] (4.1.17)). □

2.4.4 Relative exact categories

We next give some illustrations of more subtle choices for $\mathcal{C}$ or $\text{Ex}(\mathcal{C})$.

Suppose that S is a subring of R. One can then define a G-exact category $\mathcal{M}_{R,S}$, the *S-relative category*, to be the additive category $\mathcal{M}_R$, but with $\text{Ex}(\mathcal{M}_{R,S})$ consisting of all those short exact sequences in $\mathcal{M}_R$ that are split when regarded as short exact sequences of S-modules. For examples, see Exercise 2.4.5.

Similarly, it is sometimes useful to consider the full subcategory $\mathcal{D}$ of $\mathcal{M}_R$ whose objects are projective as S-modules; then all short exact sequences in $\mathcal{D}$ are split over S.

The common generalization of both these examples is given by an additive subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$ ($\mathcal{M}_S$ in the preceding examples), with $\text{Ex}(\mathcal{C})$ comprising all those short exact sequences in $\mathcal{C}$ that are split in $\mathcal{A}$.

2.4.5 On terminology

The term ‘semisimple category’ is sometimes used by K-theorists instead of ‘split exact category’. However, we feel that ‘split’ is more descriptive, since the use of 'semisimple' suggests that results about semisimple modules ought to hold in the category. However, there are many results, such as the Artinian Splitting Theorem ([BK: IRM] (4.1.17)), which hold in $\mathcal{ASS}_R$ but do not necessarily hold in a split exact category such as $\mathcal{P}_R$.

2.4.6 Exact functors

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ be G-exact categories. An *exact functor* from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which satisfies the following conditions.

EFun1. F is an additive functor.

EFun2. If

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

is a short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, then

$$F\mathbf{E} \qquad 0 \longrightarrow FM' \xrightarrow{\ F\alpha\ } FM \xrightarrow{\ F\beta\ } FM'' \longrightarrow 0$$

must belong to $\mathrm{Ex}(\mathcal{D})$.

Notice that requirement EFun2 is, in effect, two conditions – firstly, $F\mathbf{E}$ must be a short exact sequence in $\mathcal{D}$, and then it must also belong to the distinguished class $\mathrm{Ex}(\mathcal{D})$. In the literature, $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ are commonly both repletely G-exact. One then speaks of F simply as being an exact functor from $\mathcal{C}$ to $\mathcal{D}$.

2.4.7 Examples

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are G-exact categories and that both have the same underlying additive category $\mathcal{B}$. Then the identity functor on $\mathcal{B}$ will induce an exact functor from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ if and only if $\mathrm{Ex}(\mathcal{C})$ is a subclass of $\mathrm{Ex}(\mathcal{D})$. In particular, the identity functor fails to be an exact functor from $\mathcal{M}_R$ to $\mathcal{M}_R^{\oplus}$ (unless R is Artinian semisimple – see (2.3.9) and (2.4.3)).

Conversely, an exact functor between G-exact categories need not preserve all short exact sequences; examples are given by the homomorphism functors $\mathrm{Hom}(L, -)$ on $\mathcal{M}\mathit{od}_R^{\oplus}$ with L not projective (2.1.8). However, any additive functor preserves split exact sequences (2.2.20) and so will act as an exact functor on a split G-exact category. We record this fact for reference.

2.4.8 Proposition

Suppose that $\mathcal{C}$ is a split G-exact category and that $\mathcal{D}$ is a G-exact category. Then any additive functor from $\mathcal{C}$ to $\mathcal{D}$ is exact. □

The above result shows that, when $\mathcal{C}$ is split G-exact, condition EFun1 implies EFun2. More surprisingly, when $\mathcal{C}$ and $\mathcal{D}$ are abelian, EFun2 always implies EFun1, as can be seen from Exercise 2.3.11.

2.4.9 The Grothendieck group

We now trespass into K-theory to give the construction of the Grothendieck group of a G-exact category. This will at least serve to motivate our definition of a G-exact category.

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ be a G-exact category, and assume that $\mathcal{C}$ is small, or that $\mathcal{C}$ has a small skeleton (1.3.15). For an object C of $\mathcal{C}$, let $\langle C\rangle$ denote the isomorphism class of $\mathcal{C}$, so that $\langle C\rangle = \langle C'\rangle$ precisely when $C \cong C'$. Then the isomorphism classes of objects of $\mathcal{C}$ constitute a set $\mathrm{Is}(\mathcal{C})$, and we can form the free abelian group $\mathrm{Fr}(\mathcal{C})$ on $\mathrm{Is}(\mathcal{C})$. Let $\mathrm{Rel}(\mathcal{C})$ be the subgroup of $\mathrm{Fr}(\mathcal{C})$ generated by all expressions of the form

$$\langle C\rangle - \langle C'\rangle - \langle C''\rangle ,$$

one for each short exact sequence

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

belonging to $\mathrm{Ex}(\mathcal{C})$.

The *Grothendieck group* $K_0(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ of $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is the quotient group $\mathrm{Fr}(\mathcal{C})/\mathrm{Rel}(\mathcal{C})$. It is usual to omit to mention the class of exact sequences and write simply $K_0(\mathcal{C})$.

Clearly, $K_0(\mathcal{C})$ is an abelian group because $\mathrm{Fr}(\mathcal{C})$ is. The image of $\langle C\rangle$ in $K_0(\mathcal{C})$ is denoted $[C]$. By construction, a short exact sequence as above gives the equality

$$[C] = [C'] + [C''] \text{ in } K_0(\mathcal{C}).$$

It is evident that an exact functor $F : \mathcal{C} \to \mathcal{D}$ between G-exact categories gives a homomorphism

$$K_0(F) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D})$$

of abelian groups.

The Grothendieck group K_0 was introduced by A. Grothendieck in 1957 in letters to J.-P. Serre that formed the basis of [Borel & Serre 1958]. It was

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initially applied (in the notation $K(X)$) to the category of coherent sheaves on an algebraic variety X and to the category of vector bundles over a topological space X. Grothendieck chose the symbol K (for the German 'Klasse') to avoid $C(X)$, already in use as the ring of functions on X. For further history, see [Bak 1987].

Although we do not give any computations of Grothendieck groups in this text, we can give a plausibility argument to illustrate how the choice of the class $\mathrm{Ex}(\mathcal{C})$ influences $K_0(\mathcal{C})$.

Consider the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups, which is $\mathcal{A}_{\mathcal{RT}\mathbb{Z}}$ in disguise. The Primary Decomposition Theorem (2.3.20 – F) shows that the irreducible modules in $\mathcal{FA}_{\mathcal{B}}$ are the cyclic groups $\mathbb{Z}/p\mathbb{Z}$ of prime order p. Since any finite abelian group A has a composition series with factors of the form $\mathbb{Z}/p\mathbb{Z}$ for various primes p, the symbol $[A]$ in $K_0(\mathcal{FA}_{\mathcal{B}})$ can be written as a sum of symbols $[\mathbb{Z}/p\mathbb{Z}]$, and it can be shown that $K_0(\mathcal{FA}_{\mathcal{B}})$ is the free abelian group on the set

$$\{[\mathbb{Z}/p\mathbb{Z}] \mid p \text{ prime}\}.$$

However, if we work in the Grothendieck group $K_0(\mathcal{FA}_{\mathcal{B}}^{\oplus})$ of the G-exact category $\mathcal{FA}_{\mathcal{B}}^{\oplus}$, in which only split exact sequences are permitted, then the symbol $[A]$ can be reduced to symbols corresponding to indecomposable finite abelian groups, which have the form $\mathbb{Z}/p^k\mathbb{Z}$ for p prime, $k \geq 1$, but no further reduction is possible. As might be anticipated, $K_0(\mathcal{FA}_{\mathcal{B}}^{\oplus})$ is the free abelian group on the set

$$\{[\mathbb{Z}/p^k\mathbb{Z}] \mid p \text{ prime}, k \geq 1\}.$$

Granted these computations, we can see that the identity functor on the additive category $\mathcal{FA}_{\mathcal{B}}$ induces an exact functor I from the split G-exact category $\mathcal{FA}_{\mathcal{B}}^{\oplus}$ to the repletely G-exact category $\mathcal{FA}_{\mathcal{B}}$, but that

$$K_0(I) : K_0(\mathcal{FA}_{\mathcal{B}}^{\oplus}) \longrightarrow K_0(\mathcal{FA}_{\mathcal{B}})$$

is not injective, since

$$K_0(I)([\mathbb{Z}/p^k\mathbb{Z}]) = k[\mathbb{Z}/p\mathbb{Z}] = K_0(I)([(\mathbb{Z}/p\mathbb{Z})^k]).$$

2.4.10 Q-exact categories

In a fundamental paper, [Quillen 1973], Quillen has shown how to construct higher dimensional analogues, $K_1(\mathcal{C})$, $K_2(\mathcal{C}), \ldots$, of the Grothendieck group of a suitably defined exact category. Although we cannot give any indication of Quillen's construction in this text (see [Rosenberg 1996] pp. 289–297 for a quick introduction), this is a convenient place to list the additional conditions on the class $\mathrm{Ex}(\mathcal{C})$ that are required by Quillen's theory. We take our definition directly from [Quillen 1973], §2.

A *Q-exact category* is a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ in which the class $\mathrm{Ex}(\mathcal{C})$ of admissible exact sequences satisfies the following requirements in addition to GE1–3. Recall that a monomorphism or epimorphism in $\mathcal{C}$ is admissible if it occurs in a short exact sequence that belongs to $\mathrm{Ex}(\mathcal{C})$.

QE1. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible epimorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \xrightarrow{\gamma} M'' \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M \times_{M''} N' \longrightarrow M \xrightarrow{\delta\beta} N'' \longrightarrow 0.$$

QE1$^{\mathrm{op}}$. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible monomorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\gamma} N \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M' \xrightarrow{\gamma\alpha} N \longrightarrow M'' \oplus_M N \longrightarrow 0.$$

QE2. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\theta : L'' \to M''$ is a morphism in $\mathcal{C}$. Then the pull-back exact sequence (cf. Exercise 1.4.11 and [BK: IRM] (2.4.10))

$$\theta^*\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\mu} M \times_{M''} L'' \xrightarrow{\lambda} L'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under base change.)

QE2$^{\mathrm{op}}$. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\ \mu\ } M \xrightarrow{\ \lambda\ } M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\phi : M' \to N'$ is a morphism in $\mathcal{C}$. Then the push-out exact sequence

$$\phi_*\mathbf{E} \qquad 0 \longrightarrow N' \xrightarrow{\ \alpha\ } N' \oplus_{L'} M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under cobase change.)

QE3. Suppose that the morphism $\beta : M \to M''$ in $\mathcal{C}$ has a kernel in $\mathcal{C}$, and that there is a morphism $\varphi : L \to M$ such that $\beta\varphi : L \to M''$ is an admissible epimorphism. Then β is already an admissible epimorphism.

QE3$^{\mathrm{op}}$. Suppose that the morphism $\alpha : M' \to M$ in $\mathcal{C}$ has a cokernel in $\mathcal{C}$, and that there is a morphism $\psi : M \to N$ such that $\psi\alpha : M' \to N$ is an admissible monomorphism. Then α is already an admissible monomorphism.

2.4.11 Comments on the axioms

(i) Pull-backs and push-outs in an abstract category are defined by their universal properties, which are given for modules in ([BK: IRM] Proposition 2.4.9), and in Exercises 1.4.10 and 1.4.11 for categories in general. The definitions of the pull-back sequence $\theta^*\mathbf{E}$ and the push-out sequence $\phi_*\mathbf{E}$ follow those given for modules in Exercise 2.1.7.

(ii) In the statement of the axioms, we take for granted the fact that various constructions on short exact sequences yield short exact sequences. This is easily verified in a module category. In an abstract category, the verification can be made by using universal properties, or, more conveniently, by invoking the Embedding Theorems (2.3.22).

(iii) Since the underlying category $\mathcal{C}$ is assumed to be additive, rather than abelian, part of the requirement imposed by the axioms is that the various pull-backs and push-outs are objects in $\mathcal{C}$ (note Exercises 2.3.6 and 2.3.7).

(iv) It is not difficult to show that these conditions make the 'elementary' condition GE3 redundant – see Exercise 2.4.4. The fact that the Q-conditions are stronger than the G-conditions is illustrated in Exercise 2.4.11.

We call a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ *repletely Q-exact* when $\mathrm{Ex}(\mathcal{C})$ comprises all short exact sequences in $\mathcal{C}$, and *split Q-exact* when $\mathrm{Ex}(\mathcal{C})$ consists only of the split exact sequences in $\mathcal{C}$.

Although the conditions for a Q-exact category appear formidable, they are usually fairly easy to check in any given instance. For example, the checking of the claims in the following lemma is straightforward.

2.4.12 Lemma

(a) *Any split G-exact category is Q-exact.*

(b) *Let $\mathcal{C}$ be an abelian category and let* $\mathrm{Ex}(\mathcal{C})$ *be the class of all short exact sequences of $\mathcal{C}$. Then the repletely G-exact category* $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ *is Q-exact.*

□

Because of (b) above, we use the term *repletely exact abelian* to describe a repletely G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ with $\mathcal{C}$ abelian.

2.4.13 Exact subcategories

To see when a subcategory of a Q-exact category is also Q-exact, we need a preliminary definition. Let $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ be a Q-exact category. Then $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a *sub-Q-exact category* of $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ if the following conditions are satisfied.

SubQ1. $\mathcal{C}$ is a full additive subcategory of $\mathcal{A}$ with the property that whenever

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$ such that A' and A'' are isomorphic to objects of $\mathcal{C}$, then A is also isomorphic to an object of $\mathcal{C}$.

SubQ2. $\mathrm{Ex}(\mathcal{C})$ is the class of all short exact sequences in $\mathrm{Ex}(\mathcal{A})$ whose terms belong to $\mathcal{C}$.

Condition SubQ1 is often rephrased as: $\mathcal{C}$ is *closed under extensions*, or *extension closed*. Condition SubQ2 implies that the inclusion functor from $\mathcal{C}$ to $\mathcal{A}$ is exact.

In practice, it is usually obvious from the nature of $\mathcal{C}$ that the first condition is satisfied, while the canonical choice of $\mathrm{Ex}(\mathcal{C})$ is that suggested by the second condition. Thus the following theorem gives a widely applicable condition for Q-exactness. The proof is by routine verification of the axioms.

2.4.14 Theorem

Any sub-Q-exact category of a Q-exact category is again Q-exact. □

2.4.15 Corollary

Any sub-Q-exact category of a repletely exact abelian category is Q-exact.

□

[Quillen 1973], §2, observes that the above corollary has a converse: if a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is (skeletally) small, then there exists a repletely exact abelian category in which $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is sub-Q-exact. This is shown by mimicking the proofs of the earlier embedding theorems for additive categories, which were outlined in (2.3.22). The corollary and its converse enable us to define a Q-exact category as a sub-Q-exact category of a repletely exact abelian category. The new definition has the advantage that its axioms are relatively simple.

Observe that a subcategory $\mathcal{C}$ of a Q-exact category $\mathcal{A}$ may be Q-exact without being a sub-Q-exact category of $\mathcal{A}$. For example, the category $\mathcal{ASS}_R$ of Artinian semisimple modules over a ring R is an abelian category, and Q-exact; it is in fact both split and repletely exact. The inclusion functor from $\mathcal{ASS}_R$ to $\mathcal{M}_R$ is exact, but $\mathcal{ASS}_R$ is not a sub-Q-exact category of $\mathcal{M}_R$, unless R is Artinian semisimple (which makes Condition SubQ1 hold.)

The reader may wish to verify the accuracy of the following listing of Q-exact categories. Here, Q-exactness follows immediately from the corollary above.

2.4.16 Theorem

For any ring R, $\mathcal{M}\textsc{od}_R$ is a repletely exact abelian category, and the following are (repletely) sub-Q-exact categories and therefore Q-exact.

(i) *$\mathcal{P}_R$, the category of finitely generated projective modules.*

(ii) *$\mathcal{F}_R$, the category of free modules of finite rank.*

(iii) *$\mathcal{A}\textsc{rt}_R$, the category of Artinian modules.*

(iv) *$\mathcal{N}\textsc{oeth}_R$, the category of Noetherian modules.*

(v) *$\mathcal{M}_R$, the category of finitely generated modules.*

(vi) *$\mathcal{M}\textsc{od}_T$, for any extension ring T of R.* □

Exercises

Note. These exercises refer (mostly) to G-exact categories. The enthusiastic reader may wish to extend them to Q-exact categories where appropriate.

2.4.1 Let $\mathcal{C}$ be a G-exact category. Show that $\mathcal{C}^{\mathrm{op}}$ is also a G-exact category.

2.4.2 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that there are canonical ways to make $\mathcal{C} \times \mathcal{D}$ and $\mathcal{M}\textsc{or}\,\mathcal{C}$ into G-exact categories.

Prove that $\mathcal{C} \times \mathcal{D}$ is split or repletely G-exact if and only if both $\mathcal{C}$ and $\mathcal{D}$ are split or repletely G-exact respectively.

Generalize these statements for arbitrary finite direct products of G-exact categories, and for infinite direct sums of categories as in (2.3.19).

Let $\mathcal{K}$ be a field. Show that $\mathcal{M}or(\mathcal{M}_{\mathcal{K}})$ is not split G-exact, even though $\mathcal{M}_{\mathcal{K}}$ is.

2.4.3 Let $\mathcal{C}$ be a G-exact category, and let $N_1 \to N_2$ be an isomorphism in $\mathcal{C}$. Show that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathrm{Ex}(\mathcal{C})$, then the short exact sequences

$$0 \longrightarrow M' \oplus N_1 \longrightarrow M \oplus N_2 \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow M' \longrightarrow M \oplus N_1 \longrightarrow M'' \oplus N_2 \longrightarrow 0$$

are also in $\mathrm{Ex}(\mathcal{C})$.

2.4.4 Show that axioms QE1 and QE2 together imply axiom GE3, as follows. Take short exact sequences

$$(1) \qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$(2) \qquad 0 \longrightarrow N' \longrightarrow N \xrightarrow{\ \nu\ } N'' \longrightarrow 0$$

in $\mathrm{Ex}(\mathcal{C})$, construct firstly the pull-back of (1) by the admissible epimorphism

$$(id, 0) : M'' \oplus N'' \longrightarrow M'',$$

and then the pull-back of the resulting sequence by the admissible epimorphism

$$id \oplus \nu : M'' \oplus N \longrightarrow M'' \oplus N''$$

to obtain a short exact sequence in $\mathrm{Ex}(\mathcal{C})$

$$(3) \qquad 0 \longrightarrow M' \longrightarrow W \xrightarrow{\ \omega\ } M'' \oplus N \longrightarrow 0.$$

Show that $\theta = (id \oplus \nu)\omega : W \to M'' \oplus N''$ is admissible, and that the corresponding short exact sequence

$$0 \longrightarrow L \longrightarrow W \xrightarrow{\ \theta\ } M'' \oplus N'' \longrightarrow 0$$

is isomorphic to the direct sum of (1) and (2).

2.4.5 Let $\mathcal{O}$ be a commutative domain. A ring R is said to be an *$\mathcal{O}$-order* if there is an injective ring homomorphism from $\mathcal{O}$ into the centre $Z(R)$ of R, and if, as both a left and right $\mathcal{O}$-module, R is finitely generated over $\mathcal{O}$ and torsion-free. The ring $\mathcal{O}$ is called the coefficient ring. If the coefficient ring can be taken as granted, we often refer to R simply as an *order*.

Given an $\mathcal{O}$-order R, let $\mathcal{T}_{\mathcal{ORO},R}$, $\mathcal{T}_{\mathcal{O},R}$ and $\mathcal{TF}_{\mathcal{O},R}$ denote the full subcategories of $\mathcal{M}\mathrm{od}_R$ whose objects are respectively the $\mathcal{O}$-torsion, finitely generated $\mathcal{O}$-torsion and finitely generated $\mathcal{O}$-torsion-free modules.

Show that all these categories are repletely G-exact categories. In each case, show how to form an $\mathcal{O}$-relative exact category by taking as short exact sequences those which are $\mathcal{O}$-split.

2.4.6 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between G-exact categories. Show that F and G are both exact if and only if $F \oplus G : \mathcal{C} \to \mathcal{D}$ is exact, and that

$$K_0(F \oplus G) = K_0(F) + K_0(G) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D}).$$

2.4.7 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that

$$K_0(\mathcal{C} \times \mathcal{D}) \cong K_0(\mathcal{C}) \oplus K_0(\mathcal{D}).$$

Generalize this result to an arbitrary finite direct product of categories, and to infinite direct sums of categories (2.3.19).

2.4.8 **Categories with cofibrations: axioms**

In [Waldhausen 1985], Quillen's conditions QE1-QE3 (2.4.10) on the admissible exact sequences of a Q-exact category are replaced by weaker requirements which still suffice for the construction of (higher) K-theory, with the advantage that a wider range of categories can be used. This and the subsequent exercises outline the definitions and some initial calculations.

Let $\mathcal{C}$ be a category with zero object 0. The notion of an admissible monomorphism is replaced by that of a *cofibration* (which need not in fact be a monomorphism), and the counterpart of an admissible exact sequence is a *cofibration sequence*

$$M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M''$$

in which α is a cofibration and (M'', β) is a cokernel of α (as defined in (1.4.15)).

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The cofibrations are, by definition, the morphisms of a subcategory $\mathrm{co}\,\mathcal{C}$ of $\mathcal{C}$ which has the same objects as $\mathcal{C}$ and is subject to the following three axioms.

Cof1. Every isomorphism of $\mathcal{C}$ is in $\mathrm{co}\,\mathcal{C}$.

Cof2. Every morphism $0 \to C$ in $\mathcal{C}$ belongs to $\mathrm{co}\,\mathcal{C}$.

Cof3. Cofibrations are closed under cobase change, meaning that, for any cofibration $M' \xrightarrow{\alpha} M$ and any morphism $\phi : M' \to N$ in $\mathcal{C}$, the bottom row of the push-out diagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M \end{array}$$

is also in $\mathrm{co}\,\mathcal{C}$. (This implies that $N \oplus_{M'} M$ is an object of $\mathcal{C}$.)

With these axioms, the pair $(\mathcal{C}, \mathrm{co}\,\mathcal{C})$ is then called a *category with cofibrations.*

(a) Show that the push-out $0 \oplus_{M'} M$ of a cofibration $\alpha : M' \to M$ and the zero morphism $0 : M' \to 0$ is (isomorphic to) the cokernel $\mathrm{Cok}\,\alpha$ of α. Thus, by Cof3, every cofibration is part of a cofibration sequence

$$M' \xrightarrow{\alpha} M \longrightarrow \mathrm{Cok}\,\alpha.$$

The canonical morphism $M \to \mathrm{Cok}\,\alpha$ is known as a *quotient map.*

(b) Show that cofibration sequences are closed under cobase change, meaning that, for any cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

and any morphism $\phi : M' \to N$ in $\mathcal{C}$, there is a push-out diagram

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M & \xrightarrow{\overline{\beta}} & \mathrm{Cok}\,\overline{\alpha} \end{array}$$

in which the lower row is also a cofibration sequence and the morphism from M'' to $\mathrm{Cok}\,\overline{\alpha}$ is an isomorphism. (Compare with Axiom $\mathrm{QE2^{op}}$, (2.4.10).)

(c) Show that, by taking the admissible monomorphisms to be the cofibrations, any Q-exact category may be considered to be a category with cofibrations.

2.4.9 **Categories with cofibrations: constructions**

Let $\mathcal{C}$ be a category with cofibrations. For each natural number n, define $\mathcal{F}_n\mathcal{C}$ to be the category for which an object is a sequence of cofibrations

$$M_0 \longrightarrow M_1 \longrightarrow \quad \cdots \quad \longrightarrow M_n$$

in $\mathcal{C}$, and for which a morphism is a natural transformation of such diagrams. One can also form a category $\mathcal{F}_n^+\mathcal{C}$ equivalent to $\mathcal{F}_n\mathcal{C}$ by taking for an object, in addition to the above data, a quotient map for each cofibration $M_i \to M_j$, $0 \leq i < j \leq m$. The following assertions generalize readily to $n > 1$.

(a) Show that $\mathcal{F}_1\mathcal{C}$ is a category with cofibrations, where a cofibration from $\alpha : M' \to M$ to $\beta : N' \to N$ in $\operatorname{co}\mathcal{F}_1\mathcal{C}$ is, by definition, any pair γ, δ of morphisms in $\mathcal{C}$ for which

(i) there is a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\delta} \\ N' & \xrightarrow{\beta} & N \end{array}$$

(ii) both γ and the induced morphism $N' \oplus_{M'} M \to N$ are cofibrations in $\mathcal{C}$.

Deduce that δ is a cofibration.

(b) Show that an object of $\mathcal{F}_1^+\mathcal{C}$ is a cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

in which M'' is a specified cokernel of α.

Use (a) and the equivalence $\mathcal{F}_1^+\mathcal{C} \to \mathcal{F}_1\mathcal{C}$ to regard $\mathcal{F}_1^+\mathcal{C}$ as a category with cofibrations. Show that the three functors

$$s, t, q : \mathcal{F}_1^+\mathcal{C} \longrightarrow \mathcal{C}$$

which map a cofibration sequence to its respective terms

$$M', \qquad M, \qquad M''$$

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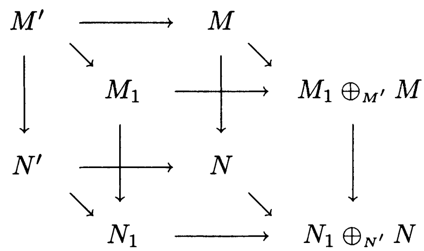
are *exact* in the sense that they preserve zero objects, cofibrations and push-out diagrams.

2.4.10 Waldhausen categories

(a) A *Waldhausen category* or *category of weak equivalences* is a category with cofibrations $\mathcal{C}$, together with a specified subcategory $\mathcal{WC}$. The morphisms in $\mathcal{WC}$ are the *weak equivalences*. The following axioms hold.

Weq1. Every isomorphism of $\mathcal{C}$ is a morphism of $\mathcal{WC}$.

Weq2. Glueing Axiom. For any commutative diagram in $\mathcal{C}$



in which each horizontal morphism is a cofibration and the three left-most vertical morphisms are weak equivalences, the fourth vertical morphism is also a weak equivalence.

Deduce that if

$$\begin{array}{ccccc} M' & \longrightarrow & M & \longrightarrow & M'' \\ \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ N' & \longrightarrow & N & \longrightarrow & N'' \end{array}$$

is a map of cofibration sequences in which γ and δ are weak equivalences, then ϵ is also a weak equivalence.

(b) Define the *Grothendieck group* $K_0(\mathcal{C})$ (which, strictly speaking, should be written $K_0(\mathcal{C}, \mathrm{co}\,\mathcal{C}, \mathcal{WC})$) of the Waldhausen category $\mathcal{C}$ to be the abelian group generated by an element $[M]$ for each object M of $\mathcal{C}$ and subject to a relation

$$[M] = [M'] + [M'']$$

for each cofibration sequence

$$M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'',$$

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and a further relation

$$[M] = [N]$$

for each weak equivalence $M \to N$ in $\mathcal{C}$.

Show that when $\mathcal{C}$ is a Q-exact category considered as a category with cofibrations, and with weak equivalences chosen to be the isomorphisms of $\mathcal{C}$, then the group just defined coincides with the Grothendieck group of $\mathcal{C}$ as defined previously (2.4.9).

2.4.11 **Q-exact is strictly stronger than G-exact**

Let $\mathcal{C}$ be the full subcategory of $\mathcal{A}_B$ whose objects are the finite abelian 2-groups. To choose $\mathrm{Ex}(\mathcal{C})$ such that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a G-exact category, let $\mathrm{Ex}(\mathcal{C})$ consist of all direct sums (as in GE3) of short exact sequences that are either split or have cyclic middle term. Thus, for example, the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\quad\sigma\quad} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\quad\pi\quad} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\sigma : (a, b) \mapsto (a, 4b) \qquad \pi : (c, d) \mapsto d$$

(using representative integers to indicate residues) is admissible, while

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\quad\alpha\quad} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\quad\beta\quad} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\alpha : x \mapsto (x, 2x) \qquad \beta : (y, z) \mapsto 2y - z$$

is not. Show that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is not a Q-exact category, because it falls foul, for example, of QE3.

Let $\phi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$, $a \mapsto 2a$. Show that $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is isomorphic to the push-out of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ over $\mathbb{Z}/2\mathbb{Z}$ with respect to ϕ, σ.

Deduce that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ also violates $\mathrm{QE2}^{\mathrm{op}}$. Thus, the admissible monomorphisms of $\mathrm{Ex}(\mathcal{C})$ fail even to make $\mathcal{C}$ a category with cofibrations.