4
Field theory ingredients

In this chapter, we shall collect some of the field theory ingredients which will often be encountered in this book. More detailed discussions and derivations can be found in classic textbooks on quantum field theories [53] and some of the QCD books in [42–46].

4.1 Wick's theorem

Let us consider free boson or fermion fields $\varphi_i(x)$ of a particle i, which one can express in terms of the creation $a^\dagger$ and annihilation a operators, and the corresponding ones $b^\dagger$ and b for the anti-particles, where a and b may (or may not) coincide:

$$\varphi_i(x) = \sum_n c_i^{(n)}(x) a_n + \sum \bar{c}_i^{(n)}(x) b_n^\dagger \, . \tag{4.1}$$

For a fermion field (u and v are Dirac spinors):

$$\psi(x) = \int d\bar{k}[a(k)u(k)\, e^{-ikx} + b^\dagger(k)v(k)\, e^{ikx}] \, , \tag{4.2}$$

and for a boson:

$$\phi(x) = \int d\bar{k}[a(k)\, e^{-ikx} + b^\dagger(k)\, e^{ikx}] \, . \tag{4.3}$$

where the phase space measure is:

$$d\bar{k} \equiv \frac{d^3k}{(2\pi)^3 2E_k} = \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k^0) \, . \tag{4.4}$$

The Wick or *normal ordered* product [95]:

$$:\varphi_1(x_1)\varphi_2(x_2): \tag{4.5}$$

is obtained by placing all creator operators to the left of all annihilation operators, and by taking care on the (anti)-commuting relations if the fields are (fermions) bosons.

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Therefore:

$$
\begin{aligned}
:\varphi_1(x_1)\varphi_2(x_2): \equiv \sum_{n,n'} \big[& c_1^{(n)}(x_1)c_2^{(n')}(x_2)a_n a_{n'} + \bar{c}_1^{(n)}(x_1)\bar{c}_2^{(n')}(x_2)b_n^\dagger b_{n'}^\dagger \\
& + \bar{c}_1^{(n)}(x_1)c_2^{(n')}(x_2)b_n^\dagger a_{n'} + (-1)^\delta c_1^{(n)}(x_1)\bar{c}_2^{(n')}(x_2)b_{n'}^\dagger a_n \big] ,
\end{aligned} \tag{4.6}
$$

where $\delta = 1(0)$ for fermions (bosons). This results can be easily generalized to more factors of fields.

4.2 Time-ordered product

A time-ordered product is obtained by rearranging the fields or operators in the natural sequence of time. At a time $t' > t$, we first create a particle at a time t with $\varphi^\dagger$ and annihilate later on at a time t' with φ. This can be encoded by the amplitude:

$$
\theta(t'-t)\varphi(t', \vec{x}')\varphi^\dagger(t, \vec{x}) \, . \tag{4.7}
$$

If, for $t' < t$, an antiparticle is produced by $\varphi(x')$, then it is annihilated by $\varphi^\dagger(x)$ at the time t, with the amplitude:

$$
\theta(t-t')\varphi^\dagger(t, \vec{x})\varphi(t', \vec{x}') \, . \tag{4.8}
$$

The sum of the two equations gives the *time-ordered product*:

$$
\mathcal{T}\varphi(x')\varphi^\dagger(x) = \theta(t'-t)\varphi(x')\varphi^\dagger(x) + (-1)^\delta \theta(t-t')\varphi^\dagger(x)\varphi(x') \, , \tag{4.9}
$$

where $\delta = 1(0)$ for fermion (boson), where one should also note that fermion-boson operators are taken to commute. The T-product is arranged from right to left with increasing times, and then the appropriate name. One can also express it in terms of the Wick product:

$$
\mathcal{T}\varphi(x')\varphi^\dagger(x) =: \varphi(x')\varphi^\dagger(x) : + \langle 0|\mathcal{T}\varphi(x')\varphi^\dagger(x)|0\rangle \, . \tag{4.10}
$$

The above results can be generalized to the $\mathcal{T}$ products of n operators/fields:

$$
\mathcal{T}\varphi_1(x_1)\cdots\varphi_n(x_n) = (-1)^\delta \varphi_{i_1}(x_{i_1})\cdots\varphi_{i_n}(x_{i_n}) \, , \tag{4.11}
$$

where in the RHS the times are ordered ($t_{i_1} > t_{i_2} > \cdots > t_{i_n}$) and δ is the number of transposition of indices of the fermion operators/fields necessary for obtaining the required form in the RHS. It can be written as:

$$
\begin{aligned}
& \mathcal{T}\varphi_1(x_1)\cdots\varphi_n(x_n) = \mathcal{T}\varphi_1(x_1)\cdots\varphi_n(x_{n-1})\varphi_n(x_n) \\
= & :\varphi_1(x_1)\cdots\varphi_n(x_{n-1}):\varphi_n(x_n) \\
& + \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)|0\rangle :\varphi_1(x_3)\cdots\varphi_n(x_{n-1}):\varphi_n(x_n) + \text{ perm.} \\
& + \langle 0|\mathcal{T}\varphi_1(x_1)\varphi_2(x_2)|0\rangle\langle 0|\mathcal{T}\varphi_3(x_3)\varphi_4(x_4)|0\rangle :\varphi_1(x_5)\cdots\varphi_n(x_{n-1}):\varphi_n(x_n) + \text{ perm.} \\
& + \cdots ,
\end{aligned} \tag{4.12}
$$

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where $\cdots$ stands for:

$$\langle 0|\mathcal{T}\varphi_1(x_1)\varphi_2(x_2)|0\rangle \cdots \langle 0|\mathcal{T}\varphi_{n-1}(x_{n-1})\varphi_n(x_n)|0\rangle + \text{permutations} , \tag{4.13}$$

if n is even, and for:

$$\langle 0|\mathcal{T}\varphi_1(x_1)\varphi_2(x_2)|0\rangle \cdots \langle 0|\mathcal{T}\varphi_{n-2}(x_{n-2})\varphi_{n-3}(x_{n-3})|0\rangle\varphi_n(x_n) + \text{permutations} , \tag{4.14}$$

if n is odd. The vacuum expectation values or contractions give rise to the field propagators.

4.3 The S-matrix

4.3.1 Generalities

In field theory, one measures S-matrix elements, which is the probability amplitudes for transition between states which contain definite numbers of particles for t ranging from $-\infty$ to $+\infty$. They are usually named 'in' and 'out' states $|\alpha, \text{in}\rangle$ and $|\beta, \text{out}\rangle$, where α, β characterize particles momenta and quantum numbers. The S-matrix can be obtained from the interaction Lagrangian:

$$S = \mathcal{T} \exp\left[i \int d^4x \; \mathcal{L}_\mathrm{I}\right] \tag{4.15}$$

which one can expand as:

$$S = 1 + i \int d^4x \; \mathcal{L}_\mathrm{I}(x) + \cdots \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \; \mathcal{T}\mathcal{L}_\mathrm{I}(x_1) \cdots \mathcal{L}_\mathrm{I}(x_n) . \tag{4.16}$$

The S-matrix is relativistically invariant:

$$S = \mathcal{U}(a, \Lambda) S \mathcal{U}^{-1}(a, \Lambda) , \tag{4.17}$$

where $\mathcal{U}(a, \Lambda)$ is a transformation under the Poincarè group. It is also unitary:

$$S^\dagger S = 1 . \tag{4.18}$$

It can be related to the transition amplitude:

$$\langle \beta, \text{out}|T|\alpha, \text{in}\rangle \tag{4.19}$$

which gives the probability that the incoming state $|\alpha\rangle$ will evolve in time to the outcoming state $|\beta\rangle$ as:

$$\mathcal{S} = 1 + iT \tag{4.20}$$

4.3.2 Applications: cross-section and decay rate

We can illustrate the discussion by considering the scattering process:

$$(p_1, J_1) + (p_2, J_2) \to (k_1, j_1) + \cdots (k_n, j_n) . \tag{4.21}$$

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of two initial particles with momenta p_1 and p_2 and spin J_1 and J_2, and n final states with momenta p_n and spin j_n. The *unpolarized* cross-section of this process can be written as:

$$\sigma = \sum \frac{W}{FD} , \tag{4.22}$$

where $\sum$ represents an averaging over initial particle polarizations; W is the transition probability per unit of time and unit of volume, F is the incident particle flux and D the target-particle density. In the laboratory frame of incident particle 1 on a target particle 2, one has the kinematic variables:

$$\begin{aligned}
\lambda\big(s, m_1^2, m_2^2\big) &= [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2] , \\
F &= 2E_1|v_1 - v_2| = \lambda^{1/2}(s)/m_2 , \\
D &= 2E_2 , && (4.23)\\
s &= (p_1 + p_2)^2 , && (4.24)
\end{aligned}$$

where v_i $(i = 1, 2)$ is the velocity of the particle i $(v_2 = 0)$. The transition probability per unit of time and unit of volume is:

$$W = \frac{1}{(2\pi)^4\delta^4(0)} \int \prod_{j=1}^{n} \frac{d^3k_j}{(2\pi)^3 2E_j} |\langle f|(S-1)|i\rangle|^2 , \tag{4.25}$$

where the sum over the helicities of different particles is understood. We have used the normalization of state $|p, \lambda\rangle$ having helicity λ and momentum p:

$$\langle p', \lambda'|p, \lambda\rangle = (2\pi)^3 2E\delta^3(p' - p)\delta_{\lambda'\lambda} . \tag{4.26}$$

Using trivial substitutions, one can deduce the well-known cross-section:

$$\begin{aligned}
\sigma &= \frac{1}{2\lambda^{1/2}\big(s, m_1^2, m_2^2\big)} \frac{\mathcal{N}}{(2J_1+1)(2J_2+1)} \\
&\quad \times \int (2\pi)^4\delta^4(P_f - P_i)|\mathcal{M}(i \to f)|^2 \prod_{j=1}^{n} \frac{d^3k_j}{(2\pi)^3 2E_j} , && (4.27)
\end{aligned}$$

where we have introduced the reduced amplitude transition $\mathcal{M}$:

$$i\langle f|T|i\rangle \equiv \delta^4(P_f - P_i)|\mathcal{M}(i \to f) . \tag{4.28}$$

Here:

$$P_f \equiv \sum_1^n p_f ; \qquad P_i = p_1 + p_2 , \tag{4.29}$$

and the statistical factor is:

$$\mathcal{N} = \prod_i \frac{1}{n_i!} , \tag{4.30}$$

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if one has n_i identical particles in the final state. Analogously, the decay rate reads for a particle of mass M at rest is:

$$\Gamma(i \to f) = \frac{\mathcal{N}}{2M} \int (2\pi)^4 \delta^4(P_f - P_i) |\mathcal{M}(i \to f)|^2 \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_j} , \qquad (4.31)$$

Repeated uses of the *reduction formula* will show that the transition matrix can be related to the Green's function of the relevant particles.

4.4 Reduction formula

Let's consider the simplest case for the elastic scattering of two scalar particles. The S-matrix of this process is:

$$\langle k_1 k_2 | S | p_1 p_2 \rangle \qquad (4.32)$$

In terms of annihilation and creation operators, which satisfy the commutation relations:

$$[a(p), a^\dagger(p')] = (2\pi)^3 2E \delta^3(p' - p) , \quad [a(p), a(p')] = 0 , \qquad (4.33)$$

the scalar field reads:

$$\phi(x) = \int \frac{d^3 k_j}{(2\pi)^3 2E_j} [a(p) e^{-ipx} + a^\dagger(p) e^{ipx}] , \qquad (4.34)$$

which can be inverted:

$$a(p) = i \int d^3 x \, e^{ipx} \overset{\leftrightarrow}{\partial}_0 \phi(x) , \qquad (4.35)$$

where:

$$f \overset{\leftrightarrow}{\partial}_0 g \equiv f(\partial_0 g) - (\partial_0 f) g . \qquad (4.36)$$

Then, after some algebra:

$$\begin{aligned} \langle k_1 k_2 \; out | p_1 p_2 \; in \rangle &= \langle k_1 | a_{out}(k_2) | p_1 p_2 \; in \rangle \\ &= i \lim_{x_0 \to +\infty} \int d^3 x \, e^{ik_2 x} \overset{\leftrightarrow}{\partial}_0 \langle k_1 | \phi(x) | p_1 p_2 \; in \rangle \\ &= \langle k_1 | a_{in}(k_2) | p_1 p_2 \; in \rangle \\ &\quad + i \int d^4 x \, \partial_0 [e^{ik_2 x} \overset{\leftrightarrow}{\partial}_0 \langle k_1 | \phi(x) | p_1 p_2 \; in \rangle] . \end{aligned} \qquad (4.37)$$

Using:

$$\begin{aligned} \partial_0 (f \partial_0 g) &= f \partial_0^2 g \\ \partial_0^2 e^{ik_2 x} &= (\nabla^2 - m^2) e^{ik_2 x} , \end{aligned} \qquad (4.38)$$

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one can replace the last term of the previous equation by:

$$i \int d^4x \, e^{ik_2x} \left(\partial_x^2 + m^2\right) \langle k_1|\phi(x)|p_1 p_2 \; in\rangle \; . \tag{4.39}$$

Using repeatedly the above manipulations, one obtains the Fourier transform of the vacuum expectation value (VEV) of the I-product of four fields:

$$\begin{aligned}\langle k_1 k_2 \; out|p_1 p_2 \; in\rangle = \langle k_1 k_2 \; in|p_1 p_2 \; in\rangle \\ &+ i^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \; e^{i[k_1x_1 + k_2x_2 - p_1y_1 - p_2y_2]} \\ &\times \left(\partial_{x_1}^2 + m^2\right)\left(\partial_{x_2}^2 + m^2\right)\left(\partial_{y_1}^2 + m^2\right)\left(\partial_{y_2}^2 + m^2\right) \langle k_1|\phi(x)|p_1 p_2 \; in\rangle \\ &\times \langle 0|\mathcal{T}[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)]|0\rangle \; ,\end{aligned} \tag{4.40}$$

where:

$$\begin{aligned}\langle k_1 k_2 \; out|p_1 p_2 \; in\rangle - \langle k_1 k_2 \; in|p_1 p_2 \; in\rangle &= \langle k_1 k_2|(S-1)|p_1 p_2\rangle \\ &= (2\pi)^4 i\delta(k_1 + k_2 - p_1 - p_2)\langle k_1 k_2|T|p_1 p_2\rangle \; ,\end{aligned} \tag{4.41}$$

and we have used the shorthand notation:

$$\partial_x \equiv \frac{\partial}{\partial x_\mu} \; . \tag{4.42}$$

Similar manipulations can be extended to spinor fields.

4.5 Path integral in quantum mechanics

The path integral method, used long time ago by Feynman [96], has been revived by Fadeev and Popov and De Witt [97] in its application to non-Abelian theory, and by 't Hooft [98] when he derives the Feynman rules for massive gauge theories, particularly for the Standard Model of the Electroweak interactions. Detailed derivation of this method are described in modern textbooks. We shall briefly outline the method here, but starting from some examples in quantum mechanics.

4.5.1 Transition matrix of quantum mechanics in one dimension

The Hermitian operator 'coordinates' Q_a and a conjuguate 'momenta' P_b, satisfy the canonical commutation relations:

$$[Q_a, P_a] = i\delta_{ab} \; , \qquad [Q_a, Q_b] = [P_a, P_b] = 0 \tag{4.43}$$

to which correspond the eigenvectors $|q\rangle$ and $|p\rangle$ and the eigenvalues q_a and p_b. In the Heisenberg picture, Q and P have a time dependence leading to:

$$|q; t\rangle = \exp(iHt)|q\rangle \; , \qquad |p; t\rangle = \exp(iHt)|p\rangle \; , \tag{4.44}$$

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for the eigenstates, which satisfy the orthonormality and completeness conditions:

$$\langle q';t|q;t\rangle = \delta(q'-q)\,, \qquad \langle p';t|p;t\rangle = \delta(p'-p)\,,$$
$$\int \prod_a dq_a |q;t\rangle\langle q;t| = 1 = \int \prod_a dp_a |p;t\rangle\langle p;t| \tag{4.45}$$

and:

$$\langle q;t|p;t\rangle = \prod_a \frac{1}{\sqrt{2\pi}} \exp(i q_a p_a)\,. \tag{4.46}$$

One should remember that, in the previous notation, the state $|q;t\rangle$ in the *Heisenberg picture* coincides with the one of the *Schrödinger picture* $|q(t)\rangle$ at a given t. Now, we wish to calculate the scalar product:

$$\langle q';t'|q;t\rangle\,, \tag{4.47}$$

which corresponds to the probability amplitude for measurements at time t' to give the state $|q';t'\rangle$, if we found that at the time t our system is in a definite state $|q;t\rangle$. This is an easy task if the time t' and t are infinitely close to each other ($t \equiv \tau;\ t' \equiv t + d\tau$ and $d\tau \to 0$) since from Eq. (4.44):

$$\langle q';\tau + d\tau|q;\tau\rangle = \langle q';\tau|\exp(-iHd\tau)|q;\tau\rangle\,. \tag{4.48}$$

Expanding $|q;\tau\rangle$ in terms of the P eigenstates $|p;\tau\rangle$ by using Eq. (4.46), one can write:

$$\begin{aligned}\langle q';\tau + d\tau|q;\tau\rangle &= \int \prod_a dp_a \langle q';\tau|\exp[-iH(Q(\tau),P(\tau))d\tau]|p;\tau\rangle\langle p;\tau|q;\tau\rangle \\ &= \int \prod_a \frac{dp_a}{2\pi} \exp\left[-iH(q',p)d\tau + i\sum_a (q'_a - q_a)p_a\right]\,,\end{aligned} \tag{4.49}$$

where each p_a is integrated from $-\infty$ to $+\infty$. One can generalize this procedure by breaking the interval $t'-t$ into $N+1$ sets of infinitesimal intervals, as shown in Fig. 4.1, and sum

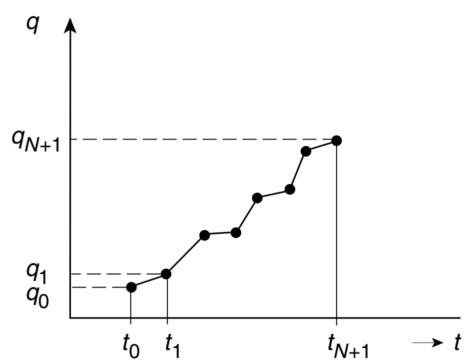

Fig. 4.1. Subdivision of the time interval $t'-t$ into $N+1$ sets of infinitesimal intervals.

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over a complete set of states $|q;\tau_k\rangle$ at each time τ_k. Then,

$$\langle q';t'|q;t\rangle = \int \left[\prod_{k=1}^{N}\prod_a dq_{k,a}\right]\left[\prod_{k=0}^{N}\prod_a \frac{dp_{k,a}}{2\pi}\right] \times \exp\left[i\sum_{k=1}^{N+1}\left\{-H(q_k,p_{k-1})d\tau + \sum_a (q_{k,a}-q_{k-1,a})p_{k-1,a}\right\}\right], \tag{4.50}$$

with $q_0 \equiv q$ and $q_{N+1} \equiv q'$. In the limit $\tau \to 0$, and then $N \to \infty$, one can assume that q_a and p_a are (to leading order in the τ-expansion) independent of τ, such that the argument of the exponential becomes an integral over τ. Making the *formal* substitutions:

$$(q_{a,k}-q_{a,k-1}) \to \dot{q}_a d\tau\ ; \qquad \sum_{k=1}^{N+1} \to \int_t^{t'}\ ; \qquad \int\left[\prod_{k=1}^{N}\prod_a dq_{k,a}\right] \to \int \prod_a dq_a(\tau)\,, \tag{4.51}$$

one, then, obtains the *path integral* ($\equiv$ integration over all paths taken by $q(\tau)$ from t to t'):

$$\langle q';t'|q;t\rangle = \int_{\substack{q_a\,\equiv\, q_a(t)\\ q'_a\,\equiv\, q_a(t')}} \prod_{\tau,a} dq_a(\tau) \prod_{\tau,b}\frac{dp_b(\tau)}{2\pi} \times \exp\left[i\int_t^{t'} d\tau\left\{-H\left(q(\tau),p(\tau)\right) + \sum_a \dot{q}_a(\tau)p_a(\tau)\right\}\right]. \tag{4.52}$$

One can perform the p integration. In order to read off the oscillating function in the exponential, it is convenient to work in the Euclidian space by formally treating $i d\tau$ to be *real*. Then, the integral has a definite norm. For a Hamiltonian of the form:

$$H(P,Q) = \frac{P^2}{2m} + V(Q)\,, \tag{4.53}$$

where $V(Q)$ is the potential, we have to perform a Gaussian integral:

$$\int_{-\infty}^{+\infty}\frac{dx}{2\pi}\exp[-ax^2+bx] = \frac{1}{\sqrt{4\pi a}}\exp[b^2/4a]\,. \tag{4.54}$$

Then, one can deduce from Eq. (4.52):

$$\langle q';t'|q;t\rangle = \int_{\substack{q_a\,\equiv\, q_a(t)\\ q'_a\,\equiv\, q_a(t')}} \prod_{\tau,a} dq_a(\tau) \times \exp\left[i\int_t^{t'} L(\tau)d\tau\right], \tag{4.55}$$

where:

$$L \equiv \frac{m}{2}\dot{q}^2 - V(q)\,, \tag{4.56}$$

is the Lagrangian.

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4.5.2 The Green's functions

One can extend the previous discussion of matrix transition to the analysis of a Green's function which is the time-ordered products of different (local) operators. This can be illustrated by the example of a quantum mechanical two-point function, which is the matrix element of the time-ordered product between ground states:

$$G(t,t') = \langle 0|\mathcal{T}Q(t_1)Q(t_2)|0\rangle\ , \quad (t_1 > t_2)\ , \tag{4.57}$$

where $|0\rangle$ denotes the ground state. By inserting complete sets of states, it can be written as:

$$G(t,t') = \int dqdq'\langle 0|q';t'\rangle\langle q';t'|\mathcal{T}Q(t_1)Q(t_2)|q;t\rangle\langle q;t|0\rangle\ , \quad (t_1 > t_2)\ . \tag{4.58}$$

Introducing the wave function of the ground state:

$$\langle 0|q;t\rangle = \phi_0(q)\exp[-iE_0t] \equiv \phi_0(q,t)\ , \tag{4.59}$$

and using an analogue of the derivation of Eq. (4.52) for the matrix element:

$$\begin{aligned}\langle q';t'|\mathcal{T}Q(t_1)Q(t_2)|q;t\rangle = \int &\langle q'|\exp[-iH(t'-t_1)]|q_1\rangle\langle q_1|Q(t_1)\exp[-iH(t_1-t_2)]|q_2\rangle\\ &\times\langle q_2|Q(t_2)\exp[-iH(t_2-t)]|q;t\rangle dq_1dq_2\ ,\end{aligned} \tag{4.60}$$

one obtains in the Schrödinger picture:

$$\begin{aligned}G(t_1,t_2) = \int_{\substack{q_a\,\equiv\, q_a(t)\\ q'_a\,\equiv\, q'_a(t')}} \prod_{\tau,a} dq_a(\tau)\prod_{\tau,b}\frac{dp_b(\tau)}{2\pi}\phi_0(q',t')\phi_0^*(q,t)q_1(t_1)q_2(t_2)\\ \times\exp\left[i\int_t^{t'}d\tau\left\{-H\left(q(\tau),p(\tau)\right)+\sum_a \dot q_a(\tau)p_a(\tau)\right\}\right].\end{aligned} \tag{4.61}$$

Now, we can remove the wave functions by introducing a complete set of states. Then:

$$\begin{aligned}\langle q';t'|Q';T'\rangle = \langle q'|\exp[-iH(t'-T')]|Q'\rangle &= \sum_n\langle q'|n\rangle\langle n|\exp\left[-iH(t'-T')\right]|Q'\rangle\\ &= \sum_n \phi_n^*(q')\phi_n(Q')\exp[-iE_n(t'-T')]\ ,\end{aligned} \tag{4.62}$$

where E_n and ϕ_n are the energy and wave functions of the state $|n\rangle$. The contribution of the ground states can be isolated by taking the limit $t' \to i\infty$ and using the fact that $E_n > E_0$ for $n \neq 0$. In this way, one gets:

$$\lim_{t'\to -i\infty}\langle q';t'|Q';T'\rangle = \phi_0^*(q')\phi_0(Q')\exp[-E_0|t'|]\exp[iE_0T']\ . \tag{4.63}$$

Similarly:

$$\lim_{t\to +i\infty}\langle Q;T|q;t\rangle = \phi_0(q)\phi_0^*(Q)\exp[-E_0|t|]\exp[\ -iE_0T]\ , \tag{4.64}$$

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from which one can deduce:

$$\mathcal{N} \equiv \lim_{\substack{t' \to -i\infty \\ t \to +i\infty}} \langle q'; t' | q; t \rangle = \phi_0^*(q')\phi_0(q) \exp[-E_0(|t| + |t'|)] \,. \tag{4.65}$$

Therefore, one can derive after some straightforward algebra:

$$\begin{aligned} \lim_{\substack{t' \to -i\infty \\ t \to +i\infty}} \langle q'; t' | \mathcal{T} Q(t_1) Q(t_2) | q; t \rangle &= \int dQ \, dQ' \langle q'; t' | Q'; T' \rangle \\ &\quad \times \langle Q'; T' | \mathcal{T} Q(t_1) Q(t_2) | Q; T \rangle \langle Q; T | q; t \rangle \\ &= \phi_0^*(q')\phi_0(q) \exp[-E_0(|t| + |t'|)] G(t_1, t_2) \,. \end{aligned} \tag{4.66}$$

Combining Eqs. (4.65) and (4.66), one can deduce the result:

$$\begin{aligned} G(t_1, t_2) = \frac{1}{\mathcal{N}} \int_{\substack{q_a \equiv q_a(t) \\ q'_a \equiv q'_a(t')}} \prod_{\tau, a} dq_a(\tau) \prod_{\tau, b} \frac{dp_b(\tau)}{2\pi} q_1(t_1) q_2(t_2) \\ \times \exp\left[i \int_t^{t'} d\tau \left\{ -H\left(q(\tau), p(\tau)\right) + \sum_a \dot{q}_a(\tau) p_a(\tau) \right\} \right] . \end{aligned} \tag{4.67}$$

This result for the two-point function can be generalized to n-point Green's function. This Green's function can be generated as:

$$G^{(n)}(\tau_1, \tau_2, \ldots \tau_n) = (-i)^n \left. \frac{\delta^n Z[J]}{\delta J(\tau_1) \cdots \delta J(\tau_n)} \right|_{J=0} . \tag{4.68}$$

by the *generating functional*:

$$Z[J] = \lim_{\substack{t' \to -i\infty \\ t \to +i\infty}} \frac{1}{\mathcal{N}} \int \mathcal{D}q \, \exp\left\{ i \int_t^{t'} d\tau \left[-\frac{m}{2}\dot{q}^2 - V(q) + J(\tau) q(\tau) \right] \right\} , \tag{4.69}$$

which corresponds to the transition amplitude from a ground state at τ to the ground state at τ' in the presence of an external source $J(\tau)$, with the normalization $Z[0] = 1$. We have introduced the symbolic notation $\mathcal{D}$ for the integration measure:

$$\mathcal{D}q \equiv \prod_{\tau, a} dq_a(\tau) \,. \tag{4.70}$$

In order to elucidate the meaning of the previous expression, we recall that, by definition, a functional is an application of the space of smooth functions $f(x)$ into complex numbers:

$$J(x) \longmapsto Z[J] \,, \tag{4.71}$$

while a functional derivative is defined as:

$$\frac{\delta Z[J]}{\delta J(y)} = \lim_{\epsilon \to 0} \frac{Z[J + \epsilon \delta_y] - Z[J]}{\epsilon} \,, \tag{4.72}$$

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where $\delta_y = \delta(x - y)$ is the δ-function at y. In the case of the functional integral:

$$Z[J] = \int dx\; K(x)J(x)\,, \tag{4.73}$$

the functional derivative is:

$$\frac{\delta Z[J]}{\delta J(y)} = K(y)\,, \tag{4.74}$$

which, after performing a Taylor expansion of the kernel $K(x)$, leads to:

$$K_n(x_1, \ldots x_n) = \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\,. \tag{4.75}$$

4.5.3 Euclidean Green's function

The unphysical limits $t' \to -i\infty,\ t \to i\infty$ can be interpreted in terms of the *Euclidean* Green's functions:

$$\begin{aligned} S^{(n)}(\tau_1, \ldots, \tau_n) &= i^n G^{(n)}(-i\tau_1, \ldots, -i\tau_n) \\ &= \left.\frac{\delta^n Z_{\mathrm{E}}[J]}{\delta J(\tau_1)\cdots\delta J(\tau_n)}\right|_{J=0}\,. \end{aligned} \tag{4.76}$$

where $Z_{\mathrm{E}}[J]$ can be deduced from $Z[J]$ by the formal change $\tau" \to i\tau$. In the Euclidean region, the path integral is well-defined, as it converges because the potential is bounded from below ($(m/2)\dot{q}^2 + V(q) > 0$), such that the exponential in Eq. (4.69) will give a damping factor.

4.6 Path integral in quantum field theory

4.6.1 Scalar field quantization

For simplicity let's consider a classical field $\phi(x)$ and the corresponding Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ to which corresponds the action:

$$\mathcal{S} = \int d^4x\; \mathcal{L}(\phi, \partial_\mu\phi)\,. \tag{4.77}$$

The field ϕ satisfies the Euler–Lagrange equation of motion:

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - \frac{\delta\mathcal{L}}{\delta\phi} = 0\,. \tag{4.78}$$

We denote by $\pi(x)$ its conjuguate momentum:

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\,(\partial_0\phi)}\,, \tag{4.79}$$

which obeys the equal-time canonical commutation relations:

$$[\pi(\vec{x},t),\phi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}')\ , \tag{4.80}$$

while the Hamiltonian density is defined as:

$$\mathcal{H} = \int d^4x[\pi(x)\partial_0\phi(x) - \mathcal{L}(x)] \tag{4.81}$$

Therefore, in order to use the previous results of quantum mechanics, one can consider a field theory as a quantum mechanical system with infinite degrees of freedom. Therefore, one can make the substitutions:

$$\begin{aligned} \mathcal{D}q\mathcal{D}p &\to \mathcal{D}\phi(x)\mathcal{D}\pi(x)\ , \\ L(q_i,\dot{q}_i) &\to \int d^3x\ \mathcal{L}(\phi,\partial_\mu\phi)\ ; \\ H(q_i,p_i) &\to \int d^3x\ \mathcal{H}(\phi,\pi)\ . \end{aligned} \tag{4.82}$$

Using the fact that the ground state in field theory is the vacuum state, the generating functional $Z[J]$ is the vacuum-to-vacuum transition amplitude in the presence of an external source $J(x)$, and read in the *Euclidian space*:[1]

$$Z[J] = \int \mathcal{D}\phi\ \exp\left\{\int d^4x[\mathcal{L}(\phi(x)) + J(x)\phi(x)]\right\}\ , \tag{4.83}$$

up to an inessential normalization factor; Here, x is the Euclidian coordinate ($\tau \to it$). In field theory, we are interested in the *connected* Green's function, which is:

$$S^{(n)}(x_1,\dots x_n) = \left\{\frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right\}\Bigg|_{J=0}\ , \tag{4.84}$$

where an extra factor of $1/Z[J]$ has been inserted in order to remove the disconnected part of the Green's function.

4.6.2 Application to $\lambda\phi^4$ theory

We can illustrate this result by working with the Lagrangian of $\lambda\phi^4$ theory:

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{I}} \tag{4.85}$$

with:

$$\begin{aligned} \mathcal{L}_{\text{free}} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\mu^2\phi\ , \\ \mathcal{L}_{\text{I}} &= -\frac{\lambda}{4!}\phi^4\ . \end{aligned} \tag{4.86}$$

[1] We shall work in the Euclidian space in this subsection, where, as mentioned previously, the integral has a definite norm, and is well defined. This space is useful for a path integral formulation of non-perturbative QCD. However, the derivation of Feynman rule can still be done in the Minkowski space.

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In *Euclidian space*, the generating functional reads:

$$Z[J] = \int \mathcal{D}\phi \exp\left\{-\int d^4x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi\right]\right\}, \tag{4.87}$$

which one can rewrite as:

$$Z[J] = \left[\exp\int d^4x\, \mathcal{L}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right] Z_0[J] \tag{4.88}$$

The free-field generating functional

$$Z_0[J] = \int \mathcal{D}\phi \exp\left[\int d^4x\, (\mathcal{L}_{\text{free}} + J\phi)\right]. \tag{4.89}$$

can be written in the form:

$$Z_0[J] = \int \mathcal{D}\phi \exp\left\{-\frac{1}{2}\int d^4x d^4y\, \phi(x)\mathbf{K}(x,y)\phi(y) + \int d^4z J(z)\phi(z)\right\}, \tag{4.90}$$

where:

$$\mathbf{K}(x,y) = \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + \mu^2\right) \tag{4.91}$$

and the identity:

$$-\left(\frac{\partial\phi}{\partial\tau}\right)^2 - (\nabla\phi)^2 = \phi\left(\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\phi \tag{4.92}$$

has been used because their divergence is a total four-divergence. Integrating the Gaussian integral:

$$\begin{aligned} &\lim_{N\to\infty}\int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_N \exp\left[-\frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j + \sum_k J_k\phi_k\right] \\ &\sim \frac{1}{\sqrt{\det\mathbf{K}}}\exp\left[\frac{1}{2}\sum_{i,j} J_i(\mathbf{K}^{-1})_{ij}J_j\right], \end{aligned} \tag{4.93}$$

one obtains:

$$Z_0[J] = \exp\left[\frac{1}{2}\int d^4x d^4y\, J(x)\Delta(x,y)J(y)\right], \tag{4.94}$$

where $\Delta(x, y)$ is the inverse of $\mathbf{K}(x, y)$:

$$\int d^4y\, \mathbf{K}(x,y)\Delta(y,z) = \delta^4(x-z), \tag{4.95}$$

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which leads to the desired expression of the *scalar propagator*:

$$\Delta(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp[ik(x-y)]}{k^2 + \mu^2} . \tag{4.96}$$

Perturbative expansion in powers of the interaction Lagrangian $\mathcal{L}_\mathrm{I}$ generates the Feynman rules for different vertices. However, in order to keep only the connected Green's function, one should expand $\ln Z$ as defined in Eq. (4.84).

4.6.3 Fermion field quantization

The quantization of the fermion field can also be done by expressing the transition amplitude as a sum over possible lines connecting the initial and final states. For a classical fermion (anti-fermion) fields ψ, $(\bar{\psi})$ and sources η, $\bar{\eta}$, the generating functional reads:

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi(x)\mathcal{D}\bar{\psi}(x) \exp\left\{i \int d^4x\, [\mathcal{L}(\psi\bar{\psi}) + \bar{\psi}\eta + \bar{\eta}\psi]\right\} , \tag{4.97}$$

where the functional integral must be taken over anti-commuting c number functions which are elements of the Grassmann algebra:

$$\begin{aligned} &\theta(x), \theta(x') = \theta(x), \bar{\theta}(x') = \bar{\theta}(x), \bar{\theta}(x') = 0 , \\ &[\theta(x)]^2 = 0 , \end{aligned} \tag{4.98}$$

where $\theta \equiv \psi$ or η. The fermion Lagrangian is:

$$\mathcal{L} \equiv \mathcal{L}_\mathrm{free} + \mathcal{L}_\mathrm{I} , \tag{4.99}$$

with:

$$\begin{aligned} \mathcal{L}_\mathrm{free}(x) &= \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) , \\ \mathcal{L}_\mathrm{I}(x) &= \bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x) . \end{aligned} \tag{4.100}$$

Since the fermion fields always enter the Lagrangian quadratically, the previous functional is a generalized Gaussian integral. Therefore, one can write:

$$Z[J] = \int \mathcal{D}\psi(x)\mathcal{D}\bar{\psi}(x) \exp\left\{\int d^4x\ \bar{\psi}A\psi\right\} = \det A , \tag{4.101}$$

where Z is the vacuum-to vacuum amplitude and the (connected) Feynman diagram generated by $\ln Z$ will be a set of single-closed fermion loops.

4.6.4 Gauge field quantization

Due to gauge invariance, gauge theories represent systems with constrained dynamic variables. Their quantization is more involved than the one of scalar field theory or of free fermion discussed previously, and so we shall leave it for discussion in the next section.

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