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CONVERGENCE TENSOR PRODUCTS AND A STRICT TOPOLOGY

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We are interested in the strict topology τ on $L_s(E, F)$, the set $L(E, F)$ of all continuous linear mappings from E into a Banach space F endowed with the topology of pointwise convergence. The T_3-completion $E \hat{\otimes}_c L_c F$ of the convergence tensor product $E \otimes_c L_c F$ is the set of all τ-continuous linear functionals on $L(E, F)$ and τ is the topology of uniform convergence on the compact subsets of $E \hat{\otimes}_c L_c F$. In the case that E is a nuclear Fréchet space, a nuclear (DF)-space or a Banach space with the bounded approximation property the topology τ agrees with the topology of $L_{co}(E, F)$.

Let E be a locally convex topological vector space and F a Banach space. We are interested in the finest locally convex vector space topology τ on the set $L(E, F)$ of all continuous linear mappings from E to F such that every filter Φ on $L(E, F)$ converges to 0 with respect to τ , if it has the following properties:

(i) Φ converges pointwise to 0 ;

(ii) Φ contains a pointwise bounded subset of $L(E, F)$.

This topology τ is called the strict topology on $L_s(E, F)$. If E is barrelled the Banach-Steinhaus theorem says that τ is finer than the

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topology of $L_{co}(E, F)$, the set $L(E, F)$ endowed with the topology of precompact convergence. The continuous convergence structure is the finest convergence structure γ on $L(E, F)$ such that every filter with the properties (i) and (ii) γ-converges to 0 . Therefore τ is the finest locally convex vector space topology coarser than the continuous convergence structure. Let us denote by $L_c(E, F)$ and $(L_c(E, F))_\tau$ the set $L(E, F)$ carrying the continuous convergence structure and the finest locally convex vector space topology coarser than the continuous convergence structure. It is known that this topology is the topology of uniform convergence on the compact subsets of $L_c(L_c(E, F))$, the dual of $L_c(E, F)$ also endowed with the continuous convergence structure. For this reason we show in Section 2: the T_3-completion $E \hat{\otimes}_c L_c F$ of the convergence tensor product $E \otimes_c L_c F$ is isomorphic to $L_c(L_c(E, F))$. In some cases we will prove $(L_c(E, F))_\tau = L_{co}(E, F)$, for example, if E is a nuclear Fréchet space, a nuclear (DF)-space or a Banach space with the bounded approximation property. But we will also prove

$$\left(L_c\left(\mathbb{R}^D, l_1\right)\right)_\tau \neq L_{co}\left(\mathbb{R}^D, l_1\right)$$

where $\mathbb{R}^D$ is an uncountable product of $\mathbb{R}$ and l_1 is the Banach space of all real sequences which are absolutely summable. This example shows that $L_{co}(E, F)$ does not always carry the finest locally convex vector space topology such that every filter with the properties (i) and (ii) converges to 0 .

Since, in this paper, the concept of convergence spaces will be used, let us say something about this concept. Every convergence space will be a convergence space in the sense of Fischer (see [7]). For a subset S of a convergence space X , the adherence $a(S)$ is defined to be the set

$$\{y : y \in X, \text{ there exists a filter } \Phi \text{ converging to } y \text{ with } S \in \Phi\} .$$

S is called dense in X if $a(S) = X$, compact if every ultra-filter Φ with $S \in \Phi$ converges in S , and relatively compact if $a(S)$ is compact. A mapping f from a convergence space X into a convergence space Y is continuous if for all $x \in X$ and all filters Φ converging to x in X

the filter $f(\Phi)$ generated by $\{f(L) : L \in \Phi\}$ converges to $f(x)$ in Y. Let us denote by $C_c(X, Y)$ the set $C(X, Y)$ of all continuous mappings from X into Y endowed with the continuous convergence structure. This convergence structure is the coarsest on $C(X, Y)$ for which the mapping $\omega : C(X, Y) \times X \to Y$ defined by $\omega(f, x) = f(x)$ for all $x \in X$ and $f \in C(X, Y)$, is continuous. This means that a filter F converges to f in $C_c(X, Y)$ if and only if for all $x \in X$ and all filters Φ converging to x in X the filter $\omega(F, \Phi)$ converges to $f(x)$ in Y. Instead of $\omega(F, \Phi)$ we will also write $F(\Phi)$.

All vector spaces considered in this paper will be $\mathbb{R}$-vector spaces. A vector space space endowed with a convergence structure is called a *convergence vector space* if the algebraic operations are continuous. For a convergence vector space E we will denote by E_τ the set E endowed with the finest locally convex vector space topology coarser than the convergence structure of E. A subspace F of a convergence vector space E will be always a vector subspace carrying the convergence structure induced by E. A T_3-convergence vector space E will be a separated convergence vector space such that for every filter Φ converging to 0 in E the filter $a(\Phi)$ generated by $\{a(V) : V \in \Phi\}$ converges also to 0. Let E, F and G be convergence vector spaces. A bijective linear mapping $T : E \to F$ is called an isomorphism if T and T^{-1} are continuous. Let us denote by $L_c(E, F)$ and $B_c(E, F; G)$ the subspaces of $C_c(E, F)$ and $C_c(E \times F, G)$ respectively consisting of the set $L(E, F)$ of all continuous linear mappings from E into F and of the set $B(E, F; G)$ of all continuous bilinear mappings from $E \times F$ into G respectively. We abbreviate $B_c(E, F; \mathbb{R})$, $B(E, F; \mathbb{R})$, $L_c(E, \mathbb{R})$ and $L(E, \mathbb{R})$ to $B_c(E, F)$, $B(E, F)$, $L_c(E)$ or L_cE and LE respectively. For every convergence vector space E there exists a natural mapping $j : E \to L_cL_cE$ defined by $[j(x)](\zeta) = \zeta(x)$ for all $x \in E$ and $\zeta \in LE$, which is always continuous. E is called L_c-embedded if j is an isomorphism from E onto $j(E)$, and c-reflexive if j is an isomorphism from E onto L_cL_cE. For every L_c-embedded convergence vector space E

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we will identify E and $j(E)$. Let us mention that every subspace of an L_c-embedded convergence vector space is also L_c-embedded.

Let X and Y be vector spaces and $\langle X, Y\rangle$ a duality (see [15]). For a subset A of X we denote by $\overline{A}^{\sigma}$ the closure of A with respect to the topology $\sigma(X, Y)$, by A^0 the polar of A which is defined as $A^0 = \{y : y \in Y, |y(a)| \leq 1 \text{ for all } a \in A\}$ and by ΓA the convex, circled hull of A . Let us mention that $\langle L(E, F), E \otimes LF\rangle$ is a duality, if E and F are separated locally convex topological vector spaces. This duality is defined by $T(x \otimes \zeta) = \zeta(T(x))$ for all $x \in E$, $\zeta \in LF$ and $T \in L(E, F)$.

I. Convergence tensor products

Since $(L_c(E, F))_\tau$ carries the topology of uniform convergence on the compact subsets of $L_c(L_c(E, F))$, we will develop a duality theory for $L_c(E, F)$. For this reason we are interested in convergence tensor products. For convergence vector spaces E and F let us denote by $E \otimes_L F$ the tensor product of E and F in the category of all convergence vector spaces. The existence of tensor products in this category was proved by several authors, for example, by Antoine [1] or Jarchow [10]. The convergence structure on $E \otimes_L F$ is the finest convergence vector space structure for which the canonical mapping $\chi : E \times F \to E \otimes F$ defined by $\chi(x, y) = x \otimes y$, is continuous. As usual we write $U \otimes V$ instead of $\chi(U, V)$ for subsets U and V of E and F respectively. Throughout this paper we assume that E and F are convergence vector spaces with point-separating duals. Therefore the mapping $j : E \otimes F \to L_cL_c(E \otimes_L F)$ is injective. Let us denote by $E \otimes_c F$ the tensor product of E and F endowed with the convergence structure induced by the mapping j .

We shall need the following lemma.

LEMMA I.1. *Let E be a convergence vector space, F an L_c-embedded convergence vector space. Denote by E_c the vector space E endowed with the convergence structure induced by L_cL_cE . Then*

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$L_c(E, F) = L_c(E_c, F)$.

Proof. $L_c(E, F)$ is a subspace of $L_c(E, L_cL_cF)$ which is isomorphic to $B_c(E, L_cF)$ (see [3]). But $B_c(E, L_cF)$ is a subspace of $C_c(E \times L_cF)$ which is c-reflexive (see [5]). Therefore $L_c(E, F)$ is L_c-embedded. Denote by E_L the vector space E considered as a subspace of $L_c(L_c(E, F), F)$. Then E_L is L_c-embedded, $L(E, F) = L(E_L, F)$ and the evaluation mapping $\omega : L_c(E, F) \times E_L \to F$ defined by $\omega(T, x) = T(x)$ for all $x \in E$ and $T \in L(E, F)$, is continuous. Since $L_c(E_L, F)$ carries the coarsest convergence structure for which ω is continuous, the identity mapping $I : L_c(E, F) \to L_c(E_L, F)$ is continuous. This implies $L_c(E, F) = L_c(E_c, F) = L_c(E_L, F)$ since the convergence structure of E_c is finer than that of E_L and coarser than that of E .

THEOREM I.2. *The convergence space* $E \otimes_c F$ *has the following properties:*

(i) $E \otimes_c F$ *carries the finest* L_c*-embedded convergence structure for which the canonical mapping* $\chi : E \times F \to E \otimes F$ *is continuous;*

(ii) $B_c(E, F; G)$ *and* $L_c(E \otimes_c F, G)$ *are isomorphic for every* L_c*-embedded convergence vector space* G *;*

(iii) if E *and* F *are locally convex topological vector spaces we have* $E \otimes_c F = E \otimes_\pi F$ *, where* $E \otimes_\pi F$ *is the projective tensor product of* E *and* F *.*

Proof. $E \otimes_c F$ carries the finest L_c-embedded convergence structure which is coarser than the convergence structure of $E \otimes_L F$. This implies *(i)*.

(ii). Let us define $\alpha : L_c(E \otimes_c F, G) \to B_c(E, F; G)$ by $\alpha(T) = T \circ \chi$. α is continuous since χ is. In [10] it is shown that α is a bijective mapping. The filters which are finite sums of filters of

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the form $\chi(x+\mathcal{U}, \mathcal{V})$ and $\chi(\mathcal{U}, y+\mathcal{V})$ where $x \in E$, $y \in F$ and $\mathcal{U}$ and $\mathcal{V}$ are filters converging to 0 in E and F respectively, generate the ideal of all filters converging to 0 in $E \otimes_L F$ (see [10]). Therefore α^{-1} is a continuous mapping from $B_c(E, F; G)$ onto $L_c(E \otimes_L F, G)$. Lemma I.1 now implies that $B_c(E, F; G)$ and $L_c(E \otimes_c F, G) = L_c(E \otimes_L F, G)$ are isomorphic.

(iii). Let $\mathcal{U}$ and $\mathcal{V}$ be the 0-neighborhood filters in E and F respectively. $\chi(\mathcal{U}, \mathcal{V})$ converges to 0 in $E \otimes_c F$ and, since $E \otimes_c F$ is L_c-embedded, the filter generated by $\{\Gamma(U \otimes V) : U \in \mathcal{U}, V \in \mathcal{V}\}$ also converges to 0. This implies $E \otimes_c F = E \otimes_\pi F$.

Since $L_c(E, L_c(F, G))$ and $B_c(E, F; G)$ are always isomorphic (see [1] and [3]) we conclude:

COROLLARY 1. *$L_c(E \otimes_c F, G)$ and $L_c(E, L_c(F, G))$ are isomorphic for all convergence vector spaces E and F and for all L_c-embedded convergence vector spaces G. If F is a c-reflexive convergence vector space, $L_c(E \otimes_c L_c F)$ and $L_c(E, F)$ are isomorphic. For convenience, let us write $L_c(E \otimes_c L_c F) = L_c(E, F)$ if F is c-reflexive.*

COROLLARY 2. *If F is c-reflexive and if $E \otimes LF$ is dense in $L_c(L_c(E, F))$, then the convergence vector space $L_c(E, F)$ is c-reflexive.*

Proof.

$$L(E, F) = L(E \otimes_c L_c F) = L(L_c L_c(E \otimes_c L_c F))$$
$$= L(L_c(L_c(E, F))) .$$

REMARK. Another way to introduce $E \otimes_c F$ is suggested by Binz: consider $E \otimes F$ as a subspace of $L_c(B_c(E, F))$. Then by Theorem I.2 this convergence vector space agrees with $E \otimes_c F$.

II. Locally convex topological vector spaces

From now on, E and F are always assumed to be separated locally convex topological vector spaces. We denote by $L_s(E, F)$ and $L_{co}(E, F)$ the set $L(E, F)$ endowed with the topology of pointwise and of precompact convergence respectively which is always coarser than the continuous convergence structure. Instead of $L_{co}(E, \mathbb{R})$ we write $L_{co}(E)$. In order to apply Corollary 1 of Theorem I.2, we also assume that E and F are complete, since F is c-reflexive if and only if it is complete (see [5]).

In [13] a complete T_3-convergence vector space $\hat{H}$ is called the T_3-completion of H, if $\hat{H}$ has the following properties:

(i) there is an isomorphism i from H onto a subspace of $\hat{H}$;

(ii) for every complete T_3-convergence vector space M and for every $T \in L(H, M)$ there exists a $\hat{T} \in L(\hat{H}, M)$ with $T = \hat{T} \circ i$.

In this section we will show that $L_c\big(L_c(E, F)\big)$ is the T_3-completion of $E \otimes_c L_c F$ if F is a Banach space. Further on we will show $\big(L_c(E, F)\big)_\tau = L_{co}(E, F)$ if and only if, for every compact subset C of $L_c\big(L_c(E, F)\big)$ there exist compact subsets K_1 and K_2 of E and $L_c F$ respectively such that $C \subseteq a\big(\Gamma\big(K_1 \otimes K_2\big)\big)$.

Let F be a Banach space, U the 0-neighborhood filter in E and Φ a filter in $L(E, F)$. Since $\Phi(U)$ converges to 0 in F if and only if Φ contains an equicontinuous set we get (see also [2]):

LEMMA II.1. *If F is a Banach space, a filter Φ converges to 0 in $L_c(E, F)$ if and only if*

(i) Φ converges to 0 in $L_s(E, F)$,

(ii) Φ contains an equicontinuous subset.

LEMMA II.2. *$E \otimes LF$ is dense in $L_c\big(L_c(E, F)\big)$ for every Banach*

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space F .

Proof. Let $\mathcal{H}$ be the family of all equicontinuous convex, circled and pointwise closed subsets of $L(E, F)$. By Lemma II.1 the continuous convergence structure and the topology of pointwise convergence agree on every $H \in \mathcal{H}$. Therefore every $\zeta \in L(L_c(E, F))$ is continuous on H with respect to the topology of pointwise convergence. This implies that for all $0 < \varepsilon \in \mathbb{R}$ one can find an $f \in L(L_s(E, F)) = E \otimes LF$ with $(\zeta - f)(H) \subseteq [-\varepsilon, \varepsilon]$ (see [15], Chapter IV, Theorem 6.2). Define $D_{H,\varepsilon} := \{f : f \in E \otimes LF, (\zeta - f)(H) \subseteq [-\varepsilon, \varepsilon]\}$. We will show that the filter $\mathcal{F}$ generated by $\{D_{H,\varepsilon} : H \in \mathcal{H}, 0 < \varepsilon \in \mathbb{R}\}$ converges to ζ in $L_c(L_c(E, F))$. Choose an $\varepsilon > 0$ and let Φ be a filter converging to T in $L_c(E, F)$. Since ζ is continuous, one can find an equicontinuous subset $G \in \Phi$ with $\zeta(G) - \zeta(T) \subseteq [-\varepsilon, \varepsilon]$. Choose an $H \in \mathcal{H}$ with $G \subseteq H$. Then we have

$$D_{H,\varepsilon}(g) - \zeta(T) \subseteq [D_{H,\varepsilon}(g) - \zeta(g)] + [\zeta(g) - \zeta(T)] \subseteq [-2\varepsilon, 2\varepsilon]$$

for all $g \in G$. Therefore $E \otimes LF$ is dense in $L_c(L_c(E, F))$.

LEMMA II.3. *Let F be a Banach space. For every filter $\mathcal{F}$ converging to 0 in $L_c(L_c(E, F))$ there exists a filter $\mathcal{G}$ in $E \otimes LF$ with the following properties:*

(i) the filter $\hat{\mathcal{G}}$ generated by $\{a(G) : G \in \mathcal{G}\}$ is coarser than $\mathcal{F}$, where $a(G)$ denotes the adherence of G in $L_c(L_c(E, F))$;

(ii) $\hat{\mathcal{G}}$ converges to 0 in $L_c(L_c(E, F))$.

Proof. For every $0 < \varepsilon \in \mathbb{R}$ and for every filter Φ converging to T in $L_c(E, F)$ there exists an $H_\Phi \in \Phi$ with $\varepsilon(H_\Phi)^0 \in \mathcal{F}$, where the polar is taken with respect to the duality $\langle L(E, F), LL_c(E, F)\rangle$. By Lemma II.1, we can assume that H_Φ belongs to the family $\mathcal{H}$ of all equicontinuous, convex, circled and pointwise closed subsets of $L(E, F)$. The filter $\mathcal{N}$ generated by the filter subbase

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$$\left\{\left(\frac{1}{\varepsilon} H_\Phi\right)^0 : 0 < \varepsilon \in \mathbb{R},\ \Phi \text{ converges in } L_c(E, F)\right\}$$

is coarser than F and converges to 0 in $L_c\big(L_c(E, F)\big)$. We now show:

$$\bigcap_{i=1}^{n} \left(\frac{1}{\varepsilon_i} H_{\Phi_i}\right)^0 \subseteq a\left(2 \bigcap_{i=1}^{n} \left(\frac{1}{\varepsilon_i} H_{\Phi_i}\right)^0 \cap (E \otimes LF)\right)$$

for all $n \in \mathbb{N}$, where the adherence is taken in $L_c\big(L_c(E, F)\big)$. The set

$$K = \left(\bigcap_{i=1}^{n} \left(\frac{1}{\varepsilon_i} H_{\Phi_i}\right)^0\right)^0$$

belongs to H. As proved in Lemma II.2, for every

$$\zeta \in K^0 = \bigcap_{i=1}^{n} \left(\frac{1}{\varepsilon_i} H_{\Phi_i}\right)^0$$

and for every $H \in H$ one can find an $x_H \in E \otimes LF$ with $\zeta - x_H \in H^0$ such that the net $(x_H)_{H \in H}$ converges to ζ in $L_c\big(L_c(E, F)\big)$. For all $H \in H$ with $K \subseteq H$ we have

$$x_H = x_H - \zeta + \zeta \in H^0 + K^0 \subseteq 2K^0 .$$

Therefore the filter G generated in $E \otimes LF$ by

$$\left\{2\left(\frac{1}{\varepsilon} H_\Phi\right)^0 \cap (E \otimes LF) : 0 < \varepsilon \in \mathbb{R},\ \Phi \text{ converges in } L_c(E, F)\right\}$$

has the desired property.

THEOREM II.4. *Let F be a Banach space. Then we have:*

(i) $L_c\big(L_c(E, F)\big)$ *is the T_3-completion $E \hat{\otimes}_c L_cF$ of $E \otimes_c L_cF$;*

(ii) $\big(L_c(E, F)\big)_\tau$ *carries the topology of uniform convergence on the compact subsets of $E \hat{\otimes}_c L_cF$.*

Proof. By definition, $E \otimes_c L_cF$ is a subspace of

$$L_cL_c\big(E \otimes_c L_cF\big) = L_c\big(L_c(E, F)\big) .$$

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Let M be a complete T_3-convergence vector space and $T \in L\left(E \otimes_c L_c F, M\right)$. For every $\zeta \in L\left(L_c(E, F)\right)$ choose a filter F on $E \otimes LF$ converging to ζ in $L_c\left(L_c(E, F)\right)$. Define $\hat{T}(\zeta)$ to be the limit of the filter $T(F)$ in M. Lemma II.3 implies the continuity of T, since for every filter G converging to 0 in $E \otimes_c L_c F$ the filter $a\left(T(G)\right)$ is coarses than $\hat{T}\left(a(G)\right)$.

To prove the second part use the results of [6].

LEMMA II.5. *Let U be a subset of E and V a closed, convex, circled subset of F. Define*

$$T_{U,V} = \{T : T \in L(E, F),\ T(U) \subseteq V\} .$$

Then we have

$$T^0_{U,V} = \overline{\Gamma\left(U \otimes V^0\right)^\sigma} \quad \text{and} \quad \left(\overline{\Gamma\left(U \otimes V^0\right)^\sigma}\right)^0 = T_{U,V}$$

where the polars of $T_{U,V}$ and V are taken with respect to the dualities $\langle LL_c(E, F), L(E, F)\rangle$ and $\langle F, LF\rangle$ respectively, and σ denotes the weak topology $\sigma\left(L\left(L_c(E, F)\right), L(E, F)\right)$.

PROPOSITION II.6. $\left(L_c(E, F)\right)_\tau = L_{co}(E, F)$ *if and only if, for every compact subset C of $L_c\left(L_c(E, F)\right)$, there exist compact subsets K_1 and K_2 of E and $L_c F$ respectively such that*

$$C \subseteq a\left(\Gamma\left(K_1 \otimes K_2\right)\right) .$$

Proof. Assume $\left(L_c(E, F)\right)_\tau = L_{co}(E, F)$ and let C be a compact subset of $L_c\left(L_c(E, F)\right)$. The polar C^0 of C, taken with respect to the duality $\langle L\left(L_c(E, F)\right), L(E, F)\rangle$ is a 0-neighborhood in $\left(L_c(E, F)\right)_\tau$ (to prove this use [6]). Therefore there exist a compact subset $K \subseteq E$ and a closed, circled, convex 0-neighborhood V in F such that

$$T_{K,V} = \{T : T \in L(E, F),\ T(K) \subseteq V\} \subseteq C^0 .$$

V^0 and $\left(T_{K,V}\right)^0$ are equicontinuous and therefore compact, topological

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subsets of L_cF and $L_c(L_c(E, F))$ respectively (see [6]). Now Lemma II.5 implies

$$C \subseteq C^{00} \subseteq (T_{K,V})^0 = \overline{\Gamma(K \otimes V^0)}^\sigma = a(\Gamma(K \otimes V^0)) .$$

To prove the converse let W be a closed, convex, circled 0-neighborhood in $(L_c(E, F))_\tau$. Since W^0 is compact in $L_c(L_c(E, F))$, one can find compact subsets K_1 and K_2 of E and L_cF respectively such that

$$W^0 \subseteq a(\Gamma(K_1 \otimes K_2)) \subseteq a\left(\Gamma\left(K_1 \otimes K_2^{00}\right)\right) .$$

Thus

$$a\left(\Gamma\left(K_1 \otimes K_2^{00}\right)\right)^0 = \left(\overline{\Gamma\left(K_1 \otimes K_2^{00}\right)}^\sigma\right)^0$$
$$= \left\{T : T \in L(E, F), T(K_1) \subseteq K_2^0\right\} \subseteq W .$$

Since K_2^0 is a 0-neighborhood in F, Proposition II.6 is proved.

The following was partially proved in [4].

THEOREM II.7. *Let E and F be Fréchet spaces. Then $L_{co}(E, L_{co}(F))$ carries the finest topology which is coarser than the convergence structures of $L_c(E, L_cF)$ and $L_c(E, L_{co}(F))$.*

Proof. Let us first show $L(E, L_cF) = L(E, L_{co}(F))$. For every $T \in L(E, L_{co}(F))$ define a bilinear mapping $v_T : E \times F \to \mathbb{R}$ by $v_T(x, y) = [T(x)](y)$. Since E and F are Fréchet spaces, v_T is continuous. Therefore one can find 0-neighborhoods U and V in E and F respectively such that

$$v_T(U, V) = [T(U)](V) \subseteq [-1, 1] ,$$

which implies that $T(U)$ is an equicontinuous subset of LF. Therefore T is also a continuous mapping from E into L_cF.

We have

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$$L_c(L_c(E, L_cF)) = L_c(L_c(E \otimes_\pi F)) = E \hat{\otimes}_\pi F ,$$

where $E \hat{\otimes}_\pi F$ is the completion of $E \otimes_\pi F$. By the Banach-Dieudonné theorem, the finest topology t on $L(E \otimes_\pi F)$ coarser than the continuous convergence structure is the topology of uniform convergence on the compact subsets of $E \hat{\otimes}_\pi F$. Grothendieck has shown that to every compact subset C of $E \hat{\otimes}_\pi F$ there exist compact, circled, convex subsets K_1 and K_2 of E and F respectively with $C \subseteq \overline{\Gamma(K_1 \otimes K_2)}^0$. Lemma II.5 implies

$$(\Gamma(K_1 \otimes K_2))^0 = T_{K_1, K_2^0}$$

which is a 0-neighborhood in $L_{co}(E, L_{co}(F))$. Since the convergence structure of $L_c(E, L_{co}(F))$ is coarser than that of $L_c(E, L_c(F))$ and finer than the topology of $L_{co}(E, L_{co}(F))$, Theorem II.7 is proved.

COROLLARY. *Let* E *be a Fréchet space and* F *a nuclear* (DF)*-space. Then* $L_{co}(E, F)$ *carries the finest topology which is coarser than the convergence structure of* $L_c(E, F)$.

Proof. Every nuclear (DF)-space is a k-space, therefore a subset of $L_{co}(F)$ is relatively compact if and only if it is equicontinuous. This implies $F = L_{co}(L_{co}(F))$. Now $L_{co}(F)$ is a Fréchet space and we can apply Theorem II.7.

LEMMA II.8. *We have* $(E \otimes_c L_cF)_\tau = E \otimes_\pi L_{co}(F)$ *if and only if for every equicontinuous subset* $H \subseteq L(E, F)$ *which is relatively compact in* $L_s(E, F)$, *there exists a* 0*-neighborhood* U *in* E *such that* $H(U)$ *is relatively compact in* F .

Proof. A subset $H \subseteq L(E, F)$ is relatively compact in $L_c(E, F)$ if and only if H is equicontinuous and relatively compact in $L_s(E, F)$ (see [6]). The topology of $(E \otimes_c L_cF)_\tau$ is the topology of uniform convergence on the equicontinuous subsets of $L((E \otimes_c L_cF)_\tau)$ which are the relatively

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compact subsets of $L_c(E \otimes_c L_c F) = L_c(E, F)$ (see [6]). Therefore we have $(E \otimes_c L_c F)_\tau = E \otimes_\pi L_{co}(F)$ if and only if for every compact subset H of $L_c(E, F)$ there exist a 0-neighborhood U in E and a compact subset $K \subseteq F$ such that $\Gamma(U \otimes K^0) \subseteq H^0$ which, by Lemma II.5, is equivalent to $H \subseteq \{T : T \in L(E, F), T(U) \subseteq K^{00}\}$.

III. Banach spaces

Throughout this section we assume that E *and* F *are Banach spaces.* For a subset $D \subseteq L(E, F)$ let us denote by $\overline{D}^s$ the closure of D in $L_s(E, F)$. We will always consider $E \otimes LF$ as a subset of $LL_c(E, F)$. For every subset $B \subseteq E$ we define

$$T_B = \{T : T \in L(E, F), \|T(b)\| \leq 1 \text{ for all } b \in B\} .$$

With the same methods used in the proof of Lemma 2.4 in [11] one can show:

LEMMA III.1. *Let* E *be a Banach space,* G *a subspace of* E *with finite dimension* k *and let* $T \in L(E, E)$ *be a compact operator. Assume that there exists an* $\varepsilon \in \mathbb{R}$ *with* $0 < \varepsilon < 1$ *and* $k\varepsilon(1-\varepsilon)^{-1} < 1$ *such that*

$$\|T(x)-x\| \leq \varepsilon\|x\| \quad \text{for all} \quad x \in G .$$

Then there exists a compact operator $S \in L(E, E)$ *such that*

$$S(x) = x \quad \text{for all} \quad x \in G \quad \text{and} \quad \|S\| \leq 2\|T\| .$$

By Lemma II.1, it is easy to prove:

PROPOSITION III.2. *The set* $K(E, E)$ *of all compact operators from* E *into* E *is dense in* $L_c(E, E)$ *if and only if there exists a net* $(T_\alpha)_{\alpha \in J}$ *in* $K(E, E)$ *with the following properties:*

(i) $(T_\alpha(x))_{\alpha \in J}$ *converges to* x *for all* $x \in E$;

(ii) *there exists an* $\eta \in \mathbb{R}$, $0 < \eta$ *with* $\|T_\alpha\| < \eta$ *for all* $\alpha \in J$.

Let us remember that, for Banach spaces E and F, a filter Φ converges to 0 in $L_c(E, F)$ if and only if it converges to 0 in

$L_s(E, F)$ and there is an $n \in \mathbb{N}$ with $nT_U \in \Phi$, where U is the closed unit ball in E. Therefore the continuous convergence structure is the finest convergence structure on $L(E, F)$ which agrees on nT_U for every $n \in \mathbb{N}$ with the topology induced by $L_s(E, F)$. This means that the topology of $(L_c(E, F))_\tau$ is the finest locally convex vector space topology agreeing on every member of $\{nT_U : n \in \mathbb{N}\}$ with the topology induced by $L_s(E, F)$. Since $\{nT_U : n \in \mathbb{N}\}$ is a countable set we can apply the following result of Roelcke (see [14]): the sets of the form

$$W = T_{B_0} \cap \bigcap_{n\in\mathbb{N}} \overline{(nT_U+T_{B_n})}^{s}$$

where B_n is a finite subset of E for all $n \in \mathbb{N} \cup \{0\}$, are a 0-neighborhood base for $(L_c(E, F))_\tau$.

THEOREM III.3. *If the set* $K(E, E)$ *of all compact operators from* E *into* E *is dense in* $L_c(E, E)$ *then we have for every Banach space* F,

$$(L_c(E, F))_\tau = L_{co}(E, F) .$$

Proof. Let

$$W = T_{B_0} \cap \bigcap_{n\in\mathbb{N}} \overline{(nT_U+T_{B_n})}^{s}$$

be a 0-neighborhood in $(L_c(E, F))_\tau$ where B_n is a finite subset of E for all $n \in \mathbb{N} \cup \{0\}$. Proposition III.2 implies the existence of a net $(T_\alpha)_{\alpha\in J}$ in $K(E, E)$ and of a real number $\eta > 0$, such that $\|T_\alpha\| \le \frac{1}{2}\eta$ for all $\alpha \in J$ and $(T_\alpha)_{\alpha\in J}$ converges to the identity mapping I in $L_{co}(E, E)$. Since $U \cap [B_n]$ is compact, where $[B_n]$ is the vector space generated by B_n in E, we conclude from Lemma III.1: for every $n \in \mathbb{N}$ there exists a compact operator $S_n \in L(E, E)$ such that $\|S_n\| \le \eta$ and $S_n(b) = b$ for all $b \in B_n$. From

$$R = R \circ S_n + R(I-S_n) \quad \text{for all} \quad R \in nT_{S_n(U)}$$

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we conclude

$$T_{(1/n)S_n(U)} \subseteq nT_U + T_{B_n} .$$

Defining

$$K = \bigcup_{n \in \mathbb{N}} (1/n)\overline{S_n(U)}$$

we get

$$T_{K \cup B_0} = T_{B_0} \cap \bigcap_{n \in \mathbb{N}} T_{(1/n)\overline{S_n(U)}} = T_{B_0} \cap \bigcap_{n \in \mathbb{N}} T_{(1/n)S_n(U)} \subseteq W .$$

Since $\overline{S_n(U)}$ is a compact subset of ηU, the set $K \cup B_0$ is also compact, and Theorem III.3 is proved.

From Lemma II.1 it follows immediately:

PROPOSITION III.4. *A Banach space* E *has the bounded approximation property if and only if the set* $LE \otimes E$ *of all operators of finite rank is dense in* $L_c(E, E)$.

Together with Theorem III.3 this proposition implies:

THEOREM III.5. *Assume* E *is a Banach space with the bounded approximation property. Then we have* $L_{co}(E, F) = \left(L_c(E, F)\right)_\tau$ *for every Banach space* F .

IV. Nuclear spaces

Let us remember that the bi-equicontinuous tensor product $L_{co}(E) \otimes_\varepsilon F$ can be considered as a subspace of $L_{co}(E, F)$. Now in some cases it is possible to show that the topology of $L_{co}(E) \otimes_\pi F$ is finer than the topology induced by $\left(L_c(E, F)\right)_\tau$. If, in addition, $LE \otimes F$ is dense in $L_c(E, F)$ and $L_{co}(E) \otimes_\varepsilon F = L_{co}(E) \otimes_\pi F$, then we have $\left(L_c(E, F)\right)_\tau = L_{co}(E, F)$. For this reason we examine nuclear spaces in this section. We will prove $\left(L_c(E, F)\right)_\tau = L_{co}(E, F)$ if E is a nuclear (DF)-space and F is a Fréchet space or if E is a nuclear Fréchet space and F is a Banach space. Finally we will show

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$$\left(L_c\left(\mathbb{R}^D, l_1\right)\right)_\tau \neq L_{co}\left(\mathbb{R}^D, l_1\right)$$

where $\mathbb{R}^D$ is an uncountable product of $\mathbb{R}$ and l_1 is the Banach space of all absolutely convergent series.

PROPOSITION IV.1. *Assume that on* $LE \otimes F$ *the topologies induced by* $\left(L_c(E, F)\right)_\tau$ *and* $L_{co}(E, F)$ *agree and assume that* $LE \otimes F$ *is dense in* $L_c(E, F)$. *Then we have*

$$\left(L_c(E, F)\right)_\tau = L_{co}(E, F) .$$

Proof. For every closed 0-neighborhood W in $\left(L_c(E, F)\right)_\tau$ there exist a precompact subset $K \subseteq E$ and a convex, circled 0-neighborhood V in F such that

$$\{g : g \in LE \otimes F, g(K) \subseteq V\} \subseteq W .$$

Let $T \in L(E, F)$ be an operator with $T(K) \subseteq \frac{1}{2}V$. Choose a filter Φ converging to T in $L_c(E, F)$ which contains $LE \otimes F$. Since Φ converges to T in $L_{co}(E, F)$ one can find an $G \in \Phi$ with $(G-T)(K) \subseteq H^{\frac{1}{2}}V$. This implies $G \cap (LE \otimes F) \subseteq W$. Since W is closed, T belongs to W and therefore we have proved $\left(L_c(E, F)\right)_\tau = L_{co}(E, F)$.

LEMMA IV.2. *The adherence* $a(LE \otimes F)$, *taken in* $L_c(E, F)$, *contains every nuclear mapping* T *from* E *into* F .

Proof. For every nuclear mapping T there exist an element $\lambda = \left(\lambda_n\right)_{n \in \mathbb{N}} \in l_1$, an equicontinuous subset $\{\zeta_n : n \in \mathbb{N}\}$ of LE and a bounded subset $\{y_n : n \in \mathbb{N}\}$ of F , such that

$$T(x) = \sum_{n \in \mathbb{N}} \lambda_n \zeta_n(x) y_n \quad \text{for all } x \in E .$$

Let V be a convex, circled 0-neighborhood in F and let x be an element of E . There exists an $r \in \mathbb{N}$ such that

$$T(x) - \sum_{n=1}^{m} \lambda_n \zeta_n(x) y_n \subseteq \tfrac{1}{2}V \quad \text{for all } m > r .$$

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Moreover there exist a 0-neighborhood U in E and a real number $\mu > 0$ such that $\zeta_n(U) \subseteq [-1, 1]$ and $\mu y_n \in \frac{1}{2}V$ for all $n \in \mathbb{N}$. For each $m > r$ we have

$$T(x) - \sum_{n=1}^{m} \lambda_n \zeta_n\left(x+\mu\|\lambda\|^{-1}U\right) y_n \subseteq \tfrac{1}{2}V - \sum_{n=1}^{m} \lambda_n \zeta_n\left(\mu\|\lambda\|^{-1}U\right) y_n$$

$$\subseteq \tfrac{1}{2}V + \mu[-1, 1] \sum_{n=1}^{m} \lambda_n \|\lambda\|^{-1} y_n \subseteq \tfrac{1}{2}V + \tfrac{1}{2}V \subseteq V .$$

Therefore the sequence $\left\{\sum_{n=1}^{m} \lambda_n \zeta_n \otimes y_n\right\}_{m\in\mathbb{N}}$ converges to T in $L_c(E, F)$.

To prove the next lemma let us first mention a result of Brauner [4]: let E be a dF-space and F a Fréchet space. Then a subset P is relatively compact in $L_{co}(F, E)$ if and only if there exists a 0-neighborhood U in F such that $P(U)$ is relatively compact in E. Recall that E is called a dF-space if and only if there exists a Fréchet space H such that E is isomorphic to $L_{co}(H)$. Together with Lemma II.8 this result implies:

LEMMA IV.3. *If E is a dF-space and F a Fréchet space then we have* $\left(F \otimes_c L_c E\right)_\tau = F \otimes_\pi L_{co}(E)$.

THEOREM IV.4. *Let E be a dF-space, F a Fréchet space and assume that E or F is nuclear. Then*

$$\left(L_c(E, F)\right)_\tau = L_{co}(E, F) .$$

Proof. The mapping $\Theta : L_c E \times F \to L_c(E, F)$ defined by $[\Theta(\zeta, y)](x) = \zeta(x)y$ for all $x \in E$, $y \in F$ and $\zeta \in LE$, is continuous. Therefore Θ induces a continuous mapping from $\left(L_c E \otimes_c F\right)_\tau = L_{co}(E) \otimes_\pi F$ into $\left(L_c(E, F)\right)_\tau$. Since E is nuclear if and only if $L_{co}(E)$ is nuclear (see [4]) we get $L_{co}(E) \otimes_\pi F = L_{co}(E) \otimes_\varepsilon F$. But $L_{co}(E) \otimes_\varepsilon F$ is a subspace of $L_{co}(E, F)$ which implies that on $LE \otimes F$ the topologies induced by $\left(L_c(E, F)\right)_\tau$ and $L_{co}(E, F)$ agree.

If F is nuclear, Grothendieck has shown in [9], Chapter II, §2,

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no. 4, that every $T \in L(E, F)$ is nuclear. If E is nuclear we argue as follows: by the preceding remark every $T \in L(E, F)$ is compact. Let U be a convex, circled 0-neighborhood in E such that $\overline{T(U)}$ is compact in F. Let B be the Banach space $\bigcup_{n \in \mathbb{N}} n\overline{T(U)}$ with unit ball $\overline{T(U)}$. Now T is a continuous mapping from E into B. Since E is nuclear, T is a nuclear mapping from E into B and therefore also a nuclear mapping from E into F. Proposition IV.1 now implies $(L_c(E, F))_\tau = L_{co}(E, F)$.

We have shown in the proof of the corollary of Theorem II.7 that a nuclear (DF)-space is a dF-space. Therefore we get:

COROLLARY. *Let E be a nuclear (DF)-space and F a Fréchet space. Then $(L_c(E, F))_\tau = L_{co}(E, F)$.*

PROPOSITION IV.5. *For every complete locally convex topological vector space E and for every Banach space F the topology of $L_{co}(E) \otimes_\pi F$ is finer than the topology induced on $LE \otimes F$ by $(L_c(E, F))_\tau$.*

Proof. Let $\mathcal{U}$ be the 0-neighborhood filter in E, $\mathcal{B}$ the family of all finite subsets of E and $V = \{y : y \in F, \|y\| \leq 1\}$. For every subset $M \subseteq E$ define $T_M = \{T : T \in L(E, F), T(M) \subseteq V\}$. Let W be a 0-neighborhood in $(L_c(E, F))_\tau$. For every $U \in \mathcal{U}$ the filter generated by $\{T_{U \cup B} : B \in \mathcal{B}\}$ converges to 0 in $L_c(E, F)$, which implies that, for every $U \in \mathcal{U}$, there exists a $B_U \in \mathcal{B}$ with $T_{U \cup B_U} \subseteq W$. Therefore the convex hull C of $\bigcup_{U \in \mathcal{U}} T_{U \cup B_U}$ is contained in W. Moreover, the convex hull D of $\bigcup_{U \in \mathcal{U}} (U \cup B_U)^0$ is a 0-neighborhood in $(L_c(E))_\tau = L_{co}(E)$ where the polars are taken with respect to the duality $\langle LE, E \rangle$. We will now show $D \otimes V \subseteq C \subseteq W$. Choose $\zeta \in D$ and $y \in V$. By definition of D we have

$$\zeta = \sum_{i=1}^{r} \lambda_i \zeta_i$$

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with $\zeta_i \in (U_i \cup B_{U_i})^0$, $0 < \lambda_i \in \mathbb{R}$ for $1 \le i \le r$ and $\lambda_1 + \ldots + \lambda_r = 1$. Since

$$(\zeta_i \otimes y)(U_i \cup B_{U_i}) = \zeta_i(U_i \cup B_{U_i})y \subseteq [-1, 1]y \subseteq V ,$$

we have $(\zeta_i \otimes y) \in T_{U_i \cup B_{U_i}}$ for $1 \le i \le r$ and

$$\zeta \otimes y = \sum_{i=1}^{r} \lambda_i(\zeta_i \otimes y) \in C .$$

THEOREM IV.6. *If E is a nuclear Fréchet space and F a Banach space then* $(L_c(E, F))_\tau = L_{co}(E, F)$.

Proof. By Proposition IV.5 the topology of $L_{co}(E) \otimes_\pi F$ is finer than the topology induced by $(L_c(E, F))_\tau$. Since $L_{co}(E)$ is nuclear we have $L_{co}(E) \otimes_\pi F = L_{co}(E) \otimes_\varepsilon F$. Therefore the topologies induced on $LE \otimes F$ by $L_{co}(E, F)$ and $(L_c(E, F))_\tau$ agree. Since E is nuclear, every $T \in L(E, F)$ is nuclear. Now Proposition IV.1 and Lemma IV.2 imply $L_{co}(E, F) = (L_c(E, F))_\tau$.

In the following example we will show that there exist a complete, nuclear, barrelled space E and a Banach space F with $(L_c(E, F))_\tau \ne L_{co}(E, F)$. Therefore $L_{co}(E, F)$ carries not always the strict topology generated by the topology of pointwise convergence on $L(E, F)$.

EXAMPLE. Let D be an uncountable set and let H be the family of all finite subsets of D . Define $R_i = \mathbb{R}$ for all $i \in D$. Let us denote by ind_L and ind_T the inductive limit taken in the category of all convergence spaces and in the category of all locally convex topological vector spaces respectively. Then we have

$$L_c\left(\prod_{i \in D} R_i\right) = \operatorname*{ind}_{H \in H} {}_L\, L_c\left(\prod_{i \in H} R_i\right)$$

and therefore

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$$L_{co}\left(\prod_{i\in D} i\right) = \left(L_c\left(\prod_{i\in D} R_i\right)\right)_\tau = \operatorname*{ind}_{H\in H}{}_T\, L_{co}\left(\prod_{i\in H} R_i\right) .$$

Since every $T \in L\left(\prod_{i\in D} R_i, l_1\right)$ has finite rank and since for every $H \in H$ we have

$$L_c\left(\prod_{i\in H} R_i, l_1\right) = L_{co}\left(\prod_{i\in H} R_i, l_1\right) = L_{co}\left(\prod_{i\in H} R_i\right) \otimes_\pi l_1$$

it is easy to see that

$$L_c\left(\prod_{i\in D} R_i, l_1\right) = \operatorname*{ind}_{H\in H}{}_L\, L_c\left(\prod_{i\in H} R_i, l_1\right) = \operatorname*{ind}_{H\in H}{}_L \left(L_{co}\left(\prod_{i\in H} R_i\right) \otimes_\pi l_1\right) .$$

This implies

$$\left(L_c\left(\prod_{i\in D} R_i, l_1\right)\right)_\tau = \operatorname*{ind}_{H\in H}{}_T \left(L_{co}\left(\prod_{i\in H} R_i\right) \otimes_\pi l_1\right)$$

$$= \left(\operatorname*{ind}_{H\in H}{}_T\, L_{co}\left(\prod_{i\in H} R_i\right)\right) \otimes_\pi l_1 = L_{co}\left(\prod_{i\in D} R_i\right) \otimes_\pi l_1 .$$

Since $L_{co}\left(\mathbb{R}^D\right)$ is not nuclear, we have

$$\left(L_c\left(\mathbb{R}^D, l_1\right)\right)_\tau = L_{co}\left(\mathbb{R}^D\right) \otimes_\pi l_1 \neq L_{co}\left(\mathbb{R}^D\right) \otimes_\varepsilon l_1 = L_{co}\left(\mathbb{R}^D, l_1\right) .$$

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