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BIPARTITE GRAPH BUNDLES WITH CONNECTED FIBRES

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Let G be a finite connected simple graph. The isomorphism classes of graph bundles and graph coverings over G have been enumerated by Kwak and Lee. Recently, Archdeacon and others characterised bipartite coverings of G and enumerated the isomorphism classes of regular $2p$-fold bipartite coverings of G, when G is nonbipartite. In this paper, we characterise bipartite graph bundles over G and derive some enumeration formulas of the isomorphism classes of them when the fibre is a connected bipartite graph. As an application, we compute the exact numbers of the isomorphism classes of bipartite graph bundles over G when the fibre is the path P_n or the cycle C_n.

1. INTRODUCTION

Let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $|X|$ denote the cardinality of a set X. The *Betti number* of G is by definition the number $\beta(G) = |E(G)| - |V(G)| + 1$, which turns out to be the number of independent cycles in G. For graph theoretic terminology not defined here, see [**3**].

Two graphs G and H are *isomorphic* if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. Such a correspondence is called an *isomorphism* between G and H. An *automorphism* of a graph G is an isomorphism of G onto itself. Thus, an automorphism of G is a permutation of the vertex set $V(G)$ which preserves adjacency. Obviously, the automorphisms of G form a permutation group $\mathrm{Aut}\,(G)$, which acts on the vertex set $V(G)$.

Now, we introduce the notion of a graph bundle [**7**]. Every edge of a graph G gives rise to a pair of oppositely directed edges. We denote the set of directed edges of G by $D(G)$. A directed edge e in $D(G)$ is denoted by uv if its initial and terminal vertices are u and v respectively, and its reverse edge is denoted by e^{-1} or vu. For a finite group Γ, a Γ*-voltage assignment* of G is a function $\phi : D(G) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in D(G)$. We denote the set of Γ-voltage assignment of G by $C^1(G;\Gamma)$. Note that the set $C^1(G;\Gamma)$ needs not be a group under pointwise multiplication.

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For a finite simple graph F, let ϕ be an $\mathrm{Aut}(F)$-voltage assignment of G. We construct a new graph $G \times^{\phi} F$ as follows: $V(G \times^{\phi} F) = V(G) \times V(F)$. Two vertices (u, s) and (v, t) in $G \times^{\phi} F$ are adjacent if either $uv \in D(G)$ and $t = \phi(uv)(s)$ in F or $u = v$ in G and the vertices s and t in F are adjacent. The graph $G \times^{\phi} F$ is called the *bundle graph associated with* ϕ. Together with the first coordinate projection $p^{\phi} : G \times^{\phi} F \to G$, the pair $(G \times^{\phi} F, p^{\phi})$ is called the *F-bundle over G associated with* ϕ, and G and F are called the *base* and the *fibre* of the F-bundle $(G \times^{\phi} F, p^{\phi})$, respectively. When there is no confusion, we often call the bundle graph $G \times^{\phi} F$ an F-bundle.

Note that for each $v \in V(G)$, the fibre $(p^{\phi})^{-1}(v) = F_v$ of v is a subgraph of the graph $G \times^{\phi} F$ which is isomorphic to F. The map p^{ϕ} maps vertices to vertices, but an image of an edge can be either an edge or a vertex. Moreover, $V(G \times^{\phi} F) = \bigcup_{v \in V(G)} V(F_v)$. If $F = \overline{K_n}$, the complement of the complete graph K_n on n vertices, then an F-bundle over G is just an n-fold covering graph of G [2]. If $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in D(G)$, then $G \times^{\phi} F$ is just the cartesian product $G \times F$ of G and F.

Two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are said to be *isomorphic* if there exists an isomorphism $\Phi : G \times^{\phi} F \to G \times^{\psi} F$ such that the diagram

$$\begin{array}{ccc} G \times^{\phi} F & \xrightarrow{\ \Phi\ } & G \times^{\psi} F \\ & {\scriptstyle p^{\phi}} \searrow \quad \swarrow {\scriptstyle p^{\psi}} & \\ & G & \end{array}$$

commutes. Such a Φ is called a *bundle isomorphism*. Notice that two isomorphic bundle graphs need not be isomorphic as graph bundles.

After the enumeration of double covers of a graph in [4] and [8], there has been much progress during the last decade in the enumeration of several graph coverings or graph bundles over a graph ([**1, 4, 5, 6**] and references there).

Kwak and Lee [6] obtained the following algebraic characterisation of isomorphism classes of F-bundles over G.

Theorem 1.1. *Let ϕ and ψ be two voltage assignments in $C^1(G; \mathrm{Aut}(F))$. Two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are isomorphic if and only if there exists $f : V(G) \to \mathrm{Aut}(F)$ such that $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for all $uv \in D(G)$.*

Corollary 1.2. *Let ϕ be a voltage assignment in $C^1(G; \mathrm{Aut}(F))$ and T a fixed spanning tree of G. Then there exists a voltage assignment ψ in $C^1(G; \mathrm{Aut}(F))$ such that $\psi(e) =$ the identity for all $e \in D(T)$ and $G \times^{\phi} F$ is isomorphic to $G \times^{\psi} F$ as bundles.*

In particular, if G is a tree, then every F-bundle graph over G is isomorphic to the cartesian product $G \times F$. For a fixed spanning tree T of G, let

$$C_T^1(G; \mathrm{Aut}(F)) = \Big\{ \phi \in C^1(G; \mathrm{Aut}(F)) \mid \phi(e) = \text{the identity for each } e \in D(T) \Big\}.$$

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It follows from Corollary 1.2 that $C^1_T\big(G;\operatorname{Aut}(F)\big)$ contains all representatives of the isomorphism classes of F-bundles over G.

Corollary 1.3. *Let ϕ and ψ be two voltage assignments in $C^1_T\big(G;\operatorname{Aut}(F)\big)$. Then two F-bundles $G\times^\phi F$ and $G\times^\psi F$ are isomorphic if and only if there exists an automorphism σ in* $\operatorname{Aut}(F)$ *such that $\psi(uv)=\sigma\phi(uv)\sigma^{-1}$ for all $uv\in D(G)-D(T)$.*

2. A characterisation of bipartite F-bundles

In this section, we consider graph bundles over G whose bundle graphs are bipartite, called *bipartite graph bundles* over G. If an F-bundle $G\times^\phi F$ is bipartite, then clearly the fibre F as a subgraph of $G\times^\phi F$ is bipartite, while the base graph G needs not be bipartite in general. In the following we discuss some characterisations of a bipartite F-bundle $G\times^\phi F$ in terms of the base graph G.

Let $\mathfrak{B}=\big\{V_1(F),V_2(F)\big\}$ be a bipartition of the vertices of a bipartite graph F. An automorphism $\sigma\in\operatorname{Aut}(F)$ is said to *preserve* the bipartition $\mathfrak{B}$ of F if $\sigma\big(V_i(F)\big)=V_i(F)$ for each $i=1,2$, and said to *reverse* the bipartition $\mathfrak{B}$ of F if $\sigma\big(V_i(F)\big)=V_j(F)$ where $i\neq j$. We put

$$\mathcal{P}_{\mathfrak{B}}(F)=\big\{\sigma\in\operatorname{Aut}(F)\mid\sigma\text{ preserves the bipartition }\mathfrak{B}\text{ of }F\big\}$$

and

$$\mathcal{R}_{\mathfrak{B}}(F)=\big\{\sigma\in\operatorname{Aut}(F)\mid\sigma\text{ reverses the bipartition }\mathfrak{B}\text{ of }F\big\}.$$

Note that, for a bipartition $\mathfrak{B}$ of F, if $\mathcal{R}_{\mathfrak{B}}(F)\neq\emptyset$, then $\big|V_1(F)\big|=\big|V_2(F)\big|$ and $\big|\mathcal{P}_{\mathfrak{B}}(F)\big|=\big|\mathcal{R}_{\mathfrak{B}}(F)\big|$. If F is a connected bipartite graph, then the bipartition of F is unique and $\operatorname{Aut}(F)$ is the disjoint union of $\mathcal{P}_{\mathfrak{B}}(F)$ and $\mathcal{R}_{\mathfrak{B}}(F)$. In this case, $\mathcal{P}_{\mathfrak{B}}(F)$ and $\mathcal{R}_{\mathfrak{B}}(F)$ are denoted by $\mathcal{P}(F)$ and $\mathcal{R}(F)$ respectively. If F is not connected, then, for a bipartition $\mathfrak{B}$ of F, $\operatorname{Aut}(F)$ needs not be the disjoint union of $\mathcal{P}_{\mathfrak{B}}(F)$ and $\mathcal{R}_{\mathfrak{B}}(F)$ even though $\mathcal{R}_{\mathfrak{B}}(F)\neq\emptyset$. From now on, we *assume* that the fibre F is a connected bipartite graph, and T is a fixed spanning tree of the base graph G of a graph bundle $G\times^\phi F$.

For each $\sigma\in\operatorname{Aut}(F)$, we define the *signature of* σ as

$$\operatorname{sig}(\sigma)=\begin{cases}+1 & \text{if }\sigma\in\mathcal{P}(F),\\ -1 & \text{if }\sigma\in\mathcal{R}(F).\end{cases}$$

An edge e in $E(G)-E(T)$ is said to be *odd* (respectively *even*) if $T+e$ has an odd (respectively even) cycle. For a directed edge e in $D(G)-D(T)$, the *signature of* e is defined as

$$\operatorname{sig}(e)=\begin{cases}+1 & \text{if the underlying edge of }e\text{ is even},\\ -1 & \text{if the underlying edge of }e\text{ is odd}.\end{cases}$$

Let $\beta_o(G,T)$ and $\beta_e(G,T)$ be the number of odd and even edges in $E(G)-E(T)$, respectively. Then $\beta(G)=\beta_e(G,T)+\beta_o(G,T)$.

Let ϕ be an $\mathrm{Aut}\,(F)$-voltage assignment of G and let $W = e_1e_2\dots e_n$ be a walk in G with length $\ell(W) = n$. The product of voltages $\phi(W) = \phi(e_n)\cdots\phi(e_2)\phi(e_1)$ is called the *net ϕ-voltage* of W.

THEOREM 2.1. *Let F be a connected bipartite graph and $\phi \in C^1(G; \mathrm{Aut}\,(F))$. Then the following are equivalent.*

(a) *$G \times^\phi F$ is bipartite.*

(b) *For each cycle C in G, $\mathbf{sig}(\phi(C)) = (-1)^{\ell(C)}$.*

Moreover, if $\phi \in C^1_T(G; \mathrm{Aut}\,(F))$, then the above statements are equivalent to

(c) *For each $uv \in D(G) - D(T)$, $\mathbf{sig}(\phi(uv)) = \mathbf{sig}(uv)$.*

PROOF: (a) $\Rightarrow$ (b) Suppose $G \times^\phi F$ is bipartite, and let C be a cycle of length n in G having vertices $v_1, v_2, \dots, v_n$ consecutively. Then (v_1, s) $(v_2, \phi(v_1v_2)(s))$ $(v_3, \phi(v_2v_3)\phi(v_1v_2)(s))$ $\cdots$ $(v_1, \phi(C)(s))$ is a path or a cycle P in $G \times^\phi F$ of length n. Let Q be a walk in $(p^\phi)^{-1}(v_1) = F_{v_1}$ connecting the vertices $(v_1, \phi(C)(s))$ and (v_1, s). Then PQ is a closed walk in $G \times^\phi F$ so that the length $\ell(PQ)$ is even, since $G \times^\phi F$ is bipartite. Note that

$$\ell(PQ) = \ell(C) + \ell(Q) = n + \ell(Q).$$

Thus the parities of $\ell(C)$ and $\ell(Q)$ are the same, that is, $(-1)^{\ell(C)} = (-1)^{\ell(Q)}$. Moreover, since F_{v_1} is bipartite, the length $\ell(Q)$ is odd if and only if $\phi(C) \in \mathcal{R}(F)$, and is even if and only if $\phi(C) \in \mathcal{P}(F)$. Therefore, $\mathbf{sig}(\phi(C)) = (-1)^{\ell(Q)} = (-1)^{\ell(C)}$.

(b) $\Rightarrow$ (a) Let u_0 be a fixed vertex in G and let the fibre F_{u_0} have a vertex bipartition $V_1(F_{u_0})$ and $V(F_{u_0}) - V_1(F_{u_0})$. For each v in $V(G)$, let P_v be the unique path in a spanning tree T which connects u_0 and v, and let

$$V_1(F_v) = \begin{cases} V(F_v) - \phi(P_v)(V_1(F_{u_0})) & \text{if } \ell(P_v) \text{ is odd,} \\ \phi(P_v)(V_1(F_{u_0})) & \text{if } \ell(P_v) \text{ is even.} \end{cases}$$

Then for each v in $V(G)$, $V_1(F_v)$ and $V(F_v) - V_1(F_v)$ form a bipartition of the bipartite graph F_v. This implies that $V_1(G \times^\phi F) = \bigcup_{v \in V(G)} V_1(F_v)$ and its complement $V_2(G \times^\phi F)$ in $V(G \times^\phi F)$ form a bipartition of the connected graph $(p^\phi)^{-1}(T)$ which is a spanning subgraph of $G \times^\phi F$. Now, if condition (b) holds, then $G \times^\phi F$ cannot have an odd cycle, and $V_1(G \times^\phi F)$ and $V_2(G \times^\phi F)$ actually form a bipartition of the connected graph $G \times^\phi F$. Hence $G \times^\phi F$ is bipartite.

(b) $\Leftrightarrow$ (c) This is clear that the length $\ell(C)$ of a cycle C in G is odd if and only if C contains an odd number of odd edges. Since $\phi \in C^1_T(G; \mathrm{Aut}\,(F))$, the conditions (b) and (c) are equivalent. ∎

COROLLARY 2.2. *Let F be a connected bipartite graph with $\mathcal{R}(F) = \emptyset$. Then the following are equivalent.*

(a) *G is bipartite.*

(b) *All bundle graphs $G \times^{\phi} F$ over G are bipartite.*

(c) *$G \times F$ is bipartite.*

PROOF: (a) $\Rightarrow$ (b) Let ϕ be an $\mathrm{Aut}\,(F)$-voltage assignment of G. Then for each cycle C in G, $\mathbf{sig}(\phi(C)) = 1$, because $\mathcal{R}(F) = \emptyset$. Since G is bipartite, $(-1)^{\ell(C)} = 1$ for each cycle C in G. Now, by Theorem 2.1, $G \times^{\phi} F$ is bipartite.

(b) $\Rightarrow$ (c) Let ϕ be the trivial voltage assignment in $C^1(G; \mathrm{Aut}\,(F))$, that is, $\phi(uv)$ is the identity for each $uv \in D(G)$. Then $G \times^{\phi} F$ is just the cartesian product $G \times F$, which is bipartite by (b).

(c) $\Rightarrow$ (a) This is clear, because G is a subgraph of $G \times F$. ∎

Corollary 2.2 implies that if $\mathcal{R}(F) = \emptyset$ and G is nonbipartite, then there is a voltage $\phi \in C^1(G; \mathrm{Aut}\,(F))$ such that $G \times^{\phi} F$ is nonbipartite. Actually, the following corollary shows that $G \times^{\phi} F$ cannot be bipartite for any voltage $\phi \in C^1(G; \mathrm{Aut}\,(F))$. In particular, if the number of vertices of F is odd, then clearly $\mathcal{R}(F) = \emptyset$, and any F-bundle over a nonbipartite graph G cannot be bipartite.

Recall that every F-bundle graph over a tree T is isomorphic to the cartesian product $T \times F$. Since the cartesian product of two bipartite graphs is also bipartite, every F-bundle graph over a tree is bipartite. Note that if $\mathcal{R}(F) \neq \emptyset$, then one can always find a nonbipartite graph bundle $G \times^{\phi} F$ for any bipartite graph G which is not a tree.

COROLLARY 2.3. *Let F be a connected bipartite graph and G a nonbipartite graph. Then there exists a bipartite F-bundle over G if and only if $\mathcal{R}(F) \neq \emptyset$.*

PROOF: Suppose that there exists a bipartite F-bundle $G \times^{\phi} F$ over a nonbipartite graph G. Since G is not bipartite, it contains at least one odd cycle C. By Theorem 2.1, $\phi(C) \in \mathcal{R}(F)$ and hence $\mathcal{R}(F) \neq \emptyset$.

Conversely, let $\sigma \in \mathcal{R}(F)$. We define $\phi : D^+(G) \to \mathrm{Aut}\,(F)$ by $\phi(uv) = \sigma$ for each $uv \in D^+(G)$, where $D^+(G)$ is a subset of $D(G)$ consisting of all positively directed edges. Then ϕ can be extended to a voltage assignment in $C^1(G; \mathrm{Aut}\,(F))$ with the property that for each cycle C $\mathbf{sig}(\phi(C)) = (-1)^{\ell(C)}$. Theorem 2.1 implies that $G \times^{\phi} F$ is a bipartite F-bundle over G. ∎

3. ENUMERATION FORMULAS

Let $\mathrm{Iso}^B(G; F)$ denote the number of isomorphism classes of bipartite F-bundles over G. The following theorem is a direct consequence of Corollaries 2.2 and 2.3.

THEOREM 3.1. *Let F be a connected bipartite graph with $\mathcal{R}(F) = \emptyset$. Then we have*

$$\mathrm{Iso}^B(G; F) = \begin{cases} 0 & \text{if } G \text{ is nonbipartite,} \\ \mathrm{Iso}\,(G; F) & \text{if } G \text{ is bipartite,} \end{cases}$$

where Iso$(G;F)$ *denotes the number of isomorphism classes of* F*-bundles over* G.

Recall that the number Iso$(G;F)$ has been computed by Kwak and Lee [**6**]. In the following, we derive an enumeration formula for the number $\mathrm{Iso}^B(G;F)$ when F is a connected bipartite graph. For a connected bipartite graph F, we first consider the following set of Aut(F)-voltage assignments:

$$BC^1_T(G;\mathrm{Aut}(F)) = \left\{\phi \in C^1_T(G;\mathrm{Aut}(F)) \mid \mathbf{sig}(\phi(e)) = \mathbf{sig}(e) \text{ for all } e \in D(G) - D(T)\right\}.$$

This set is characterised by Theorem 2.1 and Corollary 1.2 as follows.

Lemma 3.2. *Let* F *be a connected bipartite graph. Then* $BC^1_T(G;\mathrm{Aut}(F))$ *contains all representatives of the isomorphism classes of bipartite* F*-bundles over* G.

We define an action of Aut(F) on $BC^1_T(G;\mathrm{Aut}(F))$ by $(\sigma \bullet \phi)(uv) = \sigma\phi(uv)\sigma^{-1}$ for all $uv \in D(G)$. This action is well-defined, since

$$\sigma\mathcal{P}(F)\sigma^{-1} = \mathcal{P}(F) \quad \text{and} \quad \sigma\mathcal{R}(F)\sigma^{-1} = \mathcal{R}(F)$$

for each automorphism $\sigma \in \mathrm{Aut}(F)$. The following is a direct consequence of Corollary 1.3.

Lemma 3.3. *Let* F *be a connected bipartite graph, and let* ϕ *and* ψ *be two voltage assignments in* $BC^1_T(G;\mathrm{Aut}(F))$. *Then two bipartite* F*-bundles* $G \times^{\phi} F$ *and* $G \times^{\psi} F$ *are isomorphic if and only if there exists an automorphism* σ *in* Aut(F) *such that* $\sigma \bullet \phi = \psi$, *that is,* ϕ *and* ψ *lie in the same orbit of the* Aut(F)*-action on* $BC^1_T(G;\mathrm{Aut}(F))$.

By the Burnside lemma and Lemmas 3.2 and 3.3, we have the following.

Theorem 3.4. *Let* F *be a connected bipartite graph. Then the number* $\mathrm{Iso}^B(G;F)$ *of isomorphism classes of bipartite* F*-bundles over* G *is*

$$\mathrm{Iso}^B(G;F) = \frac{1}{|\mathrm{Aut}(F)|} \sum_{\sigma \in \mathrm{Aut}(F)} |\mathrm{Fix}_\sigma|,$$

where $\mathrm{Fix}_\sigma = \left\{\phi \in BC^1_T(G;\mathrm{Aut}(F)) \mid \sigma \bullet \phi = \phi\right\}$.

The following lemma shows how to compute the number $|\mathrm{Fix}_\sigma|$. For $\sigma \in \mathrm{Aut}(F)$, let $Z(\sigma) = \{\mu \in \mathrm{Aut}(F) \mid \sigma\mu = \mu\sigma\}$, $Z_{\mathcal{P}}(\sigma) = \{\mu \in \mathcal{P}(F) \mid \sigma\mu = \mu\sigma\}$, and $Z_{\mathcal{R}}(\sigma) = \{\mu \in \mathcal{R}(F) \mid \sigma\mu = \mu\sigma\}$. Clearly, if $Z_{\mathcal{R}}(\sigma) \neq \emptyset$, then $|Z_{\mathcal{P}}(\sigma)| = |Z_{\mathcal{R}}(\sigma)|$.

Lemma 3.5. *Let* F *be a connected bipartite graph. Then for each* $\sigma \in \mathrm{Aut}(F)$,

$$\begin{aligned} |\mathrm{Fix}_\sigma| &= |Z_{\mathcal{P}}(\sigma)|^{\beta_e(G,T)} |Z_{\mathcal{R}}(\sigma)|^{\beta_o(G,T)} \\ &= \begin{cases} 0 & \text{if } |Z_{\mathcal{R}}(\sigma)| = 0 \text{ and } G \text{ is nonbipartite,} \\ |Z(\sigma)|^{\beta(G)} & \text{if } |Z_{\mathcal{R}}(\sigma)| = 0 \text{ and } G \text{ is bipartite,} \\ \left(\dfrac{|Z(\sigma)|}{2}\right)^{\beta(G)} & \text{otherwise.} \end{cases} \end{aligned}$$

By applying Theorems 4 and 5 in [6] and Lemma 3.5 to Theorem 3.4, we have the following.

THEOREM 3.6. *Let F be a connected bipartite graph such that $\mathrm{Aut}(F)$ is Abelian. Then*

$$\mathrm{Iso}^B(G;F) = \begin{cases} 0 & \text{if } G \text{ is nonbipartite and } \mathcal{R}(F) = \emptyset, \\ \mathrm{Iso}(G;F) & \text{if } G \text{ is bipartite and } \mathcal{R}(F) = \emptyset, \\ \mathrm{Iso}(G;F)/2^{\beta(G)} & \text{if } \mathcal{R}(F) \neq \emptyset, \end{cases}$$

where $\mathrm{Iso}(G;F) = |\mathrm{Aut}(F)|^{\beta(G)}$ as shown in [6].

4. Applications

As an application of our results, one can compute the number $\mathrm{Iso}^B(G;F)$ of the isomorphism classes of bipartite F-bundles over G when $F = P_n$ or C_n, where P_n is the path with n vertices and C_n is the cycle of length n.

1. Let $F = P_n$. Note that the automorphism group $\mathrm{Aut}(P_n)$ of P_n is isomorphic to the cyclic group $\{1, \alpha\}$ of order 2 and hence is Abelian. Moreover, $\mathcal{P}(P_n) = \mathrm{Aut}(P_n)$ and $\mathcal{R}(P_n) = \emptyset$ if n is odd, and $\mathcal{P}(P_n) = \{1\}$ and $\mathcal{R}(P_n) = \{\alpha\}$ if n is even.

By Theorem 3.6, we have the following.

THEOREM 4.1. *The number $\mathrm{Iso}^B(G;P_n)$ of the isomorphism classes of bipartite P_n-bundles over G is*

$$\mathrm{Iso}^B(G;P_n) = \begin{cases} 0 & \text{if } G \text{ is nonbipartite and } n \text{ is odd}, \\ 2^{\beta(G)} & \text{if } G \text{ is bipartite and } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}. \end{cases}$$

The following table shows the number $\mathrm{Iso}^B(G;P_n)$ for small n and $\beta(G)$:

	bipartite G							nonbipartite G					
$\beta(G)$	0	1	2	3	4	5	$\cdots$	1	2	3	4	5	$\cdots$
$n =$ odd	1	2	4	8	16	32	$\cdots$	0	0	0	0	0	$\cdots$
$n =$ even	1	1	1	1	1	1	$\cdots$	1	1	1	1	1	$\cdots$

2. Let $F = C_n$. Note that C_n is bipartite if and only if n is even. Hence, $\mathrm{Iso}^B(G;C_n) = 0$ if n is odd. Let n be an even number. Then $\mathrm{Aut}(C_n)$ can be identified with the dihedral group $\mathbb{D}_n$, which is generated by τ and ρ, where $\tau = (1\ \ n)(2\ \ n-1)\cdots\big((n/2)\ \ (n/2)+1\big)$ and $\rho = (1\,2\,\ldots\,n)$ in the symmetric group S_n on n elements $1, 2, \ldots, n$. Moreover,

$$\mathcal{P}(C_n) = \left\{\rho^{2i}, \tau\rho^{2i+1} \mid 1 \leqslant i \leqslant \frac{n}{2}\right\} \quad \text{and} \quad \mathcal{R}(C_n) = \left\{\rho^{2i+1}, \tau\rho^{2i} \mid 1 \leqslant i \leqslant \frac{n}{2}\right\}.$$

By a simple computation, we have the following.

Lemma 4.2. *Let n be an even number and $h = 0, \dots, n-1$. Then*

(a) $$|Z_{\mathcal{P}}(\rho^h)| = |Z_{\mathcal{R}}(\rho^h)| = \begin{cases} n & \text{if } h = 0 \text{ or } h = \frac{n}{2}, \\ n/2 & \text{otherwise.} \end{cases}$$

(b) $$|Z_{\mathcal{P}}(\tau\rho^h)| = \begin{cases} 4 & \text{if } n/2 \text{ is even and } h \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

(c) $$|Z_{\mathcal{R}}(\tau\rho^h)| = \begin{cases} 0 & \text{if } n/2 \text{ is even and } h \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 3.5 and Theorem 3.4 imply the following theorem.

Theorem 4.3. *Let $n \geqslant 3$ be a natural number.*

(a) *If n is odd, $\mathrm{Iso}^B(G; C_n) = 0$.*

(b) *If n is even and not a multiple of 4, then*

$$\mathrm{Iso}^B(G; C_n) = n^{\beta(G)-1} + \frac{n-2}{4}\left(\frac{n}{2}\right)^{\beta(G)-1} + 2^{\beta(G)-1}.$$

(c) *If n is a multiple of 4, then*

$$\mathrm{Iso}^B(G; C_n) = \begin{cases} n^{\beta(G)-1} + \frac{n-2}{4}\left(\frac{n}{2}\right)^{\beta(G)-1} + 2^{\beta(G)-2}\left(1 + 2^{\beta(G)}\right) & \text{if } G \text{ is bipartite,} \\ n^{\beta(G)-1} + \frac{n-2}{4}\left(\frac{n}{2}\right)^{\beta(G)-1} + 2^{\beta(G)-2} & \text{if } G \text{ is nonbipartite.} \end{cases}$$

The following table shows the number $\mathrm{Iso}^B(G; P_n)$ for small n and $\beta(G)$:

	bipartite G							nonbipartite G					
$\beta(G)$	0	1	2	3	4	5	$\cdots$	1	2	3	4	5	$\cdots$
$n =$ odd	0	0	0	0	0	0	$\cdots$	0	0	0	0	0	$\cdots$
$n = 4$	1	3	10	36	136	528	$\cdots$	2	6	20	72	272	$\cdots$
$n = 6$	1	3	11	49	251	1393	$\cdots$	3	11	49	251	1393	$\cdots$
$n = 8$	1	4	19	106	676	4744	$\cdots$	3	15	90	612	4488	$\cdots$
$n = 10$	1	4	22	154	1258	11266	$\cdots$	4	22	154	1258	11266	$\cdots$

References

[1] D. Archdeacon, J.H. Kwak, J. Lee and M.Y. Sohn, 'Bipartite covering graphs', (preprint).

[2] J.L. Gross and T.W. Tucker, 'Generating all graph coverings by permutation voltage assignments', *Discrete Math.* **18** (1977), 273–283.

[3] J.L. Gross and T.W. Tucker, *Topological graph theory* (Wiley, New York, 1987).

[4] M. Hofmeister, 'Counting double covers of graphs', *J. Graph Theory* **12** (1988), 437–444.

[5] J.H. Kwak, J.H. Chun and J. Lee, 'Enumeration of regular graph coverings having finite abelian covering transformation groups', *SIAM J. Discrete Math.* **11** (1998), 273–285.

[6] J.H. Kwak and J. Lee, 'Isomorphism classes of graph bundles', *Canad. J. Math.* **42** (1990), 747–761.

[7] B. Mohar, T. Pisanski and M. Škoviera, 'The maximum genus of graph bundles', *European J. Combin.* **9** (1988), 215–224.

[8] D.A. Waller, 'Double covers of graphs', *Bull. Austral. Math. Soc.* **14** (1976), 233–248.

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