BADLY APPROXIMABLE FUNCTIONS AND INTERPOLATION BY BLASCHKE PRODUCTS

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A continuous function ϕ on the unit circle is called badly approximable if $\| \phi - p \|_\infty \geqq \| \phi \|_\infty$ for all polynomials p, where $\| \ \|_\infty$ is the essential supremum norm. In (4), Poreda asked whether every continuous ϕ may be written $\phi = \phi_W + \phi_B$, where ϕ_W is the uniform limit of polynomials (i.e. ϕ_W belongs to the disc algebra A) and ϕ_B is badly approximable. We call such a function ϕ *decomposable*. In (4), he characterised the badly approximable functions as those of constant non-zero modulus and negative winding number around the origin, i.e. ind $(\phi) < 0$. (See (3) for two new proofs of this result.) We show that the answer to Poreda's question is *no* in general, but give a necessary and sufficient condition for a given ϕ to have such a decomposition. Then we apply this criterion to solve an interpolation problem.

Definition. For $\phi \in C(|z| = 1)$, $\phi^\#$ is the metric projection of ϕ into H^∞. That is, $\phi^\# \in H^\infty$ and

$$\| \phi - \phi^\# \|_\infty \leqq \| \phi - g \|_\infty \quad \text{for all } g \in H^\infty.$$

It is well known and easy to prove that there is a unique function $\phi^\#$ satisfying this requirement.

Lemma. (Sarason (5)). *If ϕ is a continuous function on $|z| = 1$ then* $d(\phi, H^\infty) = d(\phi, A)$.

Theorem. *A continuous function ϕ is decomposable if and only if $\phi^\# \in A$.*

The proof is immediate once one remarks that the Lemma implies that ϕ is badly approximable if and only if ϕ is badly H^∞-approximable, i.e.

$$\| \phi - f \|_\infty \geqq \| \phi \|_\infty$$

for all $f \in H^\infty$.

Since there exists a continuous ϕ with $\phi^\# \notin A$ (see (1)), it follows that not every ϕ is decomposable.

We now apply our Theorem to prove an analogue of the following classical result.

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Theorem. (Carathéodory.) *If* $a_0, a_1, \ldots, a_n$ *are complex numbers, and if the interpolation problem*

$$F^{(j)}(0) = a_j, \quad j = 0, 1, \ldots, n \tag{1}$$

has a solution F that is bounded and analytic in the unit disc (i.e. $F \in H^\infty$*) satisfying* $|F| \leqq 1$, *then there is a Blaschke product of order* $\leqq n+1$ *that satisfies* (1).

Our result has the following statement:

Theorem. *The interpolation problem* (1) *always has a solution of the form* $F = \lambda B$, *where* λ *is a complex constant, and B is a Blaschke product of order* $\leqq n$.

Proof. Assume that the a_j are not all zero. Define

$$f(z) = a_0 + a_1 z + \frac{a_2}{2!} z^2 + \ldots + \frac{a_n}{n!} z^n.$$

Then $\phi(z) = f(z)/z^{n+1}$ is a continuous function on $\{|z| = 1\}$ that does not belong to the disc algebra A. But since ϕ satisfies a Lipschitz condition, it surely satisfies Dini's condition $\left(\int_{0+} \omega(t, \phi)/t\,dt < \infty\right)$ and so by the theorem of Carleson and Jacobs (2) the metric projection $\phi^{\#}$ of ϕ into H^∞ must belong to A. By our Theorem then, we may write

$$z^{-(n+1)}f - g = \Psi,$$

where $g \in A$ and Ψ is badly approximable. By Poreda's theorem, we may take $\Psi = c\psi$ where $c \neq 0$, Ψ is unimodular, and ind $(\psi) < 0$. But then $z^{n+1}\psi = B$ belongs to A. Further, B is unimodular, and ind $(B) = (n+1) - \text{ind}\,(\phi) \leqq n$. Hence B is a Blaschke product of degree $\leqq n$, and the result is proved.

A quite analogous argument shows that if $w_1, w_2, \ldots, w_n$ are n distinct points in $\{|z| < 1\}$, and if $b_1, b_2, \ldots, b_n$ are complex numbers, then the interpolation problem

$$F(w_j) = b_j, \quad j = 1, 2, \ldots, n \tag{2}$$

has a solution of the form $F = \lambda B$ where λ is a constant, and B is a Blaschke product of degree $\leqq n-1$. Furthermore, there is no trouble in interpolating a finite number of the derivatives of F at the points w_j.

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