Appendix E

Variation of the determinant

If M is a matrix function and δM is a small variation of M, we wish to find the variation of the determinant of M (we follow Section 4.7 of [177]).

Consider

$$\delta[\ln(\text{Det}M(x))] = \ln(\text{Det}(M + \delta M)) - \ln(\text{Det}(M))$$

$$= \ln\left[\frac{\text{Det}(M + \delta M)}{\text{Det}(M)}\right]$$

$$= \ln(\text{Det}M^{-1}\text{Det}(M + \delta M))$$

$$= \ln(\text{Det}\{M^{-1}(M + \delta M)\})$$

$$= \ln(\text{Det}\{1 + M^{-1}\delta M\})$$

$$= \ln(1 + \text{Tr}\{M^{-1}\delta M\}) + O((\delta M)^2)$$

$$= \text{Tr}\{M^{-1}\delta M\} + O((\delta M)^2) \qquad \text{(E.1)}$$

Hence

$$\delta[(\text{Det}M(x))] = \text{Tr}[M^{-1}(x)\delta M(x)]\text{Det}M(x) \qquad \text{(E.2)}$$

which is the desired result.